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# ON SCORE VECTORS AND CONNECTIVITY OF TOURNAMENTS**

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**Abstract**

An $n$-tournament $T$ is called $k$-strong $(1\leqslant k\leqslant n-2)$, if every $(n+1-k)$-subtournament of $T$ is strongly connected. This paper proves that a score vector $(s_1, s_2, \cdots, s_n)$, where $s_1\leqslant s_2\leqslant\cdots\leqslant s_n$, is the score vector of some $k$-strong tournament if and only if $\min\{t_1, t_2, \cdots, t_{n-1}\}\geqslant k$, where $t_j=s_1+s_2+\cdots+s_j-j(j-1)/2$, $j=1, 2, \cdots, n-1$.

## § 1. Introduction

Let $V(T)=\{v_1, v_2, \cdots, v_n\}$ be vertex set of an $n$-tournament $T$. A score of $v_i$ is $d_T^+(v_i)=s_i$, $i=1, 2, \cdots, n$. $S=(s_1, s_2, \cdots, s_n)$ is called a score vector of $T$. And if $s_1\leqslant s_2\leqslant\cdots\leqslant s_n$, then $S$ is called an increasing score vector (ISV) of $T$. And we say $S\in$ ISV.

An $n$-tournament $T$ is called $k$-strong $(1\leqslant k\leqslant n-2)$, if every $(n+1-k)$-subtournament of $T$ is strongly connected. If there exists a $k$-strong tournament $T$ with score vector $S$, then we say $S$ contains a $k$-strong tournament and say $S\in\mathscr{C}(k)$.

Let $X=(x_1, x_2, \cdots, x_n)$ be a vector, where $x_1, x_2, \cdots, x_n$ are nonnegative integers. We denote its dimensional by $|X|=n$. In the following denote

$$t_j=t_j(X)=x_1+x_2+\cdots+x_j-j(j-1)/2,\ j=1, 2, \cdots, n;$$

$$t(X)=\min\{t_1, t_2, \cdots, t_{n-1}\};$$

$$\max(X)=\max\{x_1, x_2, \cdots, x_n\},\ \min(X)=\min\{x_1, x_2, \cdots, x_n\}.$$

In 1953, H. G. Landau[1] proved

**Theorem A.** *If $s_1\leqslant s_2\leqslant\cdots\leqslant s_n$ is a sequence of nonnegative integers, then*

$$S=(s_1, s_2, \cdots, s_n)\in \text{ISV}\Leftrightarrow t(S)\geqslant 0,\ t_n(S)=0.$$

In 1966, F. Harary and L. Moser[2] proved

**Theorem B.** *Let $S\in ISV$. Any tournament with score vector $S$ is strong if and only if $t(S)\geqslant 1$.*

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In 1985, L. W. Beineke, K. S. Bagga[3] and G. X. Huang, J. S. Li[4] proved

**Theorem C.** *If* $S\in ISV$, $|S|>2$, *then*

$$S\in\mathscr{C}(2)\Leftrightarrow t(S)\geqslant 2.$$

In 1987 and 1988, J. S. Li and G. X. Huang proved following two theorems.

**Theorem D.** [5] *If* $S\in ISV$ *and* $|S|=n>2$, *then*

$$S\in\mathscr{C}([(n-1)/2])\Leftrightarrow t(S)=[(n-1)/2].$$

**Theorem E.** [6] *If* $S\in ISV$ *and* $|S|>3$, *then*

$$S\in\mathscr{C}(3)\Leftrightarrow t(S)\geqslant 3.$$

This paper generalizes Theorems C, D, E by the following result.

**Theorem 1.1.** *If* $S\in ISV$ *and* $|S|>k\geqslant 1$, *then*

$$S\in\mathscr{C}(k)\Leftrightarrow t(S)\geqslant k.$$

## §2. On ISV

**Definition 2.1.** *If* $X=(x_1, x_2, \cdots, x_n)$ *is a score vector, where* $1+x_1\leqslant 2+x_2\leqslant\cdots\leqslant n+x_n$, *then* $X$ *is called an almost increasing score vecter (AISV). And we say* $X\in AISV$.

**Definition 2.2.** *Let* $S=(s_1, s_2, \cdots, s_n)\in ISV$, $n\geqslant 3$. *Denote*

$$(x_1, x_2, \cdots, x_{n-1})=(s_1, s_2, \cdots, s_{j-1}, s_{j+1}, \cdots, s_n)$$

*and*

$$S-s_j=(x_1, x_2, \cdots, x_{s_j}, x_{s_j+1}-1, x_{s_j+2}-1, \cdots, x_{n-1}-1).$$

$S-s_j$ *is called a deleted node vector of* $S$.

