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BOUNDEDNESS OF SEVERAL OPERATORS ON MARTINGALE SPACES AND THE GEOMETRY OF BANACH SPACES

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Abstract

The aim of this paper is establishing some inequalities of several operators on Banach-space-valued martingales and using them to give some characterizations of geometrical properties of Banach spaces. In particular, the Φ-function inequalitities of sharp operators $f_p^\#$, $\tilde{f}_p^\#$, p-variation operators W_p, $\widetilde{W}_p$ and the martingale tranform operator T_v are discussed. It is proved that the boundedness of these operators characterizes the smoothness, convexity and UMD-property of Banach spaces.

Throughout this paper, denote by X a Banach space, (Ω, Σ, P) a probability space and $f=(f_n)$ an X-valued martingale which is adapted to some increasing sequence of sub-σ-algebras of Σ and $df=(df_n)$ its difference sequence, where $df_n=f_n-f_{n-1}$, $n\geq 1$, $f_0=0$, $\mathscr{B}_0=\{\phi, \Omega\}$. For $f=(f_n)$, we adopt the following notations:

$$\|f\|_p=\operatorname{Sup}_n \|f_n\|_p,\ 1\leq p<\infty;$$

$$f_n^*=\operatorname{Sup}_{i<n} |f_i|,\quad f^*=\operatorname{Sup}_n f_n^*;$$

$$S_n^{(p)}(f)=\Big(\sum_{i=1}^n |df_i|^p\Big)^{1/p};\quad S^{(p)}(f)=\operatorname{Sup}_n S_n^{(p)}(f);$$

$$\sigma_n^{(p)}(f)=\Big(\sum_{i=1}^n E(|df_i|^p|\mathscr{B}_{i-1})\Big)^{1/p},\quad \sigma^{(p)}(f)=\operatorname{Sup}_n \sigma_n^{(p)}(f).$$

The definitions of the p-uniform smoothness and q-uniform convexity of Banach spaces can be found in [16]. X is said to be p-smoothable (q-convexifiable) if there exists an equivalent norm on X under which X is p-uniformly smooth (q-uniformly convex). In addition, for the definition of the UMD property of Banach spaees see [2].

A function Φ: $[0, \infty)\to[0, \infty)$ with $\Phi(0)=0$ is said to be a general function if $\Phi(t)$ is an increasing (to infinite) continuous function on $[0, \infty)$; it is said to be a Young function if $\Phi(t)$ is increasing and convex on $[0, \infty)$ and $\lim_{t\to\infty} t^{-1}\Phi(t)=\infty$; Φ is said to be moderately increasing if there exists a constant $c>0$ such that

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$\Phi(2t)\leqslant c\Phi(t)$ for every $t>0$. For convex function Φ, we denote

$$p_\Phi=\operatorname*{Sup}_{t>0}\frac{t\Phi'(t)}{\Phi(t)},\quad q_\Phi=\inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)},$$

where $\Phi'(t)$ is the right continuous derivative of Φ. It is well known that Φ is moderately increasing iff $p_\Phi<\infty$, somewhere in what follows it is needed that $q_\Phi>1$. In addition, the norm of Orlicz function space is defined as usual.

§ 1. $f_p^\#$ and $\tilde{f}_p^\#$

Here

$$f_{pn}^\#=\operatorname*{Sup}_{m\geqslant n}[E(|f_m-f_{n-1}|^p|\mathscr{B}_n)]^{1/p},\ f_p^\#=\operatorname*{Sup}_n f_{pn}^\#,$$

$$\tilde{f}_{pn}^\#=\operatorname*{Sup}_{m\geqslant n}[E(|f_m-f_n|^p|\mathscr{B}_n)]^{1/p},\ \tilde{f}_p^\#=\operatorname{Sup}\tilde{f}_{pn}^\#.$$

Theorem 1. *Suppose that X is a Banach space and Φ is a moderate Young function. Then the following conditions are equivalent (TFAE):*

(i) *X is q-convexifiable ($2\leqslant q<\infty$).*

(ii) *There exists a $c=c_{q\Phi}>0$ such that*

$$\|S^{(p)}(f)^q\|_\Phi^{1/q}\leqslant c\|\tilde{f}_q^{\#q}\|_\Phi^{1/q} \tag{1}$$

for every X-valued martingale $f=(f_n)$.

(iii) *There exists a $c=c_{q\Phi}>0$ such that*

$$\|\sigma^{(q)}(f)^q\|_\Phi^{1/q}\leqslant c\|f_q^{\#q}\|_\Phi^{1/q} \tag{2}$$

for every X-valued martingale $f=(f_n)$.

Proof (i)$\Rightarrow$(ii). For $f=(f_n)$ and $\alpha>0$, $\beta>0$, $\lambda>0$, define

$$\tau=\inf\{n,\ S_n^{(q)}(f)^q>(1+\alpha)\lambda\},$$

$$\theta=\inf\{n,\ S_n^{(q)}(f)^q>\alpha\lambda\},$$

$$\mu=\inf\{n,\ f_{qn}^{\#q}>\beta\lambda\}.$$

Then τ, θ, μ are stopping times with $\theta\leqslant\tau$. It is clear that

$$P(\tau<\infty)\leqslant P(\tau<\infty,\ \theta<\mu)+P(\mu<\infty)$$

and

$$\begin{aligned}P(\tau<\infty,\ \theta<\mu)&\leqslant P(\theta<\mu,\ S_\tau^{(q)}(f)^q-S_{\theta-1}^{(q)}(f)^q>\lambda)\\&\leqslant\frac{1}{\lambda}\int_{(\theta<\mu)}(S_\tau^{(q)}(f)^q-S_{\theta-1}^{(q)}(f)^q)dP\\&=\frac{1}{\lambda}\int_{(\theta<\mu)}E\Big(\sum_{i=\theta}^{\tau}|df_i|^q|\mathscr{B}_\theta\Big)dP.\end{aligned} \tag{3}$$

From the q-convexity of X, there is[16] a $c=c_q>0$ such that every L_q-bounded X-valued martingale $f=(f_n)$ satisfy

$$E\Big(\sum_{i=n}^{m}|df_i|^q|\mathscr{B}_n\Big)\leqslant cE(|f_m-f_{n-1}|^q|\mathscr{B}_n)\quad \text{a. e.} \tag{4}$$

