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A LAW OF THE ITERATED LOGARITHM FOR NEIGHBOR-TYPE REGRESSION FUNCTION ESTIMATOR

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Abstract

Let (X, Y) be a two-dimensional random variable. A law of the iterated logarithm is established for a smoothed neighbor-type estimator of the regression function $m(x)=E(Y|X=x)$ under conditions much weaker than needed for the Nadaraya-Watson estimator. Also the sharp pointwise rates of strong consistency of this estimator is discussed in detail.

§ 1. Introduction and Background

Assume that (X, Y) is a two-dimensional random variable with finite expectation $E(Y)$. Then the regression function $m(x)=E(Y|X=x)$ of Y on X exists and is (almost surely in x) uniquely defined in view of the equation $m(X)=E(Y|X)$. Let (X_1, Y_1), (X_2, Y_2), $\cdots$ be independent random observations with the same distribution as (X, Y). It is required to estimate $m(x)$ with these observations.

Nadaraya[7] and Watson[1] independently proposed the kernel estimator

$$m_{1n}(x)=\frac{\sum_{i=1}^{n} Y_i K((x-X_i)/h)}{\sum_{i=1}^{n} K((x-X_i)/h)}, \tag{1}$$

where K is an appropriate kernel function and $h=h_n\to 0$ is a sequence of bandwidths. Since then, there have been lots of researches in literature about its consistency and rates of convergence. Härdle[3] established a law of the iterated logarithm (LIL) for m_{1n} and obtained sharp pointwise rates of strong consistency of m_{1n} under conditions requiring a) the differentiabilities of $m(x)$ and density of X and the continuity of $E(Y^2|X=x)$, b) appropriate rate on the tail of $E(Y^2)$. Hong[4], with slightly stronger condition on Y than that of Härdle, extended Härdle's results to the case of random bandwidth kernel estimator (i. e.,

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bandwidths h may be a sequence of positive random variables depending on (X_i, Y_i), $1\leqslant i\leqslant n$), and, as an application, obtainedLIL for a special nearest neighbor estimator

$$m_{2n}(x)=\sum_{i=1}^{k} Y_{R_i}/k,\ 1\leqslant k=o(n),$$

where $R_1, \cdots, R_n$ is a rearrangement of $1, \cdots, n$ in the light of

$$\|X_{R_1}-x\|\leqslant\|X_{R_2}-x\|\leqslant\cdots\leqslant\|X_{R_n}-x\|.$$

Apropos of nearest neighbor estimhtor of the general form

$$m_{3n}(x)=\sum_{i=1}^{n}\nu_{ni}Y_{R_i},$$

where $\{\nu_{ni}, 1\leqslant i\leqslant n\}$ is a sequence of appropriate nonnegative numbers, the obtainment of the LIL is more difficult than that of kernel estimator although almost all other large sample prorerties of them are comparable. Hong and Gao[6] obtained an upper bound of LIL for m_{3n} under several restricted conditions on $\{\nu_{ni}, 1\leqslant i\leqslant n\}$.

Another important estimator of $m(x)$ proposed by Yang[13] has the form (with the same meaning of K and h)

$$m_n(x)=(nh)^{-1}\sum_{i=1}^{n} Y_i K\Big(\frac{F_n(x)-F_n(X_i)}{h}\Big). \tag{2}$$

Here F_n is the empirical distribution function (e. d. f.) of $X_1, \cdots, X_n$. That m_n is in fact a (smoothed) neighbor-type estimator may be seen when $K=I_{(-\frac{1}{2},\frac{1}{2})}$. In this case $m_n(x)$ is the average number of Y_i, s for which, when $X_i\geqslant x$ (say), there exist no more than $k_n=nh/2$ X_j-values with $x\leqslant X_j$ and $X_j<X_i$. The mean square convergence and asymptotic normality of m_n to m have been studied by Yang[13] and Stute [10] respectively.

In this paper the LIL of m_n is derived under conditions much weeker than needed for kernel estimator. Specifically, only the differentiability of $E(Y|F(X)=x)$ is required (F is the distribution of X), while X need not have a density at all. On the other hand, the condition on Y is the same as that of kernel estimator. We also establish the LIL for the following estimator

$$\hat{m}_n(x)=\frac{\sum_{i=1}^{n} Y_i K((F_n(x)-F_n(X_i))/h)}{\sum_{i=1}^{n} K((F_n(x)-F_n(X_i))/h)}. \tag{3}$$

With these results of LIL, we also discuss in detail the sharp pointwise rates of strong consistency of m_n and $\hat{m}_n$ as done in [5] for nearest neighbor estimator of density function.