**Lemma 2.1.** *If* $S=(s_1, s_2, \cdots, s_n)\in ISV$, *then*

$$t(S-s_j)\geqslant t(S),\ j=1, 2, \cdots, n.$$

*Proof* Let $S-s_j=(x_1, x_2, \cdots, x_{n-1})$. When $i\leqslant s_j$, $x_i\geqslant s_i$. Thus

$$x_1+x_2+\cdots+x_i\geqslant s_1+s_2+\cdots+s_i=t_i(S)+i(i-1)/2.$$

So $t_i(S-s_j)\geqslant t_i(S)$.

When $s_j<i\leqslant n-1$, by the definition of $S-s_j$ we have

$$\begin{aligned}t_i(S-s_j)&=x_1+x_2+\cdots+x_i-i(i-1)/2\\&\geqslant(s_1+s_2+\cdots+s_{i+1})-s_j-(i-s_j)-i(i-1)/2\\&=(s_1+s_2+\cdots+s_{i+1})-(i+1)i/2=t_{i+1}(S).\end{aligned}$$

Therefore $t(S-s_j)\geqslant t(S)$.

**Lemma 2.2.** *Let* $X=(x_1, x_2, \cdots, x_n)$ *and* $x_1\leqslant x_2\leqslant\cdots\leqslant x_n$. *If*

(i) $x_1\geqslant k$, $t_{n-1}(X)\geqslant k$;

(ii) $1+x_j<x_{j+1}\ (2\leqslant j\leqslant n-2)\Rightarrow t_j(X)\geqslant k$,

*then* $t(X)\geqslant k$.

*Proof* If $t(X)<k$, then there exists $j$, $2\leqslant j\leqslant n-2$, such that

$$t_j(X)=x_1+x_2+\cdots+x_j-j(j-1)/2<k.$$

Suppose $j$ is a minimal positive integer such that above inequality holds. Thus $x_j \leqslant x_{j+1} \leqslant 1+x_j$. If $x_j \geqslant j$, then

$$x_1+x_2+\cdots+x_{j-1} \leqslant x_1+x_2+\cdots+x_j-j<j(j-1)/2+k-j<(j-1)(j-2)/2+k.$$

It is a contradiction. Thus $x_j \leqslant j-1$. Denote

$$m=\min\{n-1,\ i \mid 1+x_i<x_{i+1} \text{ and } i>j\},$$

We have $x_1+x_2+\cdots+x_m-m(m-1)/2 \geqslant k$ and

$$j-1 \geqslant x_j \geqslant x_{j+1}-1 \geqslant x_{j+2}-2 \geqslant \cdots \geqslant x_m-m+j.$$

So $x_i \leqslant i-1$, $i=j, j+1, \cdots, m$. Hence

$$\begin{aligned} x_1+x_2+\cdots+x_m &\leqslant (x_1+x_2+\cdots+x_j)+j+(j+1)+\cdots+(m-1) \\ &< j(j-1)/2+k+(m-j)(m+j-1)/2=m(m-1)/2+k. \end{aligned}$$

It is a contradiction.

**Lemma 2.3.** *Let $X=(x_1, x_2, \cdots, x_n) \in AISV$, $s_1=\min(X)$ and $s_n=\max(X)$. Then $t(X^*)=\min\{t(X), s_1, n-1-s_n\}$, where $X^*$ is an ISV which is obtained by arranging all terms of $X$.*

*Proof* Let $k=\min\{t(X), s_1, n-1-s_n\}$, $X^*=\{s_1, s_2, \cdots, s_n\}$. Obviously, $t(X^*) \leqslant t(X)$. Because

$$\begin{aligned} t(X^*) &\leqslant t_1(X^*)=s_1, \\ t(X^*) &\leqslant t_{n-1}(X^*)=s_1+s_2+\cdots+s_{n-1}-(n-1)(n-2)/2 \\ &=(s_1+s_2+\cdots+s_n)-s_n-(n-1)(n-2)/2=n-1-s_n, \end{aligned}$$

thus $t(X^*) \leqslant k$. Otherwise, we have

$$t_1(X^*)=s_1 \geqslant k, \quad t_{n-1}(X^*)=n-1-s_n \geqslant k.$$

By Definition 2.1, $1+x_1 \leqslant 2+x_2 \leqslant \cdots \leqslant n+x_n$ hold. Hence

$$\begin{aligned} 1+s_j<s_{j+1} &\Rightarrow s_1+\cdots+s_j=x_1+\cdots+x_j=j(j-1)/2+t_j(X) \geqslant j(j-1)/2+k \\ &\Rightarrow t_j(X^*) \geqslant k. \end{aligned}$$

By Lemma 2.2 we have $t(X^*) \geqslant k$. Therefore $t(X^*)=k$.