($m\geqslant n\geqslant 0$) and then (3) becomes

$$P(\tau<\infty,\ \theta<\mu)\leqslant\frac{c}{\lambda}\int_{(\theta<\mu)}E(|f_\tau-f_{\theta-1}|^q|\mathscr{B}_\theta)dP$$
$$\leqslant\frac{c}{\lambda}\int_{(\theta<\mu)}f_{q\theta}^{\#q}\,dP\leqslant c\beta P(\theta<\infty).$$

Hence

$$P(S^{(q)}(f)^q>(1+\alpha)\lambda)\leqslant c\beta P(S^{(q)}(f)^q>\alpha\lambda)+P(f_q^{\#q}>\beta\lambda),$$

this is so called good λ-inequality of $(S^{(q)}(f)^q,\ f_q^{\#q})$ and then there exists a $c>0$ such that (1) holds when β is small enough[1].

(i) $\Rightarrow$ (iii). For $f=(f_n)$ and $\alpha>0,\ \beta>0,\ \lambda>0$, define

$$\tau=\inf\{n,\ \sigma_{n+1}^{(q)}(f)^q>(1+\alpha)\lambda\},$$
$$\theta=\inf\{n,\ \sigma_{n+1}^{(q)}(f)^q>\alpha\lambda\},$$
$$\mu=\inf\{n,\ f_{qn}^{\#q}>\beta\lambda\}.$$

In this time we have

$$P(\tau<\infty,\ \theta<\mu)\leqslant P(\theta<\mu,\ \sigma_{\tau+1}^{(q)}(f)^q-\sigma_\theta^{(q)}(f)^q>\lambda)$$
$$\leqslant\frac{1}{\lambda}\int_{(\theta<\mu)}E(\sigma_{\tau+1}^{(q)}(f)^q-\sigma_\theta^{(q)}(f)^q|\mathscr{B}_\theta)dP$$
$$=\frac{1}{\lambda}\int_{(\theta<\mu)}E\Big(\sum_{i=\theta+1}^{\tau+1}|df_i|^q|\mathscr{B}_\theta\Big)dP. \tag{5}$$

From the q-convexity of X, the inequality

$$E\Big(\sum_{i=n+1}^{m}|df_i|^q|\mathscr{B}_n\Big)\leqslant cE(|f_m-f_n|^q|\mathscr{B}_n)\quad\text{a. e.} \tag{6}$$

$(m\geqslant n\geqslant 0)$ holds $(c=c_q>0)$ and then (5) becomes

$$P(\tau<\infty,\ \theta<\mu)\leqslant\frac{c}{\lambda}\int_{(\theta<\mu)}E(\|f_{\tau+1}-f_\theta\|^q|\mathscr{B}_\theta)\,dP$$
$$\leqslant\frac{c}{\lambda}\int_{(\theta<\mu)}\tilde{f}_q^{\#q}dP\leqslant c\beta P(\theta<\infty).$$

Hence

$$P(\sigma^{(q)}(f)^q>(1+\alpha)\lambda)\leqslant c\beta P(\sigma^{(q)}(f)^q>\alpha\lambda)+P(\tilde{f}_q^{\#q}>\beta\lambda)$$

and the inequality (2) follows in the similar way.

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). Let $f=(f_n)$ be an X-valued Walsh-Paley[13] martingale with $\|f\|_\infty<\infty$. Then the inequality (1) gives

$$\|S^{(q)}(f)^q\|_\Phi^{1/q}\leqslant c\|f_q^{\#q}\|_\Phi^{1/q}\leqslant c\Phi(\|f\|_\infty^q)^{1/q}<\infty$$

and then $S^{(q)}(f)<\infty$ a. e., hence X is q-convexifiable[10]. Similarly, the inequality (2) gives

$$\|\sigma^{(q)}(f)^q\|_\Phi^{1/q}\leqslant c\|\tilde{f}_q^{\#q}\|_\Phi^{1/q}\leqslant c\Phi(\|f\|_\infty^q)^{1/q}<\infty$$

and then $\sigma^{(q)}(f)<\infty$ a. e. But $\sigma^{(q)}(f)^q<\infty$ a. e. iff $S^{(q)}(f)^q<\infty$ a. e., when $E(\operatorname{Sup}_n|df_n|^q)<\infty$, the last condition holds clearly in our case $(\operatorname{Sup}_n|df_n|\leqslant 2\|f\|_\infty$ a. e.); consequently X is q-convexifiable.

Theorem 2. *Suppose that X is Banach space and Φ a moderate Young function with $q_\Phi>1$. Then TFAE:*

(i) *X is p-smoothable* $(1<p\leqslant 2)$.

(ii) *There exists a* $c=c_{p\Phi}>0$ *such that*

$$\|f_p^{\#p}\|_\Phi^{1/p}\leqslant c\|S^{(p)}(f)^p\|_\Phi^{1/p} \tag{7}$$

for every X-valued martingale $f=(f_n)$.

(iii) *There exists a* $c=c_{p\Phi}>0$ *such that*

$$\|\tilde{f}_p^{\#p}\|_\Phi^{1/p}\leqslant c\|\sigma^{(p)}(f)^p\|_\Phi^{1/p} \tag{8}$$

for every X-valued martingale $f=(f_n)$.

Proof (i)$\Rightarrow$(ii). The p-smoothness of X gives a constant $c=c_p>0$ such that

$$E(|f_m-f_{n-1}|^p|\mathscr{B}_n)\leqslant cE\left(\sum_{i=n}^m|df_i|^p|\mathscr{B}_n\right)\quad \text{a. e.} \tag{9}$$

$(m\geqslant n\geqslant 0)$, where $f=(f_n)$ is any X-valued L_p-bounded martingale. Thus we have

$$\begin{aligned}f_p^{\#p}&=\operatorname*{Sup}_n\operatorname*{Sup}_{m\geqslant n}E(|f_m-f_{n-1}|^p|\mathscr{B}_n)\\&\leqslant c\operatorname*{Sup}_n E\left(\sum_{i=n}^\infty|df_i|^p|\mathscr{B}_n\right)\\&\leqslant c\operatorname*{Sup}_n E(S^{(p)}(f)^p|\mathscr{B}_n).\end{aligned}$$

The condition $q_\Phi>1$ and Dellacherie theorem gives

$$\begin{aligned}\|f_p^{\#p}\|_\Phi&\leqslant c\|\operatorname*{Sup}_n E(S^{(p)}(f)^p|\mathscr{B}_n)\|_\Phi\\&\leqslant c\operatorname*{Sup}_n\|E(S^{(p)}(f)^p|\mathscr{B}_n)\|_\Phi\leqslant c\|S^{(p)}(f)^p\|_\Phi,\end{aligned}$$

hence (7) holds.