As a footnote, we would like to mention some related works on density estimation. These include among others Hall[2] (LIL for kernel type eitimators), Hong[5] (LIL for random bandwidth kernel estimators including nearest neighbor estimator).

§ 2. LIL and Sharp Pointwise Rates

Let $H_n(x, y)$ denote the empirical distribution function of $(X_1, Y_1), \cdots, (X_n, Y_n)$, $H(x, y)$ and $F(x)$ denote the distribution functions of (X, Y) and X respectively. Take

$$\tilde{m}_n(x) = h^{-1} \iint y K\Big(\frac{F(x_0) - F(x)}{h}\Big) H(dx, dy).$$

Now we state our main results.

Theorem 1. *Assume that F is continuous, that the support of K is $(-1, 1) = S$ (say), that K is twice continuously differentiable within S, and that $\lim_{|t|\to 1} K(t) = 0$. If the following three conditions hold*

i) $h \to 0$, $nh^3/\log\log n \to \infty$,

ii) $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sup_{m\in\{m: |m-n|\le \varepsilon n\}} \left|\frac{h_m}{h_n} - 1\right| = 0$,

iii) $\sum_{n=3}^{\infty} (h/\log\log n) EY^2 I_{(|Y| \ge a_n)} < \infty$,

where $a_n = o((nh^{-1}\log\log n)^{1/2}/(\log n)^2)$, then we have for a. e. $x_0(F)$

$$\limsup_{n\to\infty} \pm (nh/2\log\log n)^{1/2} (m_n(x_0) - \tilde{m}_n(x_0)) = \sigma \quad \text{a. s.} \tag{4}$$

and, when $b = \int K(u)\,du \ne 0$,

$$\limsup_{n\to\infty} \pm (nh/2\log\log n)^{1/2} (\hat{m}_n(x_0) - \tilde{m}_n(x_0)/b) = \sigma/b \quad \text{a. s.} \tag{5}$$

where $\sigma = \Big(\mathrm{Var}(Y \mid X = x_0) \int K^2(u)\,du\Big)^{1/2}$.

Theorem 1 implies that sharp pointwise rates of strong consistency of m_n and $\hat{m}_n$ take the forms of LIL. To show this, one needs to prove

$$(nh/\log\log n)^{1/2} (\tilde{m}_n(x_0) - m(x_0)) \to \alpha \in R^1 \text{ as } n \to \infty. \tag{6}$$

While the convergences of the "stochastic components" $m_n - \tilde{m}_n$ and $\hat{m}_n - \tilde{m}_n$ could be proved under minimal assumptions on the underlying distribution, (6) typically needs some further conditions which guarantee $m_n(x_0) \to m(x_0)$ at a satisfactory rate. In our case, smoothness of the function

$$S(u) \triangleq E(Y \mid F(X) = u)$$

in a neighborhood of $F(x_0)$ suffices. Note that in the case of the kernel estimator, the smoothness of $m(x)$ and density of X are required.

To state our results, for integer $r \ge 2$, let $H_r(x_0)$ be the set of functions m and F in which S is r-times continuously differentiable in a neighborhood of $F(x_0)$ and

$(s_0)\quad S^{(2i)}(F(x_0))=0$ for $i=1, \cdots, \left[\frac{r-1}{2}\right]$.

Theorem 2. *Under the assumptions of Theorem 1, let K be such that*

$(s_1)\ \int K(u)du=1;$

$(s_2)\ \int u^{i-1}K(u)du=0$ *for* $i=1, \cdots, \left[\frac{r}{2}\right]$.

If m and F belong to $H_r(x_0)$ and

iv) $\lim_{n\to\infty} nh^{2r+1}/\log\log n=a<\infty,$

then we have

$$\limsup_{n\to\infty} \pm(nh/2\log\log n)^{1/2}(m_n(x_0)-m(x_0))=\sigma_\pm \quad \text{a.s.} \tag{7}$$

and

$$\limsup_{n\to\infty} \pm(nh/2\log\log n)^{1/2}(\hat{m}_n(x_0)-m(x_0))=\sigma_\pm \quad \text{a.s.} \tag{8}$$

whenever (4) *and* (5) *are satisfied respectively. Here*

$$\sigma_\pm=\left(\operatorname{Var}(Y\mid X=x_0)\int K^2(u)du\right)^{1/2}\pm\frac{a^{1/2}}{\sqrt{2}\,r!}S^{(r)}(F(x_0))\int u^rK(u)du.$$

Now we give the proof of Theorem 2, that of Theorem 1 is postponed to the next section.