**Lemma 2.4.** *Let $S=(s_1, s_2, \cdots, s_n) \in ISV$. Then*

$$s_{1+s_1} \leqslant n-1-s_1, \tag{2.1}$$

*and the equality holds iff $s_1=s_n$.*

*Proof* When $s_1=s_n$, $S$ is the score vector of a regular tournament. Thus $s_1=s_n=(n-1)/2$ and (2.1) is an equality. When $s_1<s_n$, $s_1<(n-1)/2$, i. e., $s_1 \leqslant n/2-1$.

Suppose $s_{1+s_1} \geqslant n-1-s_1$. Because $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, we have

$$s_1+s_2+\cdots+s_n \geqslant s_1^2+(n-1-s_1)(n-s_1)=2s_1^2-(2n-1)s_1+n(n-1).$$

On the right hand, it is a twice polynomial for $s_1$. When $s_1 \leqslant n/2-1$, it is a decreasing function for $s_1$. Setting $s_1=n/2-1$, we obtain

$$s_1+s_2+\cdots+s_n \geqslant n(n-1)/2+1.$$

It is a contradiction.

**Lemma 2.5.**[5] *If tournament $T$ is $k$-strong, $k \geqslant 2$, then $T$ is $(k-1)$-strong.*

Now we prove the following lemma.

**Lemma 2. 6.** *Let $S=(s_1, s_2, \cdots, s_n)\in ISV$. If $S-s_j\in\mathscr{C}(k)$, where $k\leqslant s_j\leqslant n-1-k$, then $S\in\mathscr{C}(k)$.*

*Proof* We need only prove the case for $j=1$, other cases can be proved in a similar manner. Let $G$ be a $k$-strong tournament with score vector $S-s_1$, $V(G)=\{v_2, v_3, \cdots, v_n\}$. Suppose $T$ is a tournament with $V(T)=V(G)\cup\{v_1\}$ and $A(T)=A(G)\cup\{(v_1, v_i), (v_j, v_1)\,|\,2\leqslant i\leqslant 1+s_1<j\leqslant n\}$. Obviously the score vector of $T$ is $S$.

Let $U\subset V(T)$, $|U|=k-1$. If $v_1\bar{\in}U$, then $G-U$ is strong and $1\leqslant d^+_{G-U}(v_1)\leqslant n-k-1$; thus $T-U$ is strong. If $v_1\in U$, then $T-U$ is strong by Lemma 2.5.

**Lemma 2. 7.** *Let $T$ be a tournament with increasing score vector $S=(s_1, s_2, \cdots, s_n)$. The increasing score vector of its converse $T'$ is $S'=(n-1-s_n, n-1-s_{n-1}, \cdots, n-1-s_1)$. Then*

(i) $S\in\mathscr{C}(k)\Leftrightarrow S'\in\mathscr{C}(k)$;

(ii) $t_j(S)=t_{n-j}(S')$, $t(S)=t(S')$.

*Proof* (i) Because the converse of a directed path is the directed path, the conclusion holds obviously.

(ii) Let $M=\{v_1, v_2, \cdots, v_j\}$, $N=\{v_{j+1}, v_{j+2}, \cdots, v_n\}$. Denote

$$T(M\to N)=\{(u, v)\,|\,(u, v)\in A(T),\ u\in M,\ v\in N\}.$$

We have

$$t_j(S)=|T(M\to N)|=|T'(N\to M)|=t_{n-j}(S').$$

Therefore

$$t(S)=t(S').$$

## § 3. Simple Cyclic Tournaments

**Definition 3. 1.** *A tournament $F_n$ is called a simple cyclic tournament, if $V(F_n)=\{v_1, v_2, \cdots, v_n\}$, and when $n$ is odd,*

$$(v_i, v_j)\in A(F_n)\Leftrightarrow 0<j-i<n/2 \text{ or } n/2<i-j<n;$$

*when $n$ is even, $F_n=F_{n+1}-v_{n+1}$. A Hamilton cycle $(v_1v_2\cdots v_nv_1)$ of $F_n$ is called its essential cycle.*

**Lemma 3. 1.** *If $F_n$ is a simple cyclic tournament, then*

(i) *$F_n$ and its converse are isomorphic;*

(ii) *any non-strong subtournament of $F_n$ is a transitive tournament;*

(iii) *$F_n$ is $[(n-1)/2]$-strong.*

*Proof* (i) It is clear by Definition 3.1.