(i)$\Rightarrow$(iii). To prove (8), we use the similar inequality

$$E(|f_m-f_n|^p|\mathscr{B}_n)\leqslant cE\left(\sum_{i=n+1}^m|df_i|^p|\mathscr{B}_n\right)\quad \text{a. e.} \tag{10}$$

$(m\geqslant n\geqslant 0)$ when X is p-smooth, where $f=(f_n)$ is any X-valued L_p-bounded martingale. Thus we have

$$\begin{aligned}\tilde{f}_p^{\#p}&=\operatorname*{Sup}_n\operatorname*{Sup}_{m\geqslant n}E(|f_m-f_n|^p|\mathscr{B}_n)\\&\leqslant c\operatorname*{Sup}_n E\left(\sum_{i=n+1}^\infty|df_i|^p|\mathscr{B}_n\right)\\&\leqslant c\operatorname*{Sup}_n E(\sigma^{(p)}(f)^p|\mathscr{B}_n),\end{aligned}$$

and the inequality (8) follows in a similar way.

(ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i). Let $f=(f_n)$ be an X-valued Walsh-Paley martingale with $\sum_{n=1}^\infty n^{-p}|df_n|^p\in L_\infty$, define $\tilde{f}^{(n)}=(\tilde{f}_i^{(n)})$ with $d\tilde{f}_n^{(i)}=i^{-1}df_i$ $(i\geqslant n)$ and then $(S^{(p)}(\tilde{f}^{(n)}))^p=\sum_{i=n}^\infty i^{-p}|df_i|^p\downarrow 0$ as $n\to\infty$. By Fatou lemma, (7) assures that $\lim_{n\to\infty}\|\tilde{f}_p^{(n)\#p}\|_\Phi=0$ and hence $\lim_{n\to\infty}E\Phi(\tilde{f}_p^{(n)\#p})=0$. If Ψ is the complement function of the Young function Φ, we have

$$E|\tilde{f}_m^{(n)}-\tilde{f}_i^{(n)}|^p\leqslant E\tilde{f}_p^{(n)\#p}\leqslant\Psi(1)E\Phi(\tilde{f}_p^{(n)\#p})\to 0\quad(n\to\infty).$$

Hence $\sum_{n=1}^{\infty} n^{-1} df_n$ converges in L_p and a. e. The Kronecher lemma gives that $n^{-1}f_n \to 0$ a. e., this shows that X is p-smoothable by Hoffmann-Jorgensen and Pisier theorem[6]. The similar argument with a little modification gives the same consequence from condition (iii).

The following corollary follows from the Kwapien theorem[7] and Theorems 1, 2.

Corollary 1. *Banach space X is isomorphic to Hilbert space iff there exists a $c = c_\Phi > 0$ such that*

$$c^{-1}\|f_2^{\#2}\|_\Phi^{1/2} \leqslant \|S^{(2)}(f)\|_\Phi^{1/2} \leqslant c\|f_2^{\#2}\|_\Phi^{1/2} \tag{11}$$

or (and)

$$c^{-1}\|\tilde{f}_2^{\#2}\|_\Phi^{1/2} \leqslant \|\sigma^{(2)}(f)^2\|_\Phi^{1/2} \leqslant c\|\tilde{f}_2^{\#2}\|_\Phi^{1/2} \tag{12}$$

for every X-valued martingale $f=(f_n)$, where Φ is a moderate Young function with $q_\Phi > 1$.

§ 2. W_p and $\widetilde{W}_p$

For every increasing sequence (n_k) of natural numbers we define

$$W_p^{(n_k)}(f) = \Big(\sum_{k=0}^{\infty} |f_{n_{k+1}} - f_{n_k}|^p\Big)^{1/p},$$

$$\widetilde{W}_p^{(n_k)}(f) = \Big(\sum_{k=0}^{\infty} E(|f_{n_{k+1}} - f_{n_k}|^p | \mathscr{B}_{n_k})\Big)^{1/p},$$

where $n_0 = 0$ and

$$\|W_p(f)^p\|_\Phi^{1/p} = \mathrm{Sup}\,\|W_p^{(n_k)}(f)^p\|_\Phi^{1/p},$$
$$\|\widetilde{W}_p(f)\|_\Phi^{1/p} = \mathrm{Sup}\,\|\widetilde{W}_p^{(n_k)}(f)^p\|_\Phi^{1/p},$$

where the Sup runs over all such (n_k).

Theorem 3. *Suppose that X is a Banach space and Φ is a moderate Young function. Then TFAE:*

(i) *X is p-smoothable $(1 < p \leqslant 2)$.*

(ii) *There exists a $c = c_{p\Phi} > 0$ such that*

$$c^{-1}\|S^{(p)}(f)^p\|_\Phi^{1/p} \leqslant \|W_p(f)^p\|_\Phi^{1/p} \leqslant c\|S^{(p)}(f)^p\|_\Phi^{1/p} \tag{13}$$

for every X-valued martingale $f=(f_n)$.

(iii) *There exists a $c = c_{p\Phi} > 0$ such that*

$$\|f^{*p}\|_\Phi^{1/p} \leqslant c\|W_p(f)^p\|_\Phi^{1/p} \tag{14}$$

for every X-valued martingale $f=(f_n)$.