Apparently, it suffices to prove (6) holds for

$$\alpha=\frac{a^{1/2}}{r!}S^{(r)}(F(x_0))\int u^rK(u)\,du. \tag{9}$$

By Taylor's lemma and the assumptions and noting that

$$h^{-1}\int K\left(\frac{F(x_0)-F(u)}{h}\right)dF(u)=\int_{(F(x_0)-1)/h}^{F(x_0)/h}K(u)du=1$$

for $0<F(x_0)<1$ and n sufficiently large, we have

$$\begin{aligned} m_n(x_0)-m(x_0)&=h^{-1}\int(m(u)-m(x_0))K\left(\frac{F(x_0)-F(u)}{h}\right)dF(u)\\ &=\int(S(F(x_0)-uh)-S(F(x_0))K(u)du\\ &=\frac{1}{r!}h^r\int u^rS^{(r)}(\Delta)K(u)du, \end{aligned} \tag{10}$$

where Δ is between $F(x_0)$ and $F(x_0)-uh$. Hence (9) follows via the dominent convergence theorem.

Remark. If one chooses a kernel K for which $\int u^iK(u)du=0$ for $i=1, 2, \cdots, r-1$, the assumption (s_0) may be deleted. For $r>2$ this may be only achieved if one admits negative values for K.

It follows from Theorem 2 that for appropriate m, F, K, and h satisfying the conditions i)—iv), the rate of pointwise convergence of m_n to m may at most reach $(n/\log\log n)^{-r/(2r+1)}$. Then what happens when

v) $\lim_{n\to\infty} nh^{2r+1}/\log\log n=\infty$?

By Theorem 1 we have in this case

$$\lim_{n\to\infty}(n/\log\log n)^{r/(2r+1)}(m_n(x_0)-\tilde{m}_n(x_0))=0 \quad \text{a.s.}$$

On the other hand if $\int u^r K(u)du\neq 0$, we have by (10)

$$\lim_{n\to\infty}(n/\log\log n)^{r/(2r+1)}(\tilde{m}_n(x_0)-m(x_0))=\infty \quad \text{a.s.}$$

Hence the rate of pointwise convergence of m_n to m can not reach $(n/\log\log n)^{-r/(2r+1)}$.

If, however, $\int u^r K(u)du=0$, (10) guarantees that for some constant $C>0$

$$|\tilde{m}_n(x_0)-m(x_0)|=\frac{1}{r!}h^r\left|\int u^r K(u)(S^{(r)}(\Delta)-S^{(r)}(F(x_0)))du\right|$$
$$\leqslant Ch^r\sup_{|t|<h}|S^{(r)}(F(x_0)+t)-S^{(r)}(F(x_0))|\triangleq Ch^r\delta_n.$$

Note that $\lim_{n\to\infty}\delta_n=0$ in view of the continuity of $S^{(r)}$ at $F(x_0)$, there must exist h satisfing i)—iii) and v) such that

$$\limsup_{n\to\infty}(nh/\log\log n)^{1/2}|\tilde{m}_n(x_0)-m(x_1)|<\infty \quad \text{a. s.}$$

and so that

$$\limsup_{n\to\infty}(nh/\log\log n)^{1/}\ |m_n(x_0)-m(x_0)|<\infty \quad \text{a. s.} \tag{11}$$

When, for example, $S^{(r)}$ satisfies δ-Lipshitz condition $(0<\delta\leqslant 1)$ at $F(x)$, we can take $h=(n/\log\log n)^{-1/(2(r+\delta)+1)}$ such that (11) holds.

From all above, we get

Theorem 3. *Under the conditions on m, F, and K, assume that h satisfies* i)—iii). *Then we have*

1) *if* iv) *is satisfied, the rate of pointwise convergence of m_n to m may at most reach* $(n/\log\log n)^{-r/(2r+1)}$;

2) *if* v) *is satisfied, then, when $\int u^r K(u)du\neq 0$, the rate of pointwise convergence of m_n to m can not reach* $(n/\log\log n)^{-r/(r+1)}$; *and when* $\int u^r K(u)\,du=0$, *the rate may exceed* $(n/\log\log n)^{-r/(2r+1)}$ (*for example, the rate may reach* $(n/\log\log n)^{-(r+\delta)/((r+\delta)+1)}$ *if $S^{(r)}$ satisfies δ-Lipshitz condition* $(0<\delta\leqslant 1)$).

Clearly, the result of Theorem 3 is also true for $\hat{m}_n$.

§ 3. Lemmas and Proof of Theorem 1

In this section the proof of Theorem 1 is decomposed into a series of lemmas. We begin with mentioning the two inequalities.