(ii) If there exists a non-strong subtournament $T$ which is not transitive, then $T$ contains at least a 3-cycle $(v_iv_jv_kv_i)$. Without loss of generality suppose $i<j$, $i<k$, thus $i<j<k$. Let $v_m\in V(T)$. If $m<i$ or $m>k$, then $v_k\to v_m\to v_i$. If $i<m<j$, then $v_i\to v_m\to v_j$. If $j<m<k$, then $v_j\to v_m\to v_k$. Hence $T$ is strong. It is a contradiction.

(iii) The number of vertices of a maximal transitive subtournament in $F_n$ is $[n/2]+1$. For any $U \subset V(F_n)$, $|U| = [(n-1)/2]-1$, we have $n-|U| = n-[(n-1)/2]+1 > [n/2]+1$. So $F_n - U$ is not a transitive tournament. Thus $F_n - U$ is strong by (ii). Therefore $F_n$ is $[(n-1)/2]$-strong.

Let $S \in$ ISV and $S=(s_1, s_2, \cdots, s_n) \in \mathscr{C}(k)$. Obviously

$$k \leqslant s_1 \leqslant s_n \leqslant n-1-k. \tag{3.1}$$

So $k \leqslant [(n-1)/2]$.

Now we give a shorter proof for Theorem D as follows.

*Proof of Theorem D* Let $S \in \mathscr{C}([(n-1)/2])$. We choose $k=[(n-1)/2]$ in (3.1). Thus

$$[(n-1)/2] \leqslant s_1 \leqslant s_n \leqslant n-1-[(n-1)/2] = [n/2]. \tag{3.2}$$

Hence $t(S) = [(n-1)/2]$.

Otherwise, suppose $t(S) = [(n-1)/2]$. We have

$$S_1 = t_1(S) \geqslant [(n-1)/2], \quad n-1-s_n = t_{n-1}(S) \geqslant [(n-1)/2].$$

So (3.2)holds. Thus $S$ is an ISV of a simple cyclic tournament $F_n$. $F_n$ is $[(n-1)/2]$-serong by Lemma 3.1. Therefore $S \in \mathscr{C}([(n-1)/2])$.

## § 4. The Proof of Theorem 1.1

The necessity of Theorem 1.1 is clear. We prove its sufficiency by induction for the dimentionality of $S$. We may suppose $k>1$ by Theorem B.

When $|S|=3$ or 4, the conclusion holds obviously. Suppose the theorem holds when $|S|<n$ $(n \geqslant 5)$. Let $|S|=n$. We may suppose $s_1<s_n$ by Theorem D, where $S=(s_1, s_2, \cdots, s_n)$.

1. Suppose $s_1>k$.

Obviously $\min\{S-s_1\} \geqslant k$. By Lemma 2.4, we have $s_{1+s_1} < n-1-s_1 \leqslant n-2-k$. So $\max\{S-s_1\} \leqslant n-2-k$. By Lemma 2.1, $t(S-s_1) \geqslant t(S) \geqslant k$. And by Lemma 2.3, $t((S-s_1)^*) \geqslant k$, where $(S-s_1)^*$ is an ISV which is obtained by arranging all terms of $S-s_1$. By inductive hypothesis, $(S-s_1)^* \in \mathscr{C}(k)$. Because $k<s_1<n-1-k$, we have $S \in \mathscr{C}(k)$ by Lemma 2.6. In the following suppose $s_1=k$.

2. Suppose $s_n<n-1-k$.

Let $S'=(n-1-s_n, n-1-s_{n-1}, \cdots, n-1-s_1)$. Because $n-1-s_n>k$, we have $t(S') = t(S) \geqslant k$ by Lemma 2.7. Thus $S'-(n-1-s_n) \in \mathscr{C}(k)$. Therefore $S'$ and $S \in \mathscr{C}(k)$ by Lemma 2.7. In the following suppose $s_n=n-1-k$.

3. Suppose $s_{k+2}>k$ or $s_{n-1-k}<n-1-k$.

By discussing the connectivity of $S-s_1$ or $S'-(n-1-s_n)$, we have $S \in \mathscr{C}$(k) as case 1 or 2. In the following suppose $s_{k+2}=k$, $s_{n-1-k}=n-1-k$ and

$$S=(\underbrace{k,\cdots,k}_{p},s_{p+1},s_{p+2},\cdots,s_r,s_{r+1},\cdots,s_{n-q},\underbrace{n-1-k,\cdots,n-1-k}_{q}),\quad (4.1)$$

where

$$p>k+1,\ q>k+1,\ k<s_{p+1},\ s_{n-q}<n-1-k.\quad (4.2)$$

4. Suppose there exists $s_j$ such that $p\leqslant s_j\leqslant n-q-1$.

We have $t(S-s_j)\geqslant k$ by Lemma 2.1. Because $p\leqslant s_j\leqslant n-q-1$, we have

$$\min\{S-s_j\}\geqslant k,\ \max\{S-s_j\}\leqslant n-2-k.$$

Thus $t((S-s_j)')\geqslant k$. By inductive hypothesis we have $(S-s_j)'\in\mathscr{C}(k)$. Thus $S\in\mathscr{C}(k)$ by Lemma 2.6.