Proof (i)$\Rightarrow$(ii). For fix sequence (n_k), $0 = n_0 < n_1 < n_2 < \cdots$, $(f_{n_k})_{k\geq 0}$ is a martingale. From the p-smoothness of X, the inequality (9) gives

$$E(W_p^{(n_k)}(f)^p|\mathscr{B}_0)=E\Big(\sum_{k=0}^{\infty}E(|f_{n_k+1}-f_{n_k}|^p|\mathscr{B}_{n_k}|\mathscr{B}_0\Big)$$
$$\leqslant cE\Big(\sum_{k=0}^{\infty}E\Big(\sum_{i=n_k+1}^{n_{k+1}}|df_i|^p|_n\mathscr{B}_k\Big)\mathscr{B}_0\Big)$$
$$=cE(S^{(p)}(f)^p|\mathscr{B}_0). \tag{15}$$

Replace $f=(f_n)$ by $g=(g_i)=(f_i-f_{n_{k_0}-1})(i\geqslant n_{k_0})$, then (15) becomes

$$E(W_p^{(n_k)}(f)^p-W_{pn_{k_0}-1}^{(n_k)}(f)^p|\mathscr{B}_{n_{k_0}})=E(W_p^{(n_k)}(g)^p|\mathscr{B}_{n_{k_0}})$$
$$\leqslant cE(S^{(p)}(g)^p|\mathscr{B}_{n_{k_0}})$$
$$\leqslant cE(S^{(p)}(f)^p|\mathscr{B}_{n_{k_0}}). \tag{16}$$

Let $A_i=W_{pn_i}^{(n_k)}(f)^p$, $B=S^{(p)}(f)^p$ and τ be a stopping time with respect to $(\mathscr{B}_{n_k}, k\geqslant 0)$. Then (16) can be rewritten as

$$E(A_\infty-A_{\tau-1}|\mathscr{B}_\tau)\leqslant cE(B|\mathscr{B}_\tau).$$

Thus Neveu-Garsia lemma[5] implies

$$E\Phi(W_p^{(n_k)}(f)^p)\leqslant cE\Phi(S^{(p)}(f)^p) \tag{17}$$

and

$$\|W_p^{(n_k)}(f)^p\|_\Phi\leqslant c\|S^{(p)}(f)^p\|_\Phi.$$

The right hand of the inequality (13) holds.

In the other hand, taking $(n_k)=(k)$, we have

$$\|S^{(p)}(f)^p\|_\Phi^{1/p}\leqslant \mathrm{Sup}\,\|W_p^{(n_k)}(f)^p\|_\Phi^{1/p}=\|W_p(f)^p\|_\Phi^{1/p}.$$

The proof of (13) is completed.

(i) $\Rightarrow$ (iii). we proved in [10] that there exists $c>0$ such that

$$\|f^{*p}\|_\Phi^{1/p}\leqslant c\|S^{(p)}(f)^p\|_\Phi^{1/p} \tag{18}$$

for every X-valued martingale $f=(f_n)$ and Young function Φ when X is p-smoothable. Consequently, (14) follows from (18) and the left hand of (13).

(iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i). By applying the method of the proof (ii) $\Rightarrow$ (i) of Theorem 2, the p-smoothablity of X follows from (iii). To prove (ii)$\Rightarrow$(i), notice that we have

$$\||f_n|^p\|_\Phi^{1/p}=\||f_n-f_0|^p\|_\Phi^{1/p}\leqslant\|W_p(f)^p\|_\Phi^{1/p}\leqslant c\|S^{(p)}(f)^p\|_\Phi^{1/p}$$

from the right hand of (13). Applying the same method to this inequality, we obtain the p-smoothability of X.

The proof of the following corollary is similar to that of Theorem 3 and can be omitted.

Corollary 2. *Suppose that X is a Banach space and Φ is as in Theorem 3. Then TFAE:*

(i) *X is p-smoothable* $(1<p\leqslant 2)$.

(ii) *There exists a $c=c_{p\Phi}>0$ such that*

$$c^{-1}\|\sigma^{(p)}(f)^p\|_\Phi^{1/p}\leqslant\|\widetilde{W}_p(f)^p\|_\Phi^{1/p}\leqslant c\|\sigma^{(p)}(f)^p\|_\Phi^{1/p} \tag{19}$$

for every X-valued martingale $f=(f_n)$.

(iii) *There exists a $c=c_{p\Phi}>0$ such that*

$$\|f^{*p}\|_{\Phi}^{1/p}\leqslant c\|\widetilde{W}_p(f)^p\|_{\Phi}^{1/p} \tag{20}$$

for every X-valued martingale $f=(f_n)$.

Theorem 4. *Suppose that X is a Banach space and Φ a moderate Young function. Then TFAE:*

(i) *X is q-converifiable* $(2\leqslant q<\infty)$.

(ii) *There exists a* $c=c_{q\Phi}>0$ *such that*

$$\|W_q(f)^q\|_{\Phi}^{1/p}\leqslant c\|f^{*q}\|_{\Phi}^{1/q} \tag{21}$$

for every X-valued martingrle $f=(f_n)$.

(iii) *There exists a* $c=c_{q\Phi}>0$ *such that*

$$\|\widetilde{W}_q(f)^q\|_{\Phi}^{1/p}\leqslant c\|f^{*q}\|_{\Phi}^{1/q} \tag{22}$$

for every X-valued martingale $f=(f_n)$.

Proof (i)⇒(ii). For fix sequence (n_k), $0=n_0<n_1<n_2<\cdots$, $(f_{n_k}, k\geqslant 0)$ is a martingale. The q-convexity of X gives

$$E\Big(\sum_{k=1}^{\infty}\|f_{n_{k+1}}-f_{n_k}\|^q\,|\,\mathscr{B}_0\Big)\leqslant c\,\mathrm{Sup}_k E(\|f_{n_k}\|^q\,|\,\mathscr{B}_0)$$
$$\leqslant cE(f^{*q}\,|\,\mathscr{B}_0) \tag{23}$$

Replace $f=(f_n)$ by $g=(g_i)=(f_i-f_{n_{k_0-1}})\,(i\geqslant n_{k_0})$, then (23) becomes

$$E(W_q^{(n_k)}(f)^q-W_{qn_{k_0-1}}^{(n_k)}(f)^q\,|\,\mathscr{B}_{n_{k_0}})\leqslant cE(f^{*q}\,|\,\mathscr{B}_{n_{k_0}}).$$

Let $A_i=W_{qni}^{(n_k)}(f)^q$, $B=f^{*q}$ and τ be a stopping time with respect to $(\mathscr{B}_{n_k}, k\geqslant 0)$. The above inequality can be rewritten as

$$E(A_\infty-A_{\tau-1}\,|\,\mathscr{B}_\tau)\leqslant cE(B\,|\,\mathscr{B}_\tau).$$

Similar argument gives

$$E\Phi(W_q^{(n_k)}(f)^q)\leqslant cE\Phi(f^{*q})$$

and, finally, the inequality (21) holds.

