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SUFFICIENT CONDITIONS FOR LOCAL SOLVABILITY OF LEFT INVARIANT LPDOS ON THE GROUPS OF TYPE H**

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Abstract

Some sufficient conditions are established for local solvability of inhomogeneous left invariant partial differential operators on the groups of type H. First some general sufficient conditions are derived for local solvablity of left invariant operators on the groups of type H. Then by using these results the author discusses the local solvability of a special class of inhomogeneous left invariant operators on this type of groups.

§ 1. Introduction

Local solvability of left invariant linear partial differential operators on nilpotent groups has been extensively studied since 1979. Many unitary representation theoretical criteria have been obtained by many authors such as L. P. Rothschild, L. Corwin, L. P. Greenleaf, D. Tartakoff, etc. (cf [1, 4—8] for instance). The only result published concerning necessary conditions for local solvability of general left invariant differential operators is that obtained by L. Corwin and L. P. Rothschild in [6] in the homogeneous case, which was recently generalized to inhomogeneous case by the author in [12]. The most useful sufficient conditions are those obtained in [1], [5], and [7], which say that a left invariant differential operator L is locally solvable if for any infinite dimensional irreducible unitary representation π_λ the induced group-Fourier transformation $\pi_\lambda(L)$ has right inverse B_λ, and the family of operators $\{B_\lambda\}$ are bounded in certain sense. This result is indeed effective in dealing with homogeneous operators as has been shown by many papers discussing the local solvability of specific left invariant differential operators on specific groups. But as inhomogeneous operators are concerned, their application is greatly limited because very many locally solvable

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inhomogeneous operators do not satisfy the conditions in this result (e. g., the class of operators discussed in [9]). Another important advancement on this topic is the study to transversally elliptic differential operators. For this kind of inhomogeneous left invariant differential operators, some both necessary and sufficient conditions for local solvability have been obtained (cf. [8] for instance).

The present paper is devoted to studying the local solvability of inhomogeneous left invariant linear partial differential operators on the groups of type H. The paper can be divided into two parts. The first part aims at obtaining some sufficient conditions for local solvability of general inhomogeneous left invariant differential operators on this type of groups. It extends the main result of [9] on the Heisenberg group to the groups of type H. Roughly speaking, this result says that if there is a positive number M such that for all $|\lambda|\geqslant M$ the operators $\pi_\lambda(L)$ have right inverse which are bounded in certain sense, then the operator L is locally solvable (see Theorem 1 and Theorem 2 in § 3 for the exact statements). This result improves that of [1], [5] and [7] at the point that the properties of the operators $\pi_\lambda(L)$ with $|\lambda|<M$ do not require considering, and thus it is more useful in dealing with inhomogeneous operators. In the second part we first extend the concept of controllable operators introduced in [10] with respect to the Heisenberg group to the groups of type H. Then we use our results obtained in the first part to derive some sufficient conditions for local solvability of this class of operators. We prove that this class of inhomogeneous operators are locally solvable if the group-Fourier transformations of their principal parts (which are homogeneous) satisfy some invertible conditions (exact statements are given in Theorem 3 and Theorem 4 in § 4). Since this class of operators includes the transversally elliptic operators, our second part of results partially generalizes the results of [8].

The plan of the present paper is as follows: In § 2 we introduce some concepts about the groups of type H and a class of Hilbert spaces, which are the background of the whole paper. In § 3 we prove our first part of results. In § 4 we introduce the concept of controllable operators and prove the sufficient conditions for local solvability of this class of operators mentioned above.

§ 2. Backgrounds and Notations

Let G be a simply connected step 2 nilpotent Lie group. Let g be the Lie algebra of G (which we identify with the Lie algebra of all left invariant vector fields on G). Set $g_2=[g, g]=\{[X, Y]\,|\,X\in g, Y\in g\}$. It is clear that g_2 is an ideal of g, included in the centre of g. Take a linear subspace g_1 of g such that $g=g_1\oplus g_2$. It is also clear that $g_2=[g_1, g_1]$. For any linear form $\eta\in g_2^*$, let B_η be the alternating bilinear form on g_1 defined by $B_\eta(X, Y)=\eta([X, Y])$, $\forall X, Y\in g_1$. Following [3], [4] and [8], we shall say that G is of type H if the bilinear form B_η is nondegenerate for any nonzero linear functional η on g_2, or equivalently, B_η is a symplectic form on g_1 for any $\eta\in g_2^*\backslash\{0\}$.

If G is of type H, then the codimension of g_2 (ie., the dimension of g_1) is clearly an even number. Set $n=\frac{1}{2}\operatorname{codim} g_2$, $k=\dim g_2$. Denote by Γ the set $R^k\backslash\{0\}$. We shall use these notations throughout the whole paper.

Take a basis $T_1, T_2, \cdots, T_k$ of g_2 and fix it. Let $\eta_1, \eta_2, \cdots, \eta_k$ be the dual basis of g_2^*. For any $\lambda\in\Gamma$, we relate it with an element $\eta_\lambda\in g_2^*\backslash\{0\}$ by setting $\eta_\lambda=\sum_{i=1}^{k}\lambda_i\eta_i$ if $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_k)$. Let S^{k-1} be the unit sphere of R^k. Choose for every $\omega\in S^{k-1}$ a linear basis $X_1^\omega, X_2^\omega, \cdots, X_n^\omega, Y_1^\omega, Y_2^\omega, \cdots, Y_n^\omega$ of g_1 such that

$$B_{\eta_\omega}(X_i^\omega, Y_j^\omega)=\delta_{ij},\ B_{\eta_\omega}(X_i^\omega, X_j^\omega)=B_{\eta_\omega}(Y_i^\omega, Y_j^\omega)=0,\quad i, j=1, 2, \cdots, n. \tag{2.1}$$

Moreover, we need the basis $X_1^\omega, X_2^\omega, \cdots, X_n^\omega, Y_1^\omega, Y_2^\omega, \cdots, Y_n^\omega$ to vary analyticly when ω varies. Locally, this is possible. Indeed, we have