In the following suppose (see (4.1))

$$s_r<p,\ s_{r+1}>n-q-1.\quad (4.3)$$

5. Suppose $t_r(S)\geqslant 2k-1$.

We discuss the increasing score vector $S''=(s_2-1,s_3-1,\cdots,s_{n-1}-1)$. When $2\leqslant i<k$, we have $s_i-1=k-1\geqslant i$, so

$$t_1(S'')<t_2(S'')<\cdots<t_{k-1}(S'').$$

When $k\leqslant i\leqslant r$, we have $s_i+1<i$, so

$$t_k(S'')>t_{k+1}(S'')>\cdots>t_{r-1}(S'').$$

And

$$\begin{aligned}t_1(S'')&=k-1,\\ t_{r-1}(S'')&=(s_2-1)+(s_3-1)+\cdots(s_r-1)-(r-1)(r-2)/2\\ &=(s_1+s_2+\cdots s_r)-s_1-(r-1)-(r-1)(r-2)/2\\ &=t_r(S)-k\geqslant k-1.\end{aligned}$$

Thus $t_i(S'')\geqslant k-1$, $i=1,2,\cdots,r-1$.

Similarly, when $i=r,r+1,\cdots,n-2$, we have $t_i(S'')\geqslant k-1$ too. Thus $t(S'')\geqslant k-1$. By inductive hypothesis there exists a $(k-1)$-strong tournament $T_{n-2}$ with score vector $S''$. Let $V(T_{n-2})=\{v_2,v_3,\cdots,v_{n-1}\}$, $d^+_{T_{n-2}}(v_i)=s_i-1$. Denote $V_1=\{v_2,v_3,\cdots,v_k\}$, $V_2=\{v_{k+1},v_{k+2},\cdots,v_{n-1}\}$. Let $T$ be a tournament such that

$$V(T)=V(T_{n-2})\cup\{v_1,v_n\},$$

$$A(T)=A(T_{n-2})\cup\{(V_1,v_n),(V_2,v_1),(v_1,V_2),(v_n,V_1),(v_1,v_n)\},$$

where $(V_i,v_j)=\{(u,v_j)\,|\,u\in V_i\}$, $(v_j,V_i)=\{(v_j,u)\,|\,u\in V_i\}$, $i=1,2$, $j=1,n$. Obviously, the score vector of $T$ is $S$. Now we prove that $T$ is $k$-strong.

Let $U\subset V(T)$, $|U|=k-1$. If $U\cap\{v_1,v_n\}\neq\varnothing$, then $T_{n-2}-U$ is strong since $T_{n-2}$ is $(k-1)$-strong. Thus $T-U$ is strong. If $U\cap\{v_1,v_n\}=\varnothing$, then $\{v_1,v_n\}\subset V(T-U)$. We need only prove that there exist a directed path $P_1$ from $v$ to $v_1$ and a directed $P_2$ from $v_n$ to $v$ for any $v\in V(T-U)-\{v_1,v_n\}$.

Actually, because $|V_2|=n-1-k>k-1$, there exists $u\in V_2-U$. If $v\in V_1$, then choose $P_1=(vv_nuv_1)$, $P_2=(v_nuv_1v)$; if $v\in V_2$, then choose $P_1=(vv_1)$, $P_2=(v_nv)$.

Because $(v_1,v_n)\in A(T)$, $T-U$ is strong.

6. Suppose $t_r(S)\leqslant 2k-2$.

We first prove the following inequalities

$$p\leqslant 2k\leqslant r<(1+\sqrt{2})k. \tag{4.4}$$

If $p>2k$, then $t_{2k+1}(S)=k(2k+1)-(2k+1)2k/2=0$, it is a contradiction. Because $r\geqslant k+2$, we see that if $r<2k$ then

$$t_r(S)=s_1+\cdots+s_r-r(r-1)/2\geqslant kr-r(r-1)/2\geqslant 2k-1.$$

But $t_r(S)\leqslant 2k-2$, hence $r\geqslant 2k$.