(ii)⇒(iii). The inequality (22) follows from (21) and the inequality

$$E\Phi(\widetilde{W}_q^{(n_k)}(f)^q)\leqslant cE\Phi(W_q^{(n_k)}(f)^q) \tag{24}$$

which holds for every martingale $f=(f_n)$ and also is due to Neveu and Garsia[5].

(iii)⇒(i). The proof is similar to that of (iii)⇒(i) of Theorem 1.

Corollary 3. *Banach space X is isomorphic to Hilbert space iff there exists a* $c=c_\Phi>0$ *such that*

$$c^{-1}\|f^{*2}\|_{\Phi}^{1/2}\leqslant\|W_2(f)^2\|_{\Phi}^{1/2}\leqslant c\|f^{*2}\|_{\Phi}^{1/2} \tag{25}$$

or (and)

$$c^{-1}\|f^{*2}\|_{\Phi}^{1/2}\leqslant\|\widetilde{W}_2(f)^2\|_{\Phi}^{1/2}\leqslant c\|f^{*2}\|_{\Phi}^{1/2} \tag{26}$$

for every X-valued martingale $f=(f_n)$, *where Φ is a moderate Young function.*

§ 3. T_v

For a martingale $f=(f_n)$ and a real adapted R. V. sequence $v=(v_n)$, usual, define

$$T_v f=((T_v f)_n),\ (T_v f)_0=0,\ (T_v f)_n=\sum_{i=1}^{n} v_{i-1} df_i.$$

We shall consider X-valued martingale space pK_a, H_a which are similar to the classical case, for the definitions see [12].

The following proposition follows from Burkholder theorem[2] by using standard conditionallization method.

Proposition 1. *Banach space $X\in UMD$ (i. e. X has UMD-property) iff one of the following three inequalities holds for any (or some) $1<p<\infty$, every X-valued martingale $f=(f_n)$ and $v=(v_n)$ with $v^*\leqslant 1$:*

(i) $$E(|(T_v f)_m-(T_v f)_{n-1}|^p|\mathscr{B}_n)\leqslant cE(\|f_m-f_{n-1}\|^p|\mathscr{B}_n) \quad \text{a. e.} \tag{27}$$

(ii) $$(T_v f)_p^{\#}\leqslant cf_p^{\#} \quad \text{a. e.} \tag{28}$$

(iii) $$E(\sup_{m\geqslant n}(|(T_v f)_m-(T_v f)_{n-1}|^p|\mathscr{B}_n)\leqslant cE(\sup_{m\geqslant n}\|f_m-f_{n-1}\|^p|\mathscr{B}_n) \quad \text{a. e.}$$
$$(m\geqslant n\geqslant 0). \tag{29}$$

Remark. The proposition also holds if "martingale" is replaced by "dyadic martingale" in the statment. In fact, the index p of the inequality (29) can be extented to whole inteval $(0,\infty)$ if $f=(f_n)$ is a dyadic martingale[4].

Proposition 2. *Let X be a Banach space and $1<p<\infty$. Then there exists a $c>0$ such that*

$$c^{-1}E\Phi(f_p^{**})\leqslant E\Phi(f_p^{\#})\leqslant cE\Phi(f_p^{**}) \tag{30}$$

for every martingale $f=(f_n)$ and general function Φ, where

$$f_p^{**}=\sup_n \sup_{m\geqslant n}[E(|f_m|^p|\mathscr{B}_n)^{1/p}.$$

Proof The proof of the right hand of (30) is similar to the clasical case (see [11]). To the left hand, we define

$$\tau=\inf\{n,\ |f_n|>(\alpha+1)\lambda\},$$
$$\theta=\inf\{n,\ |f_n|>2^{1/p-1}\alpha\lambda\},$$
$$\mu=\inf\{n,\ f_{pn}^{\#}>\beta\lambda\}.$$

Then $\theta\leqslant\tau$ and

$$P(\tau<\infty)\leqslant P(\tau<\infty,\ \theta<\mu)+P(\mu<\infty).$$

From

$$P(\tau<\infty,\ \theta<\mu)\leqslant P(\theta<\mu,\ |f_\tau-f_{\theta-1}|^p>2^{1-p}\lambda^p)$$
$$\leqslant\frac{2^{p-1}}{\lambda^p}\int_{\{\theta<\mu\}}E(|f_\tau-f_{\theta-1}|^p|\mathscr{B}_\theta)\,dP$$
$$\leqslant\frac{2^{p-1}}{\lambda^p}\int_{\{\theta<\mu\}}f_{p\theta}^{\#p}\,dP\leqslant 2^{p-1}\beta^p P(\theta<\infty)$$

we get

$$P(f^*>(1+\alpha)\lambda)\leqslant 2^{p-1}\beta^p P(f^*>\alpha\lambda)+P(f_p^{\#}>\beta\lambda)$$

and hence $E\Phi(f^*)\leqslant cE\Phi(f_p^{\#})$ for given Φ. Now from the obvious inequality

$$[E(|f_m|^p|\mathscr{B}_n)]^{1/p}\leqslant[E(|f_m-f_{n-1}|^p|\mathscr{B}_n)^{1/p}+|f_{n-1}|\leqslant f_p^{\#}+f^*$$

the left hand of (30) follows.

Theorem 5. *Suppose that X is a Banach space, $1<p<\infty$ and Φ a moderate*

Young function. Then TFAE:

(i) $X\in UMD$.