Bernstein's inequality. Let $Z_1,\cdots,Z_n$ be independent variables such that for some $C>0$ and all $1\leqslant i\leqslant n$, $P\{|Z_i|\leqslant C\}=1$. Then we have for every $t>0$

$$P\left\{\left|\sum_{i=1}^n(Z_i-EZ_i)\right|\geqslant t\right\}\leqslant 2\exp\left\{-t^2\Big/\left(2\sum_{i=1}^n \operatorname{Var}Z_i+\frac{2}{3}Ct\right)\right\}.$$

Chen-Zhao's inequality. Suppose that Z_1, Z_2, $\cdots$ is a sequence of iid. variables, and the common distribution H is continuous everywhere on R^1. Denote by H_n the e. d. f. of $Z_1, \cdots, Z_n$. Then there exist absolute constants $C_i>0$, $i=0, 1, \cdots, 4$, such that for any $\varepsilon>0$ we have

$$P\{\sup_{A\in\mathscr{A}} |H_n(A)-H(A)|>\varepsilon\}$$
$$\leqslant C_1(\sqrt{b}/\varepsilon\sqrt{n}+1/b)\exp(-C_2n\varepsilon^2/b)+C_3\exp(-C_4n\varepsilon),$$

where $\mathscr{A}$ is a set consisting of some intervals $A\in R^1$ with

$$\sup_{A\in\mathscr{A}} H(A)\triangleq b\leqslant 1$$

and $n/\log n$ is greater than C_0/ε.

For proofs, see [11] and [1] respectively.

Lemma 1. *Suppose that F is continuous, and $c_n\to 0$ satisfying $\liminf_{n\to\infty} nc_n(\log\log n)/(\log n)^2=0$. Then we have for some $M>0$*

$$\limsup_{n\to\infty}(n/c_n\log\log n)^{1/2}\sup_{|F(x)-F(x_0)|\leqslant C_n}|F_n(x)-F_n(x_0)-F(x)+F(x_0)|\leqslant M\quad\text{a. s.}\tag{12}$$

Proof Take $U_i=F(X_i)$, $i=1, 2, \cdots$. Then U_1, U_2, $\cdots$ are iid $U[0, 1]$ variables, and

$$\sup_{|F(x)-F(x_0)|\leqslant C_n}|F_n(x)-F_n(x_0)-F(x)+F(x_0)|$$
$$=\sup_{|t-t_0|\leqslant C_n, t, t_0\in[0,1]}|U_n(t)-U_n(t_0)-t+t_0|\triangleq I_n,$$

where $U_n(t)$ is e. d. f. of U_1, U_2, $\cdots U_n$. Therefore it suffices to show for $M>0$

$$\limsup_{n\to\infty}(n/c_n\log\log n)^{1/2}I_n\leqslant M\quad\text{a. s.}\tag{13}$$

For this, suppose w. l. o. g. that $0<t_0<1$. Let

$$\alpha_n=(\log n)/\log\log n,\quad t_j=t_0+j\alpha_n^{-1}c_n,$$

where j is an integer with $|j|\leqslant 1+\alpha_n$. Then for n large enough

$$I_n\leqslant\max_{|j|\leqslant 1+\alpha_n}\sup_{t_j\leqslant t\leqslant t_{j+1}}|U_n(t)-U_n(t_0)-t+t_0|$$
$$\leqslant\max_{|j|\leqslant 1+\alpha_n}|U_n(t_j)-U_n(t_0)-t_j+t_0|+\sup_{A\in\mathscr{F}}|U_n(A)-|A||$$
$$\triangleq I_{1n}+I_{2n},\tag{14}$$

where $\mathscr{F}$ denotes the set of intervals in $[x_0-c_n, x_0+c_n]$ whose lengths are shorter than $\alpha_n^{-1}c_n$, and $|A|$ denotes the length of A.

It is apparent that for n sufficiently large

$$\sup_{A\in\mathscr{F}}|A|\leqslant\alpha_n^{-1}c_n\triangleq b_n\leqslant 1$$

and $n/\log n\geqslant\varepsilon_n^{-1}$ for all large M_1, where $\varepsilon_n=M_1(n^{-1}c_n\log\log n)^{1/2}$. So we get by Chen-Zhao's inequality

$$P\{I_{2n}\geqslant\varepsilon_n\}\leqslant C_1(\sqrt{b_n}/\varepsilon_n\sqrt{n}+1/b_n)\exp\{-C_2n\varepsilon_n^2/b_n\}+c_3\exp\{-C_4n\varepsilon_n\}$$
$$\leqslant 2C_1\alpha_nc_n^{-1}\exp\{-C_2M_1^2\log n\}+C_3\exp\{-C_4M_1(nc_n\log\log n)^{1/2}\}.$$