Lemma 1. *For every $\omega_0\in S^{k-1}$, there exists a corresponding neighborhood $S_{\omega_0}\subset S^{k-1}$ of ω_0 such that there are analytic functions $a_{ij}(\omega)$ $b_{ij}(\omega)$, $c_{ij}(\omega)$ and $d_{ij}(\omega)$ ($i, j=1, 2, \cdots, n$) on S_{ω_0} which have the property that if we set*

$$X_i^{\omega,\omega_0}=\sum_{i,j=1}^{n}(a_{ij}(\omega)X_j^{\omega_0}+b_{ij}(\omega)Y_j^{\omega_0}),$$

$$Y_i^{\omega,\omega_0}=\sum_{i,j=1}^{n}(c_{ij}(\omega)X_j^{\omega_0}+d_{ij}(\omega)Y_j^{\omega_0}),$$

$$i=1, 2, \cdots, n,$$

then $X_1^{\omega,\omega_0}, X_2^{\omega,\omega_0}, \cdots, X^{\omega,\omega_0}, Y_1^{\omega,\omega_0}, Y_2^{\omega,\omega_0}, \cdots, Y_n^{\omega,\omega_0}$ is a basis of g_1 and for any $\omega\in S_{\omega_0}$ the following holds:

$$B_{\eta_\omega}(X_i^{\omega,\omega_0}, Y_j^{\omega,\omega_0})=\delta_{ij},\ B_{\eta_\omega}(X_i^{\omega,\omega_0}, X_j^{\omega,\omega_0})=B_{\eta_\omega}(Y_i^{\omega,\omega_0}, Y_j^{\omega,\omega_0})=0,$$
$$i, j=1, 2, \cdots, n,$$

and furthermore, $a_{ij}(\omega_0)=d_{ij}(\omega_0)=\delta_{ij}$, $b_{ij}(\omega_0)=c_{ij}(\omega_0)=0$, $i, j=1, 2, \cdots, n$.

Proof Set

$$A(\omega)=\begin{bmatrix} B_{\eta_\omega}(X_i^{\omega_0}, X_j^{\omega_0}) & B_{\eta_\omega}(X_i^{\omega_0}, Y_j^{\omega_0}) \\ B_{\eta_\omega}(Y_i^{\omega_0}, X_j^{\omega_0}) & B_{\eta_\omega}(Y_i^{\omega_0}, Y_j^{\omega_0}) \end{bmatrix}.$$

This is a real matrix of order $2n\times 2n$ depending analyticly on $\omega\in S^{k-1}$. Moreover, it is obvious that $A(\omega)$ is anti-symmetric and $A(\omega_0)=J\equiv\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}$. Thus by using the standard matrix theory we conclude that there is a neighborhood $S_{\omega_0}\subset S^{k-1}$ of ω_0 and an analytic matrix $C(\omega)$ on S_{ω_0} such that $C(\omega)A(\omega)C(\omega)^t=J$, $C(\omega_0)=I$. If we write $C(\omega)=\begin{bmatrix} a_{ij}(\omega) & b_{ij}(\omega) \\ c_{ij}(\omega) & d_{ij}(\omega)\end{bmatrix}$, then it can be easily verified that $a_{ij}(\omega)$, $b_{ij}(\omega)$, $c_{ij}(\omega)$, $d_{ij}(\omega)$ are the functions required. The proof is finished.

For any $\lambda\in\Gamma$, denote by $|\lambda|$ the Euclidean norm of λ and set $\omega(\lambda)=\frac{\lambda}{|\lambda|}$. Let $\omega_0\in S^{k-1}$. Let $S_{\omega_0}\subset S^{k-1}$ be the neighborhood of ω_0 as in Lemma 1. Let Γ_{ω_0} be the open cone spreaded by S_{ω_0}, i. e., $\Gamma_{\omega_0}=\{\lambda\in\Gamma|\omega(\lambda)\in S_{\omega_0}\}$. We define a family of irreducible unitary representations $\{\pi_\lambda^{\omega_0}|\lambda\in\Gamma_{\omega_0}\}$ as follows. First, take the symplectic basis $X_1^{\omega,\omega_0}, X_2^{\omega,\omega_0},\cdots,X_n^{\omega,\omega_0}, Y_1^{\omega,\omega_0}, Y_2^{\omega,\omega_0},\cdots,Y_n^{\omega,\omega_0}$ of g_1 as in Lemma 1, where $\omega\in S_{\omega_0}$. Then for any $X\in g$ define $\pi_\lambda^{\omega_0}(\exp X)$ to be the following unitary operator on $L^2(R^n)$:

$$\pi_\lambda^{\omega_0}(\exp X)f(\xi)=\exp\Big(i\Big(\sum_{l=1}^k\lambda_lt_l+|\lambda|^{\frac12}\sum_{j=1}^n\xi_jy_j+\frac12|\lambda|\sum_{j=1}^n x_jy_j\Big)\Big)f(\xi+|\lambda|^{\frac12}x),$$
$$\forall f\in L^2(R^n),$$
$$\text{if } X=\sum_{j=1}^n(x_jX_j^{\omega,\omega_0}+y_jY_j^{\omega,\omega_0})+\sum_{l=1}^k t_lT_l.$$

It is easy to verify that the Lie-algebra representation of g induced by $\pi_\lambda^{\omega_0}$ (which we denote by the same notation) has the following property:

$$\pi_\lambda^{\omega_0}(X_j^{\omega(\lambda),\omega_0})=|\lambda|^{\frac12}\frac{\partial}{\partial\xi_j},\ \pi_\lambda^{\omega_0}(Y_j^{\omega(\lambda),\omega_0})=i|\lambda|^{\frac12}\xi_j,\ j=1,2,\cdots,n,$$
$$\pi_\lambda^{\omega_0}(T_l)=i\lambda_l,\quad l=1,2,\cdots,k. \tag{2.2}$$

In especial,

$$\pi_{\omega_0}^{\omega_0}(X_j^{\omega_0})=\frac{\partial}{\partial\xi_j},\quad \pi_{\omega_0}^{\omega_0}(Y_j^{\omega_0})=i\xi_j,\quad j=1,2,\cdots,n. \tag{2.3}$$

We shall call the induced representation of $\pi_\lambda^{\omega_0}$ on the complex universal enveloping algebra of g (i. e., the set of all left invariant partial differential operators on G) the group-Fourier transformation on G, and denote it by the same notation.

Denote by $\phi_\alpha(\xi)$ the Hermitian function on R^n with index $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in Z_+^n$, i.e.,

$$\phi_\alpha(\xi)=\prod_{j=1}^n(2^{\alpha_j}\alpha_j!)^{-\frac12}\pi^{-\frac14}e^{\frac12\xi_j^2}\Big(\frac{\partial}{\partial\xi_j}\Big)^{\alpha_j}e^{-\xi_j^2},$$

It is clear that $\phi_\alpha(\xi)\in S(R^n)$, $\forall\alpha\in Z_+^n$. As in [11], for every real number s, we introduce a function space $H^s(R^n)$ as follows:

$$H^s(R^n)=\{u(\xi)=\sum_\alpha a_\alpha\phi_\alpha(\xi)(S'(R^n))\,|\,\|u\|_{H^s}^2=\sum_\alpha(1+|\alpha|)^s|a_\alpha|^2<+\infty\}.$$

This class of spaces was systematically studied in [11]. It is proved that for any real number s, $H^s(R^n)$ is a Hilbert space, and $H^r(R^n)\subset H^s(R^n)$ if $r\geqslant s$. Moreover, we have the following conclusions:

(i) $\bigcup_s H^s(R^n)=S'(R^n)$, $\bigcap_s H^s(R^n)=S(R^n)$.