(ii) *There exists a* $c=c_{p\Phi}>0$ *such that*

$$\|(T_vf)_p^{**}\|_\Phi\leqslant c\|f_p^{\#}\|_\Phi \tag{31}$$

for every X-valued martingale $f=(f_n)$ *and* $v=(v_n)$ *with* $v^*\leqslant 1$.

(iii) *There exists a* $c=c_{p\Phi}>0$ *such that*

$$\|(T_vf)_p^{\#}\|_\Phi\leqslant c\|f_p^{**}\|_\Phi \tag{32}$$

for every X-valued martingale $f=(f_n)$ *and* $v=(v_n)$ *with* $x^*\leqslant 1$.

Proof The inequalities (28) and (30) assure that (31) and (32) hold when $X\in UMD$. Conversely, let $f=(f_n)$ be a Walsh-Paley martingale with $\|f\|_\infty<\infty$. Then (31) gives

$$\|(T_vf)_p^{**}\|_\Phi\leqslant c\|f_p^{\#}\|_\Phi\leqslant c\Phi(\|f\|_\infty)<\infty$$

and $(T_vf)_p^{**}<\infty$ a. e. Notice that $(T_vf)^*\leqslant(T_vf)_p^{**}$ a. e, hence $(T_vf)^*<\infty$ a. e. and $X\in UMD$ by Burkholder theorem [3]. On the other hand, (30) and (32) give the same consequence, the proof is completed.

Recall that the Burkholder inequality

$$\|(T_vf)^*\|_p\leqslant c\|f^*\|_p$$

$(0<p<\infty)$ shows that as an operator, every T_v: $H_p\to H_p$ (with $0<p<\infty$ and $v^*\leqslant 1$) is bounded iff $X\in UMD$. Similarly, we can prove that every T_v: ${}_pK_a\to{}_pK_a$ is bounded iff $X\in UMD$ (the proof is trivial). For dyadic martingales, the further results hold.

Theorem 6. *Suppose that X is a Banach space. Then TFAE:*

(i) $X\in UMD$.

(ii) *For any* $0<p_1, p_2<\infty$, $p_1\neq p_2$ *and every dyadic martingale* $f=(f_n)\in H_{p_1}$, *there exists an* $r=(r_n)\in\Gamma_\alpha$ (*i. e.*, r_n *is nonnegative increasing R. V. sequence and* $Er_\infty^\alpha<\infty$) *such that* $T_{vr^{-1}}f\in H_{p_2}$ *and*

$$\|T_{vr^{-1}}f\|_{H_{p_2}}^{p_2}\leqslant c\|f\|_{H_{p_1}}^{p_1},\quad Er_\infty^\alpha\leqslant c\|f\|_{H_{p_1}}^{p_1}, \tag{33}$$

where

$$(T_{vr^{-1}}f)_0=0,\quad (T_{vr^{-1}}f)_n=\sum_{i=1}^n v_{i-1}r_{i-1}^{-1/\alpha}df_i$$

and $v=(v_n)$ *with* $v_n=\pm1$, $\alpha=p_1p_2(p_2-p_1)^{-1}$.

(iii) *For any* $0<p<\infty$ *and every dyadic martingale* $f=(f_n)\in H_p$, *there exists an* $r=(r_n)\in\Gamma_p$ *such that* $T_{vr^{-1}}f\in BMO$ *and*

$$\|T_{vr^{-1}}f\|_{BMO}\leqslant c,\quad Er_\infty^p\leqslant c\|f\|_{H_p}^p, \tag{34}$$

where

$$(T_{vr^{-1}}f)_0=0,\quad (T_{vr}f)_n=\sum_{i=1}^n v_{i-1}r_{i-1}^{-1}df_i$$

and $v=(v_n)$ *with* $v_n=\pm1$.

Proof (i)⇒(ii). First of all, notice that $f\in H_p$ iff $f\in\Gamma_p$ when f is a dyadic martingale, and then there exists an $\lambda\in\Gamma_p$ such that $|f_n|\leqslant\lambda_{n-1}$ and $|f|_{p\Phi}=\|\lambda_\infty\|_p$, Denote by $r=(\lambda^{p_1/\alpha}(f))$ this sequence in the following.

Now if $0<p_1<p_2<\infty$, let $\beta=1-p_1/p_2$. From $T_{vr^{-1}}f=T_v(T_{r^{-1}}f)$ we have firstly

$$(T_{r^{-1}}f)_n=\sum_{i=1}^{n}\lambda_{i-1}(f)^{-p_1/\alpha}df_i=\frac{f_n}{\lambda_{n-1}(f)^\beta}+\sum_{i=1}^{n-1}f_i\left(\frac{1}{\lambda_{i-1}(f)^\beta}-\frac{1}{\lambda_i(f)^\beta}\right)$$

and then

$$\|(T_{r^{-1}}f)_n\|\leqslant\lambda_{n-1}^{1-\beta}+\sum_{i=1}^{n-1}\lambda_{i-1}(f)\left(\frac{1}{\lambda_{i-1}(f)^\beta}-\frac{1}{\lambda_i(f)^\beta}\right)$$
$$\leqslant\sum_{i=1}^{n}\frac{1}{\lambda_{i-1}(f)^\beta}(\lambda_{i-1}(f)-\lambda_{i-2}(f))$$
$$=\int_0^{\lambda_{i-1}(f)}\frac{dt}{t^\beta}\leqslant\frac{p_2}{p_1}\lambda_{n-1}(f)^{p_1/p_2}.$$

Hence $T_{r^{-1}}f\in H_{p_2}$ and

$$\|T_{r^{-1}}f\|_{H_{p_2}}^{p_2}\leqslant c\|f\|_{H_{p_1}}^{p_1}. \tag{35}$$

From the UMD-property of X, the Burkholder inequality gives

$$\|T_{vr^{-1}}f\|_{H_{p_2}}^{p_2}\leqslant c\|T_{r^{-1}}f\|_{H_{p_2}}^{p_2}\leqslant c\|f\|_{H_{p_1}}^{p_1},$$

this is (33).