Noting that $\liminf_{n\to\infty} nc_n(\log\log n)/(\log n)^2>0$, we see that the right hand side of above inequality is bounded by $O(n^{-2})$ for M_1 sufficiently large. It then follows from Borel-Cantelli's lemma that

$$\limsup_{n\to\infty} (n/c_n \log\log n)^{1/2} I_{2n} \leqslant M_1 \quad \text{a. s.} \tag{15}$$

For $a,\ b\in[0,\ 1]$, $i\geqslant 1$, define

$$B_i(a,\ b)=I_{(\min(a,b)<U_i<\max(a,b))}-EI_{(\min(a,b)<U_i<\max(a,b))},$$

where I_A denotes the indicator function of set A. Then

$$I_{1n}=n^{-1}\max_{|j|<1+\alpha_n}\left|\sum_{i=1}^{n} B_i(t_0,\ t_j)\right|.$$

Suppose w. l. o. g. that $c_n\downarrow$, $nc_n\uparrow$. Let $n_m=[e^{\sqrt{m}}]$, $d_m=m^{-1}c_{n_m}$ for $m\geqslant 1$, and $\varepsilon'_n=M_1(nc_n\log\log n)^{1/2}$. Denote for $n_m<n\leqslant n_{m+1}$

$$A_m=\left\{\max_{|j|\leqslant 2m}\left|\sum_{i=1}^{n_m} B_i(t_0,\ t_0+jd_m)\right|\geqslant\varepsilon'_{n_m}\right\},$$

$$B_n=\left\{\max_{|j|\leqslant 2m}\left|\sum_{i=n_m+1}^{n} B_i(t_0,\ t_0+jd_m)\right|\geqslant\varepsilon'_{n_m}\right\},$$

$$D_n=\left\{\max_{|j|\leqslant 1+\alpha_n}\left|\sum_{i=1}^{i} B_i(t_0+[j\alpha_n^{-1}c_nd_m^{-1}]d_m,\ \ t_0+j\alpha_n^{-1}c_n)\right|\geqslant\varepsilon'_n\right\}.$$

Since for any $0\leqslant a<b<c\leqslant 1$

$$B_i(a,\ b)+B_i(b,\ c)=B_i(a,\ c)$$

and $[j\alpha_n^{-1}c_nd_m^{-1}]\leqslant 2m$ for $|j|\leqslant 1+\alpha_n$, we have for $n_m<n\leqslant n_{m+1}$

$$\{I_{1n}\geqslant 3M_1(n^{-1}c_n\log\log n)^{1/2}\}\subset A_m\cup B_n\cup D_n.$$

Therefore, if we can prove

$$\sum_{m=1}^{\infty} P\left\{A_m\cup\bigcup_{n=n_m+1}^{n_{m+1}}(B_n\cup D_n)\right\}<\infty, \tag{16}$$

then

$$\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty}(I_{1n}\geqslant 3M_1(n^{-1}c_n\log\log n)^{1/2})\right\}=0,$$

that is

$$\limsup_{n\to\infty}(n/c_n\log\log n)^{1/2} I_{1n}\leqslant 3M_1 \quad \text{a. s.}$$

which, together with (14) and (15), is tantamount to (13). Thus, to prove (13), it suffices to verify (16). Now the verification of (16) is the same as that of (3.8) of [8].

Lemma 2. *If h satisfies the condition* i), *then there exist a sequence of sets A_n satisfying $\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty} A_n\right\}=0$ such that for $\omega\overline{\in} A_n$, $|F_n(x)-F_n(x_0)|<hy$ ields $|F(x)-F(x_0)|<(1+c_1\theta_n)h$, and $|F_n(x)-F(x_0)|\geqslant h$ and $|F(x)-F(x_0)|<h$ yield $|F(x)-F(x_0)|\geqslant(1-c_2\theta_n)h$, where $\theta_n=((nh)^{-1}\log\log n)^{1/2}$, $c_1,\ c_2>0$ are constants.*