(ii) If m is a nonnegative integer, then

$$H^m(R^n)=\{u(\xi)\in S'(R^n)\,|\,\xi^\alpha D^\beta u(\xi)\in L^2(R^n),\ \forall|\alpha+\beta|\leqslant m\},$$

and $H^m(R^n)$ has an equivalent norm $\|u\|'_{H^m}=(\sum_{|\alpha+\beta|\leqslant m}\|\xi^\alpha D^\beta u(\xi)\|_{L^2}^2)^{\frac12}$.

For every real number s, we introduce an operator Λ^s: $S'(R^n)\to S'(R^n)$ as follows:

$$\Lambda^s u(\xi)=\sum_{\alpha}(1+|\alpha|)^{\frac{s}{2}}a_\alpha\phi_\alpha(\xi),\quad \text{if } u(\xi)=\sum_{\alpha}a_\alpha\phi_\alpha(\xi).$$

This is a linear topological isomorphism on $S'(R^n)$. Its restriction on $S(R^n)$ is also a linear topological isomorphism. Furthermore, the restriction of Λ^s on every $H^r(R^n)$ is an isomorphism of $H^r(R^n)$ onto $H^{r-s}(R^n)$. Set

$$E_j^+=\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\xi_j}-\xi_j\Big),\quad E_j^-=-\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\xi_j}+\xi_j\Big),$$

$$N_j=\frac{1}{2}\Big(-\frac{\partial^2}{\partial\xi_j^2}+\xi_j^2-1\Big)=E_j^+E_j^-=E_j^-E_j^+-I,\quad j=1,2,\cdots,n. \tag{2.4}$$

One may easily verify

$$E_j^+\phi_\alpha=\sqrt{1+\alpha_j}\,\phi_{\alpha+e_j},\ E_j^-\phi_\alpha=\sqrt{\alpha_j}\,\phi_{\alpha-e_j},\ N_j\phi_\alpha=\alpha_j\phi_\alpha, \tag{2.5}$$

for any $j=1, 2, \cdots, n$. Here e_j denotes the index whose j-th coordinate is 1 and the other coordinates are zero. We have used the notation $\phi_{\alpha-e_j}=0$ if $\alpha_j=0$. Denote $E_+^\alpha=\prod_{j=1}^n(E_j^+)^{\alpha_j}$, $E_-^\alpha=\prod_{j=1}^n(E_j^-)^{\alpha_j}$. We have

Lemma 2. *For every positive number m, there exist corresponding continuous linear operators $F_{\alpha\beta}^{(m)}$ and $\bar{F}_{\alpha\beta}^{(m)}(|\alpha+\beta|\leq m)$, whose restrictions on $H^r(R^n)$ are bounded for any $r\in(-\infty, +\infty)$, such that*

$$\Lambda^m=\sum_{|\alpha+\beta|\leq\infty}F_{\alpha\beta}^{(m)}E_+^\alpha E_-^\beta=\sum_{|\alpha+\beta|\leq m}E_+^\alpha E_-^\beta\bar{F}_{\alpha\beta}^{(m)}.$$

Proof One may easily prove that the following holds:

$$\Lambda^2=I+\sum_{j=1}^n E_j^+E_j^-=-(n-1)I+\sum_{j=1}^n E_j^-E_j^+,$$

$$[E_i^+, E_j^-]=-\delta_{ij}I,\qquad i, j=1, 2, \cdots, n.$$

Moreover, if we denote by Q_j^+ and Q_j^- the continuous linear operators defined by

$$Q_j^+u=\sum_{\alpha}\lambda_\alpha a_\alpha\phi_{\alpha+e_j},\ Q_j^-u=\sum_{\alpha\geq e_j}\lambda_\alpha a_\alpha\phi_{\alpha-e_j},\ \forall u=\sum_{\alpha}a_\alpha\phi_\alpha,$$

where $\lambda_\alpha=\sqrt{1+|\alpha|}\Big/\sum_{j=1}^n\sqrt{1+\alpha_j}$, then we have

$$\Lambda^1=\sum_{j=1}^n Q_j^-E_j^+=\sum_{j=1}^n E_j^-Q_j^+.$$

Noticing that the restrictions of Q_j^+ and Q_j^- on every $H^r(R^n)$ are bounded, we come to our conclusion immediately.

To end this section, we remind the reader to notice that for any $j=1, 2, \cdots, n$, the following equalities hold:

$$\pi_{\alpha_0}^{\omega_0}(X_j^{\omega_0}+iY_j^{\omega_0})=\sqrt{2}\,E_j^+,\ \pi_{\alpha_0}^{\omega_0}(X_j^{\omega_0}-iY_j^{\omega_0})=-\sqrt{2}\,E_j^-,$$

$$\pi_{\alpha_0}^{\omega_0}((X_j^{\omega_0})^2+(Y_j^{\omega_0})^2-i[X_j^{\omega_0}, Y_j^{\omega_0}])=\frac{\partial^2}{\partial\xi_j^2}-\xi_j^2+1=-2N_j,$$

which are the immediate consequence of (2.3) and (2.4).

§ 3. Some General Results for Local Solvability

Let the notations S_{ω_0} and Γ_{ω_0} be as before. Let K be the Hilbert space of all Hilbert–Schmidt operators on $L^2(R^n)$ with norm $\|A\|_{HS}=\sqrt{\operatorname{tr}(AA^*)}$. Denote by $L^0(\Gamma_{\omega_0}, K)$ the set of all maps A: $\Gamma_{\omega_0}\to K$ with property that for any φ, $\psi\in S(R^n)$, the function $f(\lambda)=(A(\lambda)\varphi, \psi)$ is a measurable function on Γ_{ω_0}, where $(\cdot, \cdot)$ denotes the inner product on $L^2(R^n)$. For a given linear map P: $S(R^n)\to S'(R^n)$, we denote by $\operatorname{tr}P$ the sum $\sum_\alpha (P\phi_\alpha, \phi_\alpha)$ if this summation is convergent, where (u, v) denotes the dual operation of $u\in S'(R^n)$ to the function $v\in S(R^n)$.

Lemma 3. *Let $A\in L^0(\Gamma_{\omega_0}, K)$. Suppose that there exist constants $C\geqslant 0$, $M>0$ and $p\in(-\infty, +\infty)$ such that*

$$\|A(\lambda)\|_{HS}\leqslant C|\lambda|^p, \quad \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Then for any $\psi(\lambda)\in C_0^\infty(R^k)$ taking the value 1 identically when $|\lambda|\leqslant M+1$ and any $\varphi(\lambda)\in C^\infty(\Gamma_{\omega_0})$ homogeneous of degree $r<-p-\frac{1}{2}(n+k)$ whose restriction on S_{ω_0} has compact support, there exists a unique function $u\in L^2(G)$ such that

$$\langle u, v\rangle=\int_{\Gamma_{\omega_0}}(1-\psi(\lambda))\varphi(\lambda)\operatorname{tr}(\pi_\lambda^{\omega_0}(v)A(\lambda))|\lambda|^n d\lambda, \quad \forall v\in L^1(G)\cap L^2(G),$$

where $\langle u, v\rangle=\int_G u(\sigma)v(\sigma)d\sigma$, and $d\sigma$ is the Haar measure on G.