If $0<p_2<p_1<\infty$, a similar computation gives

$$\|(T_{r^{-1}}f)_n\|\leqslant|f_n|\lambda_{n-1}(f)^{-p_1/\alpha}+\sum_{i=1}^{n}|f_i|(\lambda_i(f)^{-p_1/\alpha}-\lambda_{i-1}(f)^{-p_1/\alpha})$$
$$\leqslant2\lambda_{n-1}(f)^{1-p_1/\alpha}=2\lambda_{n-1}{}^{p_1/p_2}.$$

Then $T_{r^{-1}}f\in H_{p_2}$ and

$$\|T_{r^{-1}}f\|_{H_{p_2}}^{p_2}\leqslant c\|f\|_{H_{p_1}}^{p_1}.$$

Consequently, (12) follows from the UMD-property of X and the same argument.

The proof of converse is easy.

(i)⇒(iii). Let $\lambda=(\lambda_n(f))$ as above, take $p_0<\min(1,p)$ and define $r=(r_i)$, where

$$r_i=\operatorname*{Sup}_{j\leqslant i}E(\lambda_\infty(f)^{p_0}|\mathscr{B}_j)^{1/p_0},\ i\geqslant1.$$

Consider $g=(g_n)$, where

$$g_0=0,\ g_n=\sum_{i=1}^{n}r_{i-1}^{-1}df_i(n\geqslant1).$$

Then

$$g_m-g_{n-1}=\sum_{i=n}^{m}r_{i-1}^{-1}df_i=\frac{f_m-f_n}{\gamma_{m-1}}+\sum_{i=n}^{m-1}(f_i-f_{i-1})\left(\frac{1}{r_{i-1}}-\frac{1}{r_i}\right),$$
$$\|g_m-g_{n-1}\|\leqslant\frac{2\lambda_{m-1}}{r_{m-1}}+2\sum_{i=n}^{m-1}\lambda_{i-1}\left(\frac{1}{r_{i+1}}-\frac{1}{r_i}\right)$$
$$\leqslant2+2\sum_{i=n}^{m-1}(\lambda_i-\lambda_{i-1})\frac{1}{r_i}. \tag{36}$$

Since

$$1\leqslant E(\lambda_\infty(f)^{p_0}|\mathscr{B}_i)E\left(\frac{1}{\lambda_\infty(f)^{p_0}}\middle|\mathscr{B}_i\right)\leqslant r_i^{p_0}E\left(\frac{1}{\lambda_\infty(f)^{p_0}}\middle|\mathscr{B}_i\right),$$
$$\frac{1}{r_i}\leqslant E\left(\frac{1}{\lambda_\infty(f)^{p_0}}\middle|\mathscr{B}_i\right)^{1/p_0}\leqslant E\left(\frac{1}{\lambda_\infty(f)}\middle|\mathscr{B}_i\right),$$

(36) becomes

$$|g_m-g_{n-1}|\leq 2+2\sum_{i=n}^{m-1}(\lambda_i-\lambda_{i-1})E\left(\frac{1}{\lambda_\infty(f)}\Big|\mathscr{B}_i\right)$$

and then

$$E(\operatorname{Sup}_{m\geq n}|g_m-g_{n-1}|\,|\mathscr{B})_n\leq 2+2E\left(\sum_{i=n}^{\infty}(\lambda_i-\lambda_{i-1})E\left(\frac{1}{\lambda_\infty(f)}\Big|\mathscr{B}_i\right)\Big|\mathscr{B}_n\right)$$

$$\leq 2+2E\left(\sum_{i=n}^{\infty}\frac{\lambda_i-\lambda_{i-1}}{\lambda_\infty(f)}\Big|\mathscr{B}_n\right)\leq 4.$$

Notice that g is also a dyadic martingale, the UMD-property of X and the remark assure

$$E(\operatorname{Sup}_{m\geq n}\|(T_{vr^{-1}}f)_m-(T_{vr^{-1}}f)_{n-1}\|\,|\mathscr{B}_n)\leq E(\operatorname{Sup}_{m\geq n}\|g_m-g_{n-1}\|\,|\mathscr{B}_n)\leq 4.$$

On the other hand, it is clear that

$$Er_\infty^p=E\operatorname{Sup}_{j\geq 1}E(\lambda_\infty(f)^{p_0}|\mathscr{B}_j)^{p/p_0}\leq cE\lambda_\infty(f)^p<\infty,$$

this is required. The converse is also easy.

Remark. For the classical case, see [12].

Theorem 7. *Suppose that X is a Banach space. Then TFAE:*

(i) $X\in UMD$.

(ii) *For any $0<\alpha,\ \beta<\infty$ and $1<1/\gamma=1/\alpha+1/\beta$, there exists a $c=c_{\alpha,\beta,\gamma}>0$ such that*

$$E((T_{vr}f)^{*\gamma}|\mathscr{B}_0)^{1/\gamma}\leq cE(f^{*\alpha}|\mathscr{B}_0)^{1/\alpha}E(r_\infty^\beta|\mathscr{B}_0)^{1/\beta}\quad a.\ e. \tag{37}$$

for every $f=(f_n)\in H_\alpha$ and $r\in\Gamma_\beta$.

(iii) *For every $0<\alpha,\ \beta<\infty$ and $1<1/\gamma=1/\alpha+1/\beta$, there exists a $c=c_{\alpha,\beta,\gamma}>0$ such that*

$$\|T_{vr}f\|_{H_\gamma}\leq c\|f\|_{H\alpha}\|r_\infty\|_\beta \tag{38}$$

for every $f=(f_n)\in H_\alpha$ and $r\in\Gamma_\beta$, where $T_{vr}f=((T_{vr}f)_n,\ (T_{vr}f)_0=0,\ (T_{vr}f)_n=\sum_{i=1}^n v_{i-1}r_{i-1}df_i\ (n\geq 1)$ and $v=(v_n)$ with $v^*\leq 1$.