By (4.1) and $s_r\leqslant p-1$, we have

$$t_r(S)-k\leqslant kp+(p-1)(r-p)-r(r-1)/2-k$$
$$=-p^2+(k+r+1)p-r(r+1)/2-k.$$

On the right hand, it is a twice polynomal for $p$, its discriminant

$$\Delta=(k+r+1)^2-2r(r+1)-4k=-r^2+2kr+k^2-2k+1.$$

If $r>k+(2k^2-2k+1)^{1/2}$, $\Delta<0$. Because $(1+\sqrt{2})k>k+(2k^2-2k+1)^{1/2}$, we see that if $r>(1+\sqrt{2})k$ then $t_r(S)-k<0$. But $t(S)\geqslant k$, hence (4.4) holds. Denote

$$x_i=\begin{cases} s_i-1 & \text{if } 1\leqslant i\leqslant t_r(S),\\ s_i & \text{if } t_r(S)<i\leqslant r.\end{cases}$$

We now prove that $X=(x_1, x_2, \cdots, x_r)\in \mathrm{ISV}$.

Denote $t_i=t_i(S)$, $t'_i=t_i(X)$. Obviously

$$0\leqslant (t_{i+1}-t'_{i+1})-(t_i-t'_i)\leqslant 1.$$

Thus if $t_i<t_{i+1}$, then $t'_i\leqslant t'_{i+1}$; if $t_i\geqslant t_{i+1}$, then $t'_i\geqslant t'_{i+1}$. By the definitions of $S$ and $t_i$ we have

$$k=t_1<t_2<\cdots<t_k,\ t_k=t_{k+1}>t_{k+2}>\cdots>t_r.$$

Thus

$$k-1=t'_1\leqslant t'_2\leqslant\cdots\leqslant t'_k,\ t'_k\geqslant t'_{k+1}\geqslant t'_{k+2}\geqslant\cdots\geqslant t'_r=0.$$

So $X\in \mathrm{ISV}$.

Because $t_r(S)\leqslant 2k-2<r$, we have $s_r=x_r$. Let

$$h=\begin{cases} p-1 & \text{if } p \text{ is even},\\ p & \text{if } p \text{ is odd}.\end{cases}$$

Obviously $h$ is odd, $k+1\leqslant h\leqslant 2k-1$ and $x_1=k-1\geqslant (h-1)/2$. Denote $d=[(r+h)/2]$. When $r\geqslant p+1$, we have $d\geqslant[2p/2]=p>s_r$. When $r=p$, $r=p=2k$ by (4.4). Thus $s_r=k$, $h=2k-1$, $d=[(2k+2k-1)/2]=2k-1>k=s_r$. Hence $d>s_r=x_r$ holds.

We discuss the following three sequences:

$$x_1-(h-1)/2,\ x_2-(h-1)/2, \cdots, x_h-(h-1)/2; \tag{4.5}$$

$$d-x_{h+1},\ d-x_{h+2}, \cdots, d-x_r \text{ (when } r+h \text{ is odd)}; \tag{4.6}$$

$$d-x_{h+1},\ d-x_{h+2}, \cdots, d-x_d,\ d-x_{d+1}-1, \cdots, d-x_r-1 \text{ (when } r+h \text{ is even)}. \tag{4.7}$$

By simple computing, we have

(i) the sums of all terms of (4.5), (4.6), (4.7) are equal;

(ii) the difference of the maximal and minimal terms in (4.5) equals zero or one.

Now we prove

(iii) the maximal terms $d-x_{h+1}<h$ in (4.6) and (4.7).

Actually, $r<(1+\sqrt{2})k<3k-1$ by (4.4). Because $h\geqslant k+1$, we have $r<3k-1\leqslant h+2k-2$. Thus

$$d-x_{h+1}=[(r+h)/2]-x_{h+1}\leqslant(r+h)/2-(k-1)<h.$$

Hence (iii) holds.

By (i), (ii) and (iii), there exists a bipartite graph $D$, $V(D)=\{M; N\}$, where $M=\{v_1, v_2, \cdots, v_h\}$, $N=\{v_{h+1}, v_{h+2}, \cdots, v_r\}$, such that the degree sequence of $M$ is (4.5), the degree sequence of $N$ is (4.6) or (4.7) according as $r+h$ is odd or even respectively, and $v_{h+1}$ is adjacent to the last $d-x_{h+1}$ vertices in $M$.

Because $h<2k$, there exists a sequence $n_1, n_2, \cdots, n_h$, which is the arrangement of $\{1, 2, \cdots, h\}$ such that if $n_i>k$, then $n_{i-1}$ and $n_{i+1}\leqslant k$, $i=1, 2, \cdots, h$, where $n_0=n_h$, $n_1=n_{h+1}$.