The proof of this lemma is similar to that of Lemma 2 of [9], with a little modification. We omit it here.

Let m be a nonnegative integer. Denote by $W^m(G)$ the Hilbert space of all distributions u on G such that $Lu\in L^2(G)$ for any left invariant linear partial differential operator L of order $\leqslant m$. Take a basis $Z_1, Z_2, \cdots, Z_{2n}$ of g_1, and set $\mathscr{A}_0=\{I\}$, $\mathscr{A}_m=\mathscr{A}_0\cup\{Z_{j_1}Z_{j_2}\cdots Z_{j_k}|1\leqslant j_1, j_2, \cdots, j_k\leqslant 2n;\ 1\leqslant k\leqslant m\}$. Then $W^m(G)$ has an inner product as follows: $(u, v)=\sum_{L\in\mathscr{A}_m}(Lu, Lv)$, $\forall u, v\in W^m(G)$, where $(\cdot, \cdot)$ denotes the inner product of $L^2(G)$.

Lemma 4. *Let m be a nonnegative integer. Let $A\in L^0(\Gamma_{\omega_0}, K)$. Suppose that for any partial differential operator P of the form $P=\sum_{|\alpha+\beta|\leqslant m} a_{\alpha\beta}\xi^\alpha D^\beta$ ($a_{\alpha\beta}$ are independent of ξ), the composition operator $PA(\lambda)$ is a Hilbert–Schmidt operator on $L^2(R^n)$ for λ a. e. Γ_{ω_0}, and moreover, there exist constants $p\in(-\infty, +\infty)$ and $M>0$ independent of P, and another constant $C_P\geqslant 0$ depending on P, such that*

$$\|PA(\lambda)\|_{HS}\leqslant C_P|\lambda|^p, \ \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Then for any $\psi(\lambda)\in C_0^\infty(R^k)$ taking the value 1 identically when $|\lambda|\leqslant M+1$ and any $\varphi(\lambda)\in C^\infty(\Gamma_{\omega_0})$ homogeneous of degree $r<-\frac{m}{2}-p-\frac{1}{2}(n+k)$ and whose restriction *on S_{ω_0} has compact support, there exists a unique $u\in W^m(G)$ such that*

$$\langle u, v\rangle=\int_{\Gamma_{\omega_0}}(1-\psi(\lambda))\varphi(\lambda)\mathrm{tr}(\pi_\lambda^{\omega_0}(v)A(\lambda))|\lambda|^n d\lambda,\quad \forall v\in L^1(G)\cap L^2(G).$$

The proof of this lemma is similar to that of Lemma 3 of [9], with a little modification. We omit it here.

Remark. For $s\in(-\infty,+\infty)$, let K^s denote the set of bounded linear operators B: $L^2(R^n)\to H^s(R^n)$ such that $\Lambda^s B$ is a Hilbert-Schmidt operator on $L^2(R^n)$, and let $L^0(\Gamma_{\omega_0}, K^s)$ be the set of all maps A: $\Gamma_{\omega_0}\to K^s$ with the property that for any φ, $\psi\in S(R^n)$, the function $f(\lambda)=(A(\lambda)\varphi, \psi)$ is measurable on Γ_{ω_0}. Then the condition in Lemma 4 is equivalent to the following:

$A\in L^0(\Gamma_{\omega_0}, K^m)$ and there exist constants $C\geqslant 0$, $M>0$ and $p\in(-\infty, +\infty)$ such that

$$\|\Lambda^m A(\lambda)\|_{HS}\leqslant C|\lambda|^p,\ \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M). \tag{3.1}$$

Indeed, if $A\in L^0(\Gamma_{\omega_0}, K^m)$ then $A\in L^0(\Gamma_{\omega_0}, K)$ since $A(\lambda)=\Lambda^{-m}\Lambda^m A(\lambda)$ for λ a. e. Γ_{ω_0} and Λ^{-m} is bounded on $L^2(R^n)$. Furthermore, if (3.1) holds, then for any partial differential operator $P=\sum_{|\alpha+\beta|\leqslant m} a_{\alpha\beta}\xi^\alpha D^\beta$ we have

$$\|PA(\lambda)\|_{HS}\leqslant\|P\Lambda^{-m}\|\,\|\Lambda^m A(\lambda)\|_{HS}\leqslant C'|\lambda|^p,\ \ \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

One may easily prove the converse by using Lemma 2.

In the following, we denote by $W^{-m}(G)$ the dual space of $W^m(G)$ for nonnegative integer m.

Lemma 5. *Let m be a nonnegative integer. Let $A\in L^0(\Gamma_{\omega_0}, K^{-m})$. Suppose that there exist constants $C\geqslant 0$, $M>0$ and $p\in(-\infty, +\infty)$ such that*

$$\|\Lambda^{-m}A(\lambda)\|_{HS}\leqslant C|\lambda|^p,\quad \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Then for any $\psi(\lambda)\in C_0^\infty(R^k)$ taking the value 1 identically when $|\lambda|\leqslant M+1$ and any $\varphi(\lambda)\in C^\infty(\Gamma_{\omega_0})$ homogeneous of degree $r<\frac{m}{2}-p-\frac{1}{2}(n+k)$ and whose restriction on S_{ω_0} has compact support, the equality

$$\langle u, v\rangle=\int_{\Gamma_{\omega_0}}(1-\psi(\lambda))\varphi(\lambda)\mathrm{tr}(\pi_\lambda^{\omega_0}(v)A(\lambda))|\lambda|^n d\lambda,\quad \forall v\in L^1(G)\cap W^m(G) \tag{3.2}$$

defines a element $u\in W^{-m}(G)$, if the neighborhood $S_{\omega_0}\subset S^{k-1}$ of ω_0 is sufficiently small.