Proof By the well-known Smirnov-Chung's theorem we have

$$\limsup_{n\to\infty}(n/\log\log n)^{1/2}\sup_x|F_n(x)-F(x)|=c_0 \quad \text{a. s.}$$

for some $c_0>0$. It follows that $\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty} A_{1n}\right\}=0$, and that for $\omega\overline{\in} A_{1n}$, $|F_n(x)-F_n(x_0)|<h$ yields $|F(x)-F(x_0)|<2h$ for large n, where

$$A_{1n}=\{\omega;\ (n/\log\log n)^{1/2}\sup_x |F_n(x)-F(x)|\geqslant 2c_0\}.$$

Hence, note that $nh(\log\log n)/(\log n)^2\to\infty$ via the condition i), Lemma 1 guarantees that for some $M_2>0$

$$\limsup_{n\to\infty} (n/h\log\log n)^{1/2} \sup_{|F_n(x)-F_n(x_0)|<h} |F_n(x)-F_n(x_0)-F(x)+F(x_0)|<M_2 \quad \text{a. s.}$$

which, if taking

$$A_{2n}=\{\omega;\ (n/h\log\log n)^{1/2} \sup_{|F_n(x)-F_n(x_0)|<h} |F_n(x)-F_n(x_0)-F(x)+F(x_0)|\geqslant 2M_2,$$

ensures that $\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty} A_{2n}\right\}=0$, and that for $\omega\,\overline{\in}\,A_{2n}$, $|F_n(x)-F_n(x_0)|<h$ yields $|F(x)-F(x_0)|<(1+4M_2\theta_n)h$.

On the other hand, if we write

$$A_{3n}=\{\omega;\ (n/h\log\log n)^{1/2} \sup_{|F(x)-F(x_0)|<h} |F_n(x)-F_n(x_0)-F(x)+F(x_0)|\geqslant 2M\}$$

we have by Lemma 1 that $\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty} A_{3n}\right\}=0$, and that for $\omega\,\overline{\in}\,A_{3n}$, $|F_n(x)-F_n(x_0)|\geqslant h$ and $|F(x)-F(x_0)|<h$ yield that

$$\begin{aligned}|F(x)-F(x_0)|&\geqslant |F_n(x)-F_n(x_0)|-2\sup_{|F(x)-F(x_0)|<h}|F_n(x)-F_n(x_0)-F(x)+F(x_0)|\\ &\geqslant(1-4M\theta_n)h.\end{aligned}$$

Now taking $A_n=A_{2n}\cup A_{3n}$, we complete the proof of Lemma 2.

Since A_n satisfy $\lim_{k\to\infty} P\left\{\bigcup_{n=k}^{\infty} A_n\right\}=0$, which have no influence on the following discussion, we can and will assume that all A_n are empty sets.

Now define

$$B_1=\{x;\ |F_n(x)-F_n(x_0)|<h,\ |F(x)-F(x_0)|<h\},$$
$$B_2=\{x;\ |F_n(x)-F_n(x_0)|<h,\ h\leqslant|F(x)-F(x_0)|<(1+c_1\theta_n)h\}.$$

Since the support of K is $(-1, 1)$ on which K is twice continuously differentiable, we have by Taylor expansion and Lemma 2 (guaranteeing that $|F(x)-F(x_0)|<(1+c_1\theta_n)h$ if $|F_n(x)-F_n(x_0)|<h$)

$$\begin{aligned}m_n(x_0)&=h^{-1}\iint yK\left(\frac{F_n(x_0)-F_n(x)}{h}\right)H_n(dx,\ dy)\\
&=h^{-1}\iint_{B_1} yK\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx,\ dy)\\
&\quad+h^{-2}\iint_{B_1} y(F_n(x_0)-F_n(x)-F(x_0)+F(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx,\ dy)\\
&\quad+h^{-3}\iint_{B_1}\frac{1}{2}y(F_n(x_0)-F_n(x)-F(x_0)+F(x))^2K''(\Delta)H_n(dx,\ dy)\\
&\quad+h^{-1}\iint_{B_2} yK\left(\frac{F_n(x_0)-F_n(x)}{(1+c_1\theta_n)h}\right)H_n(dx,\ dy)\\
&\quad+h^{-2}\iint_{B_2} y\left(F_n(x_0)-F_n(x)-\frac{F(x_0)-F(x)}{(1+c_1\theta_n)h}\right)K'(\Delta_1)H_n(dx,\ dy)\end{aligned}$$

$$\triangleq \sum_{j=1}^{5} T_{jn}, \tag{17}$$

where Δ is between $(F_n(x_0)-F(x))/h$ and $(F(x_0)-F(x))/h$, and Δ_1 is between $(F_n(x_0)-F_n(x))/h$ and $(F(x_0)-F(x))/(1+c_1\theta_n)h$.

In the sequel denote by C an absolute constant whose value may be different from appearance to appearance.