Let $E$ and $F$ be two simple cyclic tournaments such that $V(E)=\{v_{n_1}, v_{n_2}, \cdots, v_{n_h}\}$, $V(F)=N$. The essential cycle in $E$ is $(v_{n_1}, v_{n_2}, \cdots, v_{n_h}, v_{n_1})$.

Let $G$ be an $r$-tournament such that

(i) $E$ and $F$ are two subtournaments of $G$ and $V(E)\cap V(F)=\varnothing$;

(ii) if $v_i\in M$ and $v_j\in N$, then

$$(v_i, v_j)\in A(G)\Leftrightarrow v_i \text{ and } v_j \text{ are adjacent in } D.$$

Obviously $(v_{h+1}, v_i)\in A(G)$, $i=1, 2, \cdots, h-d+x_{h+1}$, and $X=(x_1, x_2, \cdots, x_r)$ is a score vector of $G$.

Let $S'=(n-1-s_n, n-1-s_{n-1}, \cdots, n-1-s_{r+1}, n-1-s_r, \cdots n-1-s_1)$. By Lemma 2.7 we have $t_{n-r}(S')=t_r(S)$. Denote

$$y_i=\begin{cases} n-2-s_{n+1-i} & \text{if } 1\leqslant i\leqslant t_r(S), \\ n-1-s_{n+1-i} & \text{if } t_r(S)<i\leqslant n-r. \end{cases}$$

Obviously $Y=(y_1, y_2, \cdots, y_{n-r})\in \mathrm{ISV}$. Similarly, we have $(n-r)$-tournament $H$, $V(H)\{=v_n, v_{n-1}, \cdots, v_{r+1}\}$ and its increasing score vector is $Y$, where $d_H(v_{n+1-j})=y_j$, $j=1, 2, \cdots, n-r$. Suppose the converse of $H$ is $H'$.

Now let $T$ be an $n$-tournament such that

(i) $G$ and $H'$ are two subtournaments of $T$ and $V(G)\cap V(H')=\varnothing$;

(ii) when $v_i\in V(G)$ and $v_j\in V(H')$,

$$(v_i, v_j)\in A(T)\Leftrightarrow 1\leqslant i=n+1-j\leqslant t_r(S).$$

Obviously the increasing score vector of $T$ is $S$. We will prove $T$ is $k$-strong.

Let $U\subset V(T)$, $|U|=k-1$. If $u\in V(G-U)$, $u'\in V(H'-U)$, $(u, u')\in A(T)$, then $u$(or $u'$) is called a bridge node of $G-U$ (or $H'-U$). First we prove there exists a Hamilton path of $G-U$, whose terminal vertex is a bridge node. Obviously we need only prove that for any non-bridge node $v\in G-U$ there exists a directed path from $v$ to a bridge node $u$ of $G-U$ (i. e., $u$ is a reachable node of $v$).

Obviously there exists at least one bridge node $u_0$ of $E-U$. Then $d_G^+(u_0)=k-1$, $d_G^-(u_0)=r-k$, $d_E^-(u_0)=(h-1)/2$, $d_F^-(u_0)=d_G^-(u_0)-d_E^-(u_0)=r-k-(h-1)/2$. The discussing can divided into four cases.

Case 1. $E-U$ and $F-U$ are all strong.

Obviously $u_0$ is a reachable node of all non-bridge nodes of $E-U$. Suppose $|U\cap F|<r-k-(h-1)/2$. Because $d_F^-(u_0)=r-k-(h-1)/2$, there exists $v\in V(F)$ such that $(v, u_0)\in A(G)$. Thus $u_0$ is a reachable node of every vertex of $F-U$. Suppose $|U\cap F|\geqslant r-k-(h-1)/2$. We have $|U\cap E|\leqslant k-1-(r-k-(h-1)/2)=2k-r+(h-3)/2$. Let $v\in V(F-U)$. Because $d_G^+(v)\geqslant k-1$, $d_F^+(v)\leqslant[(r-h)/2]$, $r\geqslant 2k$, we have

$$d_E^+(v)=d_G^+(v)-d_F^+(v)\geqslant k-1-[(r-h)/2]>2k-r+(h-3)/2\geqslant|U\cap E|.$$

Thus there exists $w\in V(E-U)$ such that $(v, w)\in A(G)$. Therefore $u_0$ is a reachable node of $v$.

Case 2. $E-U$ is strong, $F-U$ is non-strong.

Because $F$ is a simple cyclic tournament, $F-U$ is transitive by Lemma 3.1. Let $z$ be a terminal vertex of $F-U$. If $z$ is a bridge node, then $z$ is a reachable node of other vertices of $F-U$. If $z$ is a non-bridge node, then $d_G^+(z)\geqslant k$. So there exists $w\in V(E-U)$ such that $(z, w)\in A(G)$. Because $E-U$ is strong, $u_0$ is a reachable node of every vertex of $F-U$.