Proof Let $f\in H_\alpha$ and $r\in\Gamma_\beta$. From

$$(T_rf)_n=r_{n-1}f_n-\sum_{i=1}^n f_i(r_i-r_{i-1})$$

we have

$$(T_rf)_n^*\leq 2f_n^*r_{n-1}\quad\text{and}\quad(T_rf)^*\leq 2f^*r_\infty,$$

and then the conditional Holder inequality gives

$$E((T_rf)^{*\gamma}|\mathscr{B}_0)\leq cE(f^{*\alpha\frac{\gamma}{\alpha}}r_\infty^{\beta\frac{\gamma}{\beta}}|\mathscr{B}_0)\leq cE(f^{*\alpha}|\mathscr{B}_0)^{\gamma/\alpha}E(r_\infty^\beta|\mathscr{B}_0)^{\gamma/\beta}.$$

The UMD-property of X and (29) assure

$$E((T_{vr}f)^{*\gamma}|\mathscr{B}_0)^{1/\gamma}\leq cE((T_rf)^{*\gamma}|\mathscr{B}_0)^{1/\gamma}.$$

Hence (37) holds.

Take expectation in two hands of (37) and use the Holder inequality, the inequality (38) is obtained.

The UMD-property of X follows from the Walsh-Paley martingale arguments.

Theorem 8. *Suppose that X is a Banach space and $1<p<\infty$. Then TFAE:*

(i) $X \in UMD$.

(ii) *For any* $0<\alpha, \beta<\infty$ *and* $1/\gamma=1/\alpha+1/\beta$ *with* $\gamma p>1$, *there exists a* $c=c_{\alpha,\beta,\gamma,p}>0$ *such that*

$$E((T_{vr}f)^{\#p\gamma}_{p\gamma}|\mathscr{B}_0)^{1/p\gamma} \leqslant cE(f^{\#p\alpha}_{p\alpha}|\mathscr{B}_0)^{1/p\alpha}E(r_\infty^{p\beta}|\mathscr{B}_0)^{1/p\beta} \quad \text{a. e.} \tag{39}$$

for every $f=(f_n)\in H_{p\alpha}$ *and* $r=(r_n)\in\Gamma_{p\beta}$.

(iii) *For any* $0<\alpha, \beta<\infty$ *and* $1/\gamma=1/\alpha+1/\beta$ *with* $\gamma p>1$, *there exists a* $c=c_{\alpha,\beta,\gamma,p}>0$ *such that*

$$\|(T_{vr}f)^{\#}_{p\gamma}\|_{p\gamma} \leqslant c\|f^{\#}_{p\alpha}\|_{p\alpha}\|r_\infty\|_{p\beta} \tag{40}$$

for every $f=(f_n)\in H_{p\alpha}$ *and* $r=(r_n)\in\Gamma_{p\beta}$, *where* $T_{vr}f$ *and* $v=(v_n)$ *are as in Theorem 7.*

Proof Firstly, we have

$$(T_rf)_m-(T_rf)_{n-1}=r_{m-1}(f_m-f_{n-1})+\sum_{i=n}^{m-1}(f_i-f_{n-1})(r_{i-1}-r_i)$$

and then

$$\operatorname*{Sup}_{m\geqslant n}|(T_rf)_m-(T_rf)_{n-1}| \leqslant 2\operatorname*{Sup}_{m\geqslant n}|f_m-f_{n-1}|r_\infty.$$

The conditional Holder inequality gives

$$\begin{aligned}
&[E(\operatorname*{Sup}_{m\geqslant n}|(T_rf)_m-(T_rf)_{n-1}|^{p\gamma}|\mathscr{B}_n)]^{1/p\gamma}\\
&\leqslant c[E(\operatorname*{Sup}_{m\geqslant n}|f_m-f_{n-1}|^{p\gamma\frac{\gamma}{\alpha}}r_\infty^{p\beta\frac{\gamma}{\beta}}|\mathscr{B}_n)]^{1/p\gamma}\\
&\leqslant c[E(\operatorname*{Sup}_{m\geqslant n}|f_m-f_{n-1}|^{p\alpha}|\mathscr{B}_n)]^{1/p\alpha}[E(r_\infty^{p\beta}|\mathscr{B}_n)]^{1/p\beta}
\end{aligned}$$

and the conditional Doob inequality gives

$$\begin{aligned}
(T_rf)^{\#}_{p\gamma n}&=\operatorname*{Sup}_{m\geqslant n}[E(|(T_rf)_m-(T_rf)_{n-1}|^{p\gamma}\mathscr{B}|_n)]^{1/p\gamma}\\
&\leqslant[E(\operatorname*{Sup}_{m\geqslant n}|(T_rf)_m-(T_rf)_{n-1}|^{p\gamma}|\mathscr{B}_n)]^{1/p\gamma}\\
&\leqslant c[E(\operatorname*{Sup}_{m\geqslant n}|f_m-f_{n-1}|^{p\alpha}|\mathscr{B}_n)]^{1/p\alpha}[E(r_\infty^{p\beta}|\mathscr{B}_n)]^{1/p\beta}\\
&\leqslant c\operatorname*{Sup}_{m\geqslant n}[E(|f_m-f_{n-1}|^{p\alpha}|\mathscr{B}_n)^{1/p\alpha}[E(r_\infty^{p\beta}|\mathscr{B}_n)]^{1/p\beta}\\
&=cf^{\#}_{p\alpha n}[E(r_\infty^{p\beta}|\mathscr{B}_n)]^{1/p\beta}
\end{aligned}$$

and

$$(T_rf)^{\#}_{p\gamma}\leqslant cf^{\#}_{p\alpha}\operatorname*{Sup}_n[E(r_\infty^{p\beta}|\mathscr{B}_n)]^{1/p\beta}.$$

Hence

$$\begin{aligned}
[E((T_rf)^{\#p\gamma}_{p\gamma}|\mathscr{B}_0)]^{1/p\gamma}&\leqslant c[E(f^{\#p\alpha}_{p\alpha}|\mathscr{B}_0)]^{1/p\alpha}[E(\operatorname*{Sup}_n E(r_\infty^{p\beta}|\mathscr{B}_n)|\mathscr{B}_0)^{1/p\beta}\\
&=c[E(f^{\#p\alpha}_{p\alpha}|\mathscr{B}_0)]^{1/p\alpha}[E(r_\infty^{p\beta}|\mathscr{B}_0)]^{1/p\beta}.
\end{aligned}$$

The UMD-property of X and (29) assure (39) holds. Take expectation in two hands in (39), the inequality (40) is obtained by Holder inequality.

Conversely, the UMD-property of X follows from Theorem 5 and the arguments of Walsh-Paley martingales as above.

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