Lemma 3. *We have*

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T_{in}=0 \quad \text{a. s.} \quad i=3,\ 5.$$

Proof Consider T_{3n}. Clearly

$$|T_{3n}|\leqslant Ch^{-3}\sup_{|F(x)-F(x_0)|<h}(F_n(x)-F_n(x_0)-F(x)+F(x_0))^2\iint|y|H_n(dx,dy)$$

and

$$\iint|y|H_n(dx,dy)=\frac{1}{n}\sum_{i=1}^{n}|Y_i|\to E|Y|<\infty \quad \text{a. s.} \quad \text{as } n\to\infty.$$

Thus, by Lemma 1 and the condition i)

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T_{3n}=0 \quad \text{a. s.}$$

Apropos of T_{5n}, since K' is bounded, we have

$$|T_{5n}|\leqslant Ch^{-2}\sup_{|F(x)-F(x)|<(1+c_1\theta_n)h}\left|F_n(x_0)-F_n(x)-\frac{F(x_0)-F(x)}{(1+c_1\theta_n)h}\right|\iint_{B_2}|y|H_n(dx,\ dy)$$

$$\leqslant Ch^{-2}\sup_{|F(x)-F(x_0)|<2h}|F_n(x_0)-F_n(x)-F(x_0)+F(x)|\iint_{B_2}|y|H_n(dx,\ dy)$$

$$+Ch^{-1}\theta_n\iint_{B_2}|y|H_n(dx,dy),$$

which, together with Lemma 1, implies that it suffices to show

$$\lim_{n\to 8}h^{-1}\iint_{B_2}|y|H_n(dx,\ dy)=0 \quad \text{a. s.} \tag{18}$$

For this, taking $X_{ni}=I_{(h\leqslant|F(x_i)-F(X_0)|<(1+c_1\theta_n)h)}$, we have

$$\iint_{B_2}|y|H_n(dx,dy)\leqslant\frac{1}{n}\sum_{i=1}^{n}|Y_i|I_{(h\leqslant|F(X_i)-F(x_0)|<(1+c_1\theta_n)h)}$$

$$\leqslant\Big(\frac{1}{n}\sum_{j=1}^{n}Y_j^2\Big)^{1/2}\Big(\frac{1}{n}\sum_{i=1}^{n}X_{ni}\Big)^{1/2}. \tag{19}$$

Condition iii) implies that $EY^2<\infty$, and so that

$$\frac{1}{n}\sum_{i=1}^{n}Y_i^2\to EY^2 \quad \text{a. s.} \tag{20}$$

Also, it is easily seen that

$$\text{Var } X_{ni}\leqslant EX_{ni}=P\{h\leqslant|F(X)-F(x_0)|<(1+c_1\theta_n)h\}=2c_1\theta_n h.$$

Hence $EX_{n1}=o(h^2)$ via the condition i), and then for any $\varepsilon>0$ and n large enough Bernstein's inequality shows

$$P\Big\{(nh^2)^{-1}\sum_{i=1}^{n}X_{ni}>2\varepsilon\Big\}\leqslant P\left|\sum_{i=1}^{n}(X_{ni}-EX_{ni})\right|>nh^2\varepsilon\Big\}$$

$$\leqslant 2\exp\{-n^2h^4\varepsilon^2/2(2c_1n\theta_n h+nh^2\varepsilon)\}$$

$$\leq 2\exp\left\{-(nh\varepsilon)^2/2\left(2c_1\left(\frac{\log\log n}{nh^3}\right)^{1/2}+\varepsilon\right)\right\}=O(n^{-2}).$$

Therefore, we obtain by Borel-Cantelli's lemma and letting $\varepsilon\to 0$

$$\lim_{n\to\infty}\frac{1}{nh^2}\sum_{i=1}^{n}X_{ni}=0\quad \text{a.s.},$$

which, combined with (19) and (20), yields (18).

Denote

$$B_3=\{x: |F_n(x)-F_n(x_0)|\geq h,\ |F(x)-F(x_0)|<h\}.$$

Noting that K and K' vanish outside $(-1, 1)$, we have

$$T_{1n}=h^{-1}\iint yK\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx, dy)-h^{-1}\iint_{B_3} yK\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx, dy)$$
$$\triangleq T'_{1n}-T''_{1n},$$

$$T_{2n}=h^{-2}\iint y(F_n(x_0)-F_n(x)-F(x_0)-F(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx, dy)$$
$$-h^{-2}\iint_{B_3} y(F_n(x_0)-F_n(x)-F(x_0)+F(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx, dy)$$
$$\triangleq T'_{2n}-T''_{2n}.$$

Lemma 4. *We have for a. e. $x_0(F)$*

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T''_{in}=0\quad \text{a. s. } i=1, 2, \tag{21}$$

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T_{4n}=0\quad \text{a. s.} \tag{22}$$