Proof We claim that for some neighborhood $S'_{\omega_0}\subset S_{\omega_0}$, there exist finite many homogeneous left invariant differential operators $Q_j(j=1, 2, \cdots, N)$ of degree m and corresponding bounded linear operators B_j: $H^{-m}(R^n)\to L^2(R^n)(j=1, 2, \cdots, N)$ depending on $\omega\in S'_{\omega_0}$ analyticly, such that

$$\sum_{j=1}^N \pi_\omega^{\omega_0}(Q_j)B_j^\omega=I,\quad \forall\omega\in S'_{\omega_0}. \tag{3.3}$$

(Notice that $\pi_\omega^{\omega_0}(Q_j)$ maps $L^2(R^n)$ into $H^{-m}(R^n)$ boundedly). If this claim is proved, then from the equalities

$$\pi_\lambda^{\omega_0}({}^tQ_jv)=\pi_\lambda^{\omega_0}(v)\pi_\lambda^{\omega_0}(Q_j)=|\lambda|^{\frac{1}{2}m}\pi_\lambda^{\omega_0}(v)\pi_{\omega(\lambda)}^{\omega_0}(Q_j).$$

We get the following

$$\begin{aligned}|\mathrm{tr}(\pi_\lambda^{\omega_0}(v)A(\lambda))|&=|\lambda|^{-\frac{1}{2}m}\Big|\sum_{j=1}^N\mathrm{tr}(\pi_\lambda^{\omega_0}({}^tQ_jv)B_j^{\omega(\lambda)}A(\lambda))\Big|\\&\leqslant|\lambda|^{-\frac{1}{2}m}\sum_{j=1}^N\|\pi_\lambda^{\omega_0}({}^tQ_jv)\|_{HS}\cdot\|B_j^\omega A(\lambda)\|_{HS}\quad(\text{here }\omega=\omega(\lambda))\\&\leqslant|\lambda|^{-\frac{1}{2}m}\sum_{j=1}^N\|\pi_\lambda^{\omega_0}({}^tQ_jv)\|_{HS}\cdot\|B_j^\omega\Lambda^m\|_{L^2}\cdot\|\Lambda^{-m}A(\lambda)\|_{HS}\\&\leqslant C'|\lambda|^{-\frac{1}{2}m+p}\sum_{j=1}^N\|\pi_\lambda^{\omega_0}({}^tQ_jv)\|_{HS},\quad\forall v\in L^1(G)\cap W^m(P).\end{aligned}$$

Therefore, since $\varphi(\lambda)$ belongs to $C^\infty(\Gamma_{\omega_0})$ and is homogeneous of degree r, we get the following

$$\begin{aligned}&\int_{\Gamma_{\omega_0}}|1-\psi(\lambda)||\varphi(\lambda)||\mathrm{tr}(\pi_\lambda^{\omega_0}(v)A(\lambda))||\lambda|^nd\lambda\\&\leqslant C'\sum_{j=1}^N\int_{\Gamma_{\omega_0}}|1-\psi(\lambda)||\lambda|^{-\frac{1}{2}m+p+r+\frac{1}{2}n}\cdot\|\pi_\lambda^{\omega_0}({}^tQ_jv)\|_{HS}|\lambda|^{\frac{1}{2}n}d\lambda\\&\leqslant C'\Big(\int_{\Gamma_{\omega_0}}|1-\psi(\lambda)|^2|\lambda|^{-m+2p+2r+n}d\lambda\Big)^{\frac{1}{2}}\cdot\sum_{j=1}^N\Big(\int_{\Gamma_{\omega_0}}\|\pi_\lambda^{\omega_0}({}^tQ_jv)\|_{HS}^2|\lambda|^nd\lambda\Big)^{\frac{1}{2}}\\&\leqslant C''\sum_{j=1}^N\|{}^tQ_jv\|_{L^2}\leqslant C'''\|v\|_{W^m},\quad\forall v\in L^1(G)\cap W^m(G).\end{aligned}$$

From this and the denseness of $L^1(G)\cap W^m(G)$ in $W^m(G)$, we see that equality (3.2) defines a unique continuous linear functional u on $W^m(G)$, and the conclusion of Lemma 4 is then proved. Hence, to finish the proof, we need only to prove our claim on (3.3).

By Lemma 2, we see that there exist bounded linear operators $\bar F_{\alpha\beta}$: $L^2(R^n)\to L^2(R^n)$ $(|\alpha+\beta|\leqslant m)$ such that

$$\Lambda^m=\sum_{|\alpha+\beta|\leqslant m}E_+^\alpha E_-^\beta\bar F_{\alpha\beta},$$

or equivalently,

$$I=\sum_{|\alpha+\beta|\leqslant m}E_+^\alpha E_-^\beta\bar F_{\alpha\beta}\Lambda^{-m}.\tag{3.4}$$

Set

$$Q_{\alpha\beta}=(-1)^{|\beta|}\Big(\frac{1}{\sqrt{2}}\Big)^{|\alpha+\beta|}\prod_{j=1}^n(X_j^{\omega_0}+iY_j^{\omega_0})^{\alpha_j}\prod_{j=1}^n(X_j^{\omega_0}-iY_j^{\omega_0})^{\beta_j},$$

$$B_{\alpha\beta}=\bar F_{\alpha\beta}\Lambda^{-m}.$$

Then we see that $B_{\alpha\beta}(|\alpha+\beta|\leqslant m)$ are bounded linear operators mapping $H^{-m}(R^n)$ into $L^2(R^n)$, and furthermore, by applying equalities (2.6) and (3.4) we get

$$\sum_{|\alpha+\beta|\leqslant m}\pi_{\omega_0}^{\omega_0}(Q_{\alpha\beta})\circ B_{\alpha\beta}=I.$$

Denote

$$R(\omega)=\sum_{|\alpha+\beta|\leqslant m}(\pi_\omega^{\omega_0}(Q_{\alpha\beta})-\pi_{\omega_0}^{\omega_0}(Q_{\alpha\beta}))\circ B_{\alpha\beta},\quad\forall\omega\in S_{\omega_0}.$$

Since $\pi_\omega^{\omega_0}(Q_{\alpha\beta})$ is a partial differential operator of the form

$$\pi_{\omega}^{\omega_0}(Q_{\alpha\beta})=\sum_{|\alpha'+\beta'|<m} a_{\alpha'\beta'}^{(\alpha\beta)}(\omega)\xi^{\alpha'}D^{\beta'}$$

with coefficients $a_{\alpha'\beta'}^{(\alpha\beta)}(\omega)$ being analytic functions on S_{ω_0}, we see that $R(\omega)$ is a bounded linear operator on $H^{-m}(R^n)$ and depends on $\omega\in S_{\omega_0}$ analyticly. Hence, since $R(\omega_0)=0$, we conclude that for some neighborhood $S'_{\omega_0}\subset S_{\omega_0}$ of ω_0, $\|R(\omega)\|_{H^{-m}}\leqslant\frac{1}{2}$, $\forall\omega\in S'_{\omega_0}$, and thus the inverse operator $V(\omega)=(I+R(\omega))^{-1}$ exists when $\omega\in S'_{\omega_0}$ and moreover, $V(\omega)$ depends on $\omega\in S'_{\omega_0}$ analyticly and

$$\|V(\omega)\|_{H^{-m}}\leqslant(1-\|R(\omega)\|_{H^{-m}})^{-1}\leqslant 2,\quad \forall\omega\in S'_{\omega_0}.$$

Now set $B_{\alpha\beta}^{\omega}=B_{\alpha\beta}\circ V(\omega)$, for $\omega\in S'_{\omega_0}$. Then we see that $B_{\alpha\beta}^{\omega}$ maps $H^{-m}(R^n)$ into $L^2(R^n)$ boundedly and depends on $\omega\in S'_{\omega_0}$ analyticly. Furthermore, one may easily verify the following

$$\sum_{|\alpha+\beta|<m}\pi_{\omega}^{\omega_0}(Q_{\alpha\beta})B_{\alpha\beta}^{\omega}=I,\quad \forall\omega\in S'_{\omega_0}.$$

This proves our claim on (3.3). The proof of Lemma 5 is ended.