Case 3. $E-U$ is non-strong, $F-U$ is strong.

Because $E$ is a simple cyclic tournament and $E-U$ is non-strong, we have $|U\cap E|\geqslant(h-1)/2$ by Lemma 3.1. Thus $|U\cap F|\leqslant k-1-(h-1)/2<r-k-(h-1)/2$ $=d_F^-(u_0)$ and $d_{F-U}^-(u_0)\geqslant 1$. Because $F-U$ is strong, $u_0$ is a reachable node of every vertex of $F-U$.

Because $E-U$ is transitive, let $w$ be its terminal. If $w$ is a bridge node, then the conclusion holds. If $w$ is a non-bridge node, then $d_{F-U}^+(w)\geqslant 1$. Let $v\in V(F-U)$ and $(w, v)\in A(G)$. Because $u_0$ is a reachable node of $v$, $u_0$ is a reachable node of $w$.

Case 4. $E-U$ and $F-U$ are all non-strong.

Because $E$ and $F$ are two simple cyclic tournaments, we have $|U\cap E|\geqslant(h-1)/2$ and $|U\cap F|\geqslant[(r-h-1)/2]$. Thus

$$k-1=|U|\geqslant|U\cap E|+|U\cap F|\geqslant(h-1)/2+[(r-h-1)/2]=[r/2]-1\geqslant k-1.$$

Hence we have $U=(U\cap E)\cup(U\cap F)$, $|U\cap E|=(h-1)/2$, $|U\cap F|=[(r-h-1)/2]=k-1-(h-1)/2<r-k-(h-1)/2$.

If $v_i\in V(E-U)$, $1\leqslant i\leqslant k$, then $d_G^+(v_i)=k-1$. Thus $v_i$ is a bridge node of $G-U$. Let $P_0$ be a longest path of $E-U$. Then $P_0$ is contained in the essential cycle of $E$. By the order of vertex of $E$, at least one of the adjacent two vertices of $P_0$ is a bridge node.

Let $z$ be the terminal vertex of the transitive tournament $F-U$. If $z$ is a bridge node, then $z$ is a reachable node of other vertices of $F-U$. And every vertex of $V(E-U)$ is a bridge node. Thus the conclusion holds. In the following suppose $z$ is a non-bridge node. Thus $d^+_{E-U}(z)\geqslant 1$. If a terminal vertex $w$ of $P_0$ is a bridge node, then $w$ is a reachable node of every vertex of $V(G-U)$. In the following suppose $w$ is a non-bridge node and $(u, w)\in A(P_0)$. Thus $u$ is a bridge node.

If $(w, z)\in A(G)$, then $u$ is a reachable node of $z$ by $d^+_{E-U}(z)\geqslant 1$. The conclusion holds.

If $(z, w)\in A(G)$, we prove $d^+_G(z)>k$. Conversely, $d^+_G(z)=k$. Because there exists a non-bridge node in $E$, we have $t_r(S)<p$. Thus $|\{v\in V(F)\,|\,d^+_G(v)=k\}|=1$, $z=v_{h+1}=v_p$. By the construction of $G$, if $v\in V(E-U)$, $(z, v)\in A(G)$, then $d^+_G(v)=k-1$, i. e., $v$ is a bridge node. But $(z, w)\in A(G)$ and $w$ is a non-bridge node. It is a contradiction.

By $d^+_G(z)>k$, we have $d^+_{E-U}(z)>1$. Thus there exists $v\in V(E-U)$, $v\neq w$, such that $(z, v)\in A(G)$. So $u$ is a reachable node of $z$. Because $d^+_{F-U}(W)\geqslant 1$, $z$ is a reachable node of $w$. Therefore $u$ is a reachable node of other vertices of $G-U$.

As mentioned above, there exists a Hamilton path $P=(u_1, u_2\cdots u_j)$ in $G-U$, where $u_j$ is a bridge node. Let $y\in V(H'-U)$, $(u_j, y)\in A(G)$. Because $d^+_G(y)=r-1$, we have $(y, u_1)\in A(T)$. Thus there exists a cycle $C_1=(yu_1u_2\cdots u_jy)$ in $T-U$, $C_1$ contains all vertices of $G-U$ and one vertex of $H'-U$. Similarly, we can find out cycle $C_2$ in $T-U$, $C_2$ contains all vertices of $H'-U$ and one vertex of $G-U$. Therefore $T-U$ is strong.

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