The first main result is the following

Theorem 1. *Let L be a left invariant differential operator on G, which is a group of type H. Suppose that there are positive number M and integer s such that for any $\omega_0\in S^{k-1}$ there exists corresponding $A_{\omega_0}\in L^0(\Gamma_{\omega_0}, K^s)$ satisfying the following two conditions:*

(i) $\pi_{\lambda}^{\omega_0}(L)A_{\omega_0}(\lambda)f=f$, $\forall f\in S(R^n)$, λ a.e. $\Gamma_{\omega_0}(|\lambda|\geqslant M)$.

(ii) *There are constants $C\geqslant 0$ and $p\in(-\infty,+\infty)$ such that*

$$\|\Lambda^s A_{\omega_0}(\lambda)\|_{HS}\leqslant C|\lambda|^p,\quad \lambda \text{ a. e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Then L is locally solvable. More precisely, there is a distribution $E\in W^m(G)$ such that $LE=\delta$ in a neighborhood of the unit element, where $m=\min(s,-[2p]-n-h)$.

Proof First we select finite many points $\omega_j\in S^{k-1}(j=1, 2, \cdots, N)$ such that $\bigcup_{j=1}^{N}S_{\omega_j}=S^{k-1}$. If $m<0$, we need S_{ω_j} to be chosen sufficienely small so that the conclusion of Lemma 5 holds. Let $\{\varphi_j(\omega)\,|\,j=1, 2, \cdots, N\}$ be a partition of unit of S^{k-1} corresponding to the covering $\{S_{\omega_j}|j=1, 2, \cdots, N\}$. Denote $A_j=A_{\omega_j}$, $j=1, 2, \cdots, N$. Choose a function $\psi(\lambda)\in C_0^\infty(R^k)$ such that it is identically 1 in the region $|\lambda|\leqslant M+1$. Then by Lemma 4 and Lemma 5, we see that there exist distributions $E_j\in W^m(G)(j=1, 2, \cdots, N)$ such that

$$\langle E_j, v\rangle=\int_{\Gamma_{\omega_j}}(1-\psi(\lambda))\varphi_j\Big(\frac{\lambda}{|\lambda|}\Big)\mathrm{tr}(\pi_{\lambda}^{\omega_j}(v)A_j(\lambda))|\lambda|^n d\lambda,$$
$$\forall v\in C_0^\infty(G),\ j=1, 2, \cdots, N.$$

Let $E_0=\sum_{j=1}^{N}E_j$. Then $E_0\in W^m(G)$ and one may easily prove that E_0 satisfies the following:

$$\langle LE_0, v\rangle=\int_{\Gamma}(1-\psi(\lambda))\Big(\sum_{j=1}^{N}\varphi_j\Big(\frac{\lambda}{|\lambda|}\Big)\mathrm{tr}\,\pi_\lambda^{\omega_j}(v)\Big)|\lambda|^n d\lambda$$
$$=v(0)-\int_{R^k}\psi(\lambda)\hat{v}(0,\lambda)d\lambda,\quad \forall v\in C_0^\infty(G),$$

where $\hat{v}(0, \lambda)=\int_{g_2} v(\exp T)e^{i\eta_\lambda(T)}dT$, $\forall\lambda\in\Gamma$. Let $G_2=\exp g_2$, $G_1=\exp g_1$. Since g_2 is a commutative Lie algebra and the exponential map of g onto G is injection, we see that G_2 is a simply connected commutative Lie subgroup of G, i. e., it can be identified with R^k. Furthermore, G_1 is a submanifold of G and we have $G=G_1\otimes G_2$ in the topological sense. Since $\psi\in C_0^\infty(R^k)$, we see by using the inverse Fourier transformation that there is $w\in S(G_2)$ such that $\psi(\lambda)=\int_{g_2} w(\exp T)e^{i\eta_\lambda(T)}dT$, $\forall\lambda\in\Gamma$. Thus the above equalities imply the following:

$$LE_0=\delta-\delta_1\otimes w \text{ on } G,$$

where δ_1 is the Dirac's δ-distribution on G_1 with support at the unit element. From this, we immediately come to our conclusion by following a similar deduction as in the proof of Theorem 1 in [9].

For any real number s, let B^s be the set of all linear maps mapping $L^2(R^n)$ into $H^s(R^n)$ boundedly, and let $L^0(\Gamma_{\omega_0}, B^s)$ be the set of all maps A: $\Gamma_{\omega_0}\to B^s$ such that $f(\lambda)=(A(\lambda)\varphi, \psi)$ is measurable on Γ_{ω_0} for any $\varphi, \psi\in S(R^n)$. Our second result can be satated as follows:

Theorem 2. *Let L and G be as in Theorem 1. Suppose that there are positive number M and integer s such that for any $\omega_0\in S^{k-1}$ there exists corresponding $A_{\omega_0}\in L^0(\Gamma_{\omega_0}, B^s)$ satisfying the following:*

(i) $\pi_\lambda^{\omega_0}(L)A_{\omega_0}(\lambda)f=f$, $\forall f\in S(R^n)$, λ a.e. $\Gamma_{\omega_0}(|\lambda|\geqslant M)$.

(ii) *There are constants $C\geqslant 0$ and $p\in(-\infty, +\infty)$ such that*

$$\|\Lambda^s A_{\omega_0}(\lambda)\|\leqslant C|\lambda|^p,\quad \lambda \text{ a. e. } \Gamma(|\lambda|\geqslant M).$$

Then there exists a distribution $E\in W^m(G)$, where $m=\min(s-n-1, -[2p]-n-k)$, such that $LE=\delta$ in a neighborhood of the unit element, and thus L is locally solvable.

Proof In fact, one may easily prove that Λ^{-n-1} is a Hilbert-Schmidt operator on $L^2(R^n)$. Thus by applying Theorem 1 and the fact that $\|\Lambda^{s-n-1}A_{\omega_0}(\lambda)\|_{HS}\leqslant\|\Lambda^{-n-1}\|_{HS}\cdot\|\Lambda^s A_{\omega_0}(\lambda)\|$, we get Theorem 2.

§ 4. Controllable Operators

In this section we are going to use Theorem 1 and Theorem 2 to discuss the local solvability of a special class of inhomogeneous left invariant partial differential operators on the groups of type H. Let us first introduce

Definition 1. *Let Ω be a subset of R^k. Let $P_\lambda=P_\lambda(\xi, D)$ and $Q_\lambda=Q_\lambda(\xi, D)$ be* *two families of partial differential operators on R^n with polynomial coefficients, $\lambda\in\Omega$. Denote by R_λ the orthogonal projection of $L^2(R^n)$ onto* Ker P_λ *(here we regard P_λ as a map of $L^2(R^n)$ into $H^{-m}(R^n)$ with $m=\deg P_\lambda(\xi,\eta)$). If there exists a constant $C\geqslant 0$ independent of $\lambda\in\Omega$ such that*

$$\|Q_\lambda f\|_{L^2}\leqslant C(\|P_\lambda f\|_{L^2}+\|R_\lambda f\|_{L^2}),\quad \forall f\in S(R^n),$$

then we say that Q_λ is weaker than P_λ uniformly for $\lambda\in\Omega$.

By a similar deduction as in Lemma 2 of [10], we can get

Lemma 6. *Q_λ is weaker than P_λ uniformly for $\lambda\in\Omega$ if and only if there exist bounded linear operators $B_{\lambda 1}$ and $B_{\lambda 2}$ on $L^2(R^n)$ with bounds independent of λ such that*

$$Q_\lambda f=B_{\lambda 1}P_\lambda f+B_{\lambda 2}R_\lambda f,\quad \forall f\in S(R^n).$$

Definition 2. *Let L be an inhomogeneous left invariant differential operator on the group G having the form $L=L_m+L_{m-1}+\cdots+L_0$, where L_j is homogeneous of degree j, $j=0, 1, \cdots, m$. We say that L is of the type that the lower order terms are controllable, or simply, L is a controllable operator, if for any $\omega_0\in S^{k-1}$ there exists a corresponding neighborhood $S_{\omega_0}\subset S^{k-1}$ of ω_0 such that for every $j=0, 1, \cdots, m-1$, $\pi^\omega_{\omega_0}(L_j)$ is weaker than $\pi^\omega_{\omega_0}(L_m)$ uniformly for $\omega\in S_{\omega_0}$.*

Let $Z_1, Z_2, \cdots, Z_{2n}$ be an arbitrary basis of g_1. Let $T_1, T_2, \cdots, T_k$ be an arbitrary basis of g_2. Then every inhomogeneous left invariant linear partial differential operator L on G can be written in the form $L=L_m+L_{m-1}+\cdots+L_0$ with

$$L_j=L_j(Z,T)=\sum_{|\alpha|+2|\beta|=j}a^{(j)}_{\alpha\beta}Z^\alpha T^\beta,\quad j=0,1,\cdots,m, \tag{4.2}$$

where $Z^\alpha=Z_1^{\alpha_1}Z_2^{\alpha_2}\cdots Z_{2n}^{\alpha_{2n}}$, $T^\beta=T_1^{\beta_1}T_2^{\beta_2}\cdots T_k^{\beta_k}$, and $a^{(j)}_{\alpha\beta}$'s are constant complex numbers, $j=0, 1, \cdots, m$. The function $a_j(\zeta,\lambda)\equiv L_j(i\zeta, i\lambda)(\zeta\in R^{2n}, \lambda\in R^k)$ is called the amplitude of the operator L_j, $j=0, 1, \cdots, m$.

Lemma 7. *Let L be as above. Then L is a controllable operator if the following conditions are satisfied:*

(i) *There exist a real number m' and positive constants C, C' and M such that*

$$C'|\zeta|^{m'}\leqslant|a_m(\zeta,\omega)|\leqslant C|\zeta|^m,\ \forall\zeta\in R^{2n}(|\zeta|\geqslant M),\ \forall\omega\in S^{k-1}.$$

(ii) *There is a real number $\rho\in(0, 1]$ and there are some positive constants $C_{\alpha j}$ ($\alpha\in Z^{2n}_+$, $j=0, 1, m$) such that for any $\alpha\in Z^{2n}_+$ and every $j=0, 1, \cdots, m$, the following holds:*

$$|D^\alpha_\zeta a_j(\zeta,\omega)|\leqslant C_{\alpha j}|a_m(\zeta,\omega)|(1+|\zeta|)^{-\rho|\alpha|},\ \forall\zeta\in R^{2n}(|\zeta|\geqslant M),\ \forall\omega\in S^{k-1}.$$

Proof Let ω_0 be an arbitrary point on S^{k-1}. Let $X_1^{\omega_0}, X_2^{\omega_0}, \cdots, X_n^{\omega_0}, Y_1^{\omega_0}, Y_2^{\omega_0}, \cdots, Y_n^{\omega_0}$ be a symplectic basis associated to the symplectic form $B_{\eta_{\omega_0}}$. Then there exists an invertible linear transformation on g_1 mapping this basis into the basis $Z_1, Z_2, \cdots, Z_{2n}$. Using this transformation, (4.2) can be rewritten as

$$L_j=L_j(X^{\omega_0}, Y^{\omega_0}, T)=\sum_{|\alpha+\beta|+2|\gamma|=j}\tilde a^{(j)}_{\alpha\beta\gamma}Y^\alpha_{\omega_0}X^\beta_{\omega_0}T^\gamma,$$

where $\bar{a}^{(j)}_{\alpha\beta\gamma}$ are some constant complex numbers, $X^{\beta}_{\omega_0}=(X^{\omega_0}_1)^{\beta_1}(X^{\omega_0}_2)^{\beta_2}\cdots(X^{\omega_0}_n)^{\beta_n}$, $Y^{\alpha}_{\omega_0}=(Y^{\omega_0}_1)^{\alpha_1}(Y^{\omega_0}_2)^{\alpha_2}\cdots(Y^{\omega_0}_n)^{\alpha_n}$, $T^{\gamma}=T^{\gamma_1}_1T^{\gamma_2}_2\cdots T^{\gamma_k}_k$. Denote $a_j(\xi,\eta,\lambda)=L_j(i\xi,i\eta,i\lambda)$, $j=0,1,\cdots,m$. Then since the linear transformation we used is invertible, one may easily prove that the functions $a_j(\xi,\eta,\lambda)$ satisfy the same conditions as $a_j(\zeta,\lambda)$ if we denote $\zeta=(\xi,\eta)$, $j=0,1,\cdots,m$. Noticing that $\pi^{\omega_0}_{\omega_0}(L_j)=a_j(\xi,D,\omega_0)$, and $\omega^{\omega_0}_{\omega}(L_j)$ depends on ω analyticly, $j=0,1,\cdots,m$, by a similar deduction as in the proof of Proposition 5 in [10] we conclude that there is a neighborhood $S_{\omega_0}\subset S^{k-1}$ of ω_0 such that $\pi^{\omega_0}_{\omega}(L_j)$ is weaker than $\pi^{\omega_0}_{\omega}(L_m)$ uniformly for $\omega\in S_{\omega_0}$, for each $j=0,1,\cdots,m-1$. We omit the details here.

From this lemma, we see that transversally elliptic operators are controllable operators.

Theorem 3. *Let L be a controllable operator with principal part L_m. Suppose that L_m satisfies the following conditions:*

(i) *For some real number s and any $\omega_0\in S^{k-1}$, there exists corresponding $A_{\omega_0}\in L^0(S_{\omega_0},B^s)$ such that*

$$\pi^{\omega_0}_{\omega}(L_m)A_{\omega_0}(\omega)f=f,\ \forall f\in S(R^n),\ \omega \text{ a. e. } S_{\omega_0},$$

and moreover, there is a corresponding constant $C_{\omega_0}\geqslant 0$ such that

$$\|A^sA_{\omega_0}(\omega)\|\leqslant C_{\omega_0},\ \omega \text{ a. e. } S_{\omega_0}.$$

(ii) *The orthogonal projection $R_{\omega_0}(\omega)$: $L^2(R^n)\to\operatorname{Ker}\pi^{\omega_0}_{\omega}(L_m)$ can be extended as a bounded linear map of $H^s(R^n)$ into $H^s(R^n)$. Moreover, there exi ts a corresponding constant $C'_{\omega_0}\geqslant 0$ such that*

$$\|A^sR_{\omega_0}(\omega)A^{-s}\|\leqslant C'_{\omega_0},\ \omega \text{ a. e. } S_{\omega_0}.$$

Then there exists a distribution $E\in W^r(G)$, where $r=\min(s-n-1,\ m-n-k)$, such that $LE=\delta$ in a neighborhood of the unit element, and thus L is locally solvable.

Proof We have

$$\pi^{\omega_0}_{\lambda}(L)=|\lambda|^{\frac{m}{2}}\pi^{\omega_0}_{\omega}(L_m)+|\lambda|^{\frac{m-1}{2}}\pi^{\omega_0}_{\omega}(L_{m-1})+\cdots+\pi^{\omega_0}_{\omega}(L_0),$$

where $\omega=\omega(\lambda)$. By Lemma 6 we see that there exist bounded linear operators $B^{\omega_0\omega_0}_{j1}$ and $B^{\omega_0\omega_0}_{j2}$ on $L^2(R^n)$, $j=0,1,\cdots,m-1$, $\omega\in S_{\omega_0}$, such that

$$\pi^{\omega}_{\omega_0}(L_j)f=B^{\omega_j\omega_0}_{j1}\pi^{\omega}_{\omega_0}(L_m)f+B^{\omega_j\omega_0}_{j2}R_{\omega_0}(\omega)f,\quad \forall f\in S(R^n),\tag{4.4}$$

and furthermore, for some positive number C independent of ω,

$$\|B^{\omega_j\omega_0}_{j1}\|\leqslant C,\ \|B^{\omega_j\omega_0}_{j2}\|\leqslant C,\ \omega \text{ a. e. } S_{\omega_0},\ j=0,1,\cdots,m-1.$$

By using the conditions (i) and (ii), we get from (4.4) the following

$$\pi^{\omega}_{\omega_0}(L_j)(I-R_{\omega_0}(\omega))A_{\omega_0}(\omega)f=B^{\omega_j\omega_0}_{j1}f,\quad \forall f\in S(R^n).$$

Hence we have

$$\begin{aligned}\pi^{\omega_0}_{\lambda}(L)(I-R_{\omega_0}(\omega))A_{\omega_0}(\omega)f&=|\lambda|^{\frac{m}{2}}f+\sum_{j=0}^{m-1}|\lambda|^{\frac{j}{2}}B^{\omega_j\omega_0}_{j1}f\\&=|\lambda|^{\frac{m}{2}}\Big(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B^{\omega_j\omega_0}_{j1}\Big)f,\\&\qquad\forall f\in S(R^n),\ \forall\lambda\in\Gamma_{\omega_0},\end{aligned}\tag{4.5}$$

where $\omega=\omega(\lambda)$. Noticing that if $|\lambda|\geqslant 1$ then

$$\left\|\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right\|\leqslant|\lambda|^{-\frac{1}{2}}\sum_{j=0}^{m-1}\|B_{j1}^{\omega_1\omega_0}\|\leqslant mC|\lambda|^{-\frac{1}{2}},$$

we see that if we take $M>0$ sufficiently large then we have

$$\left\|\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right\|<\frac{1}{2}, \quad \text{if } |\lambda|\geqslant M.$$

Therefore, the inverse $\left(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}$ exists when $|\lambda|\geqslant M$. Moreover, we have

$$\left\|\left(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}\right\|\leqslant\left(1-\left\|\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right\|\right)^{-1}\leqslant 2, \quad \text{if } |\lambda|\geqslant M.$$

For any $f\in S(R^n)$, take a sequence of functions $f_j^\omega\in S(R^n)$, $j=1, 2, \cdots$, such that $f_j^\omega\to\left(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}f$ in $L^2(R^n)$. Then by using (4.5) to f_j and then taking the limit, we get

$$\pi_\lambda^{\omega_0}(L)\circ|\lambda|^{-\frac{m}{2}}(I-R_{\omega_0}(\omega))A_{\omega_0}(\omega)\left(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}f=f,$$
$$\forall\lambda\in\Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Furthermore, using the conditions (i) and (ii) once more we get

$$\left\|\Lambda^s(I-R_{\omega_0}(\omega))A_{\omega_0}(\omega)\left(I+\sum_{j=1}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}\right\|$$
$$\leqslant\|\Lambda^s(I-R_{\omega_0}(\omega))\Lambda^{-s}\|\cdot\|\Lambda^{-s}A_{\omega_0}(\omega)\|\cdot\left\|\left(I+\sum_{j=0}^{m-1}|\lambda|^{-\frac{m-j}{2}}B_{j1}^{\omega_1\omega_0}\right)^{-1}\right\|$$
$$\leqslant C', \quad \lambda \text{ a.e. } \Gamma_{\omega_0}(|\lambda|\geqslant M).$$

Thus by using Theorem 2 we see the conclusions of the present theorem hold. This finishes the proof.

Theorem 4. *Let L be a left invariant differential operator satisfying the conditions of Lemma 6. Then L is locally solvable if the following two equivalent conditions are satisfied:*

(i) *For any $\omega_0\in S^{k-1}$ and almost all $\omega\in S_{\omega_0}$,* Ker $\pi_\omega^{\omega_0}(L_m)^*=\{0\}$.

(ii) *For any $\omega_0\in S^{k-1}$ and almost all $\omega\in S_{\omega_0}$, $\pi_\omega^{\omega_0}(L_m)$ is a surjection on $S'(R^n)$.*

Proof In fact, from the pseudodifferential operator theory of Shubin [13], we see that any operator satisfying the conditions in Lemma 7 always satisfies the codition (ii) of Theorem 3. Moreover, for this class of operators, the condition (i) in Theorem 3 is equivalent to the two equivalent conditions of the present theorem. Thus the conclusion is a corollary of Theorem 3.

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