Proof First, by Lemma 2 we have

$$B_3\subset\{x: (1-c_2\theta_n)h\leq|F(x)-F(x_0)|<h\}\triangleq B_4.$$

Then

$$|T''_{2n}|\leq Ch^{-2}\sup_{|F(x)-F(x_0)|<h}|F((x_0)-F_n(x)-F(x_0)+F(x)|\iint_{B_4}|x|H_n(dx, dy),$$

and hence similarly to T_{5n}

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T''_{2n}=0\quad \text{a. s.}$$

Next, take

$$B_5=\{x: h\leq|F(x)-F(x_0)|<(1+c_1\theta_n)h\},$$

$$K_n(x)=\left|K\left(\frac{F(x_0)-F(x)}{(1+c_1\theta_n)h}\right)\right|I_{(h\leq|F(x_0)-F(x)|<(1+c_1\theta_n)h)}.$$

$$T_{4n}\leq h^{-1}\iint_{B_5}|y|\left|K\left(\frac{F(x_0)-F(x)}{(1+c_1\theta_n)h}\right)\right|H_n(dx, dy)$$
$$=\frac{1}{nh}\sum_{i=1}^{n}(K_nX_i)|Y_i|-EK_n(X_i)|Y_i|)+h^{-1}EK_n(X)|Y|$$
$$\triangleq T'_{4n}+T''_{4n}$$

$$T''_{4n}=h^{-1}\iint |y|K_n(x)H(dx,\ dy)$$
$$=h^{-1}\int \widetilde{m}(x)K_n(xF(dx)$$
$$=\int_{1\le t\le 1+c_1\theta_n}\widetilde{m}(F^{-1}(F(x_0)-ht))\left|K\left(\frac{t}{1+c_1\theta_n}\right)\right|dt$$
$$\le c_1\theta_n\sup_{1\le t\le 1+c_1\theta_n}\widetilde{m}(F^{-1}(F(x_0)-ht))\sup_{1\le t\le 1+c_1\theta_n}\left|K\left(\frac{t}{1+c_1\theta_n}\right)\right|,$$

where $\widetilde{m}(x)=E(|Y|\,|X=x)$. Since $h\to 0$ and

$$F(\{x_0;\ 0<F(x_0)<1,\ F^{-1}(F(x_0))=x_0\})=1,$$

we have for a. e. $x_0(F)$ and n large enough

$$\sup_{1\le t\le 1+c_1\theta_n}\widetilde{m}(F^{-1}(F(x_0)-ht))\le C<\infty.$$

Then by the condition $\lim_{|t|\to 1}K(t)=0$

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T''_{4n}=0 \quad \text{a. s. for a. e. } x_0(F).$$

As for T'_{4n}, Theorem 2 of [3] guarantees

$$\limsup_{n\to\infty}(2n\sigma_n^2\log\log n)^{-1/2}T'_{4n}=1 \quad \text{a. s.} \tag{23}$$

where, taking $m^*(x)=E(Y^2|X=x)$,

$$\sigma_n^2=(nh)^{-2}\operatorname{Var}(K_n(X)|Y|)$$
$$\le\frac{1}{n^2h}\int_{1\le q\le 1+c_1\theta_n}m^*(F^{-1}(F(x_0)-ht)K^2\left(\frac{t}{1+c_1\theta_n}\right)dt$$
$$\le\frac{c_1\theta_n}{n^2h}\sup_{1\le t\le 1+c_1\theta_n}m^*(F^{-1}(F(x_0)-ht)K^2\left(\frac{t}{1+c_1\theta_n}\right)=o(1)/n^2h$$

for a. e. $x_0(F)$ and n large enough. Here $o(1)\to 0$ as $n\to\infty$. Hence (23) implies for a. e. $x_0(F)$

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T'_{4n}=0 \quad \text{a. s.}$$

Finally, observing that

$$|T''_{1n}|\le h^{-1}\iint_{B_3}|y|\left|K\left(\frac{F(x_0)-F(x)}{h}\right)\right|H_n(dx,\ dy)$$

we get similarly to T_{4n}

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}T_n=0 \quad \text{a. s.} \quad \text{for a. e. } x_0(F).$$

Thus (21) and (22) hold.

Now write $\alpha_n(x)=F_n(x)-F(x)$. Then

$$T_{2n}=h^{-2}\iint (y-m(x))(\alpha_n(x_0)-\alpha_n(x)K'\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx,\ dy)$$
$$+h^{-2}\int (m(x)-m(x_0))(\alpha_n(x_0)-\alpha_n(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)F_n(dx)$$
$$+h^{-2}m(x_0)\int (\alpha_n(x_0)-\alpha_n(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)\alpha_n(dx)$$
$$+h^{-2}m(x_0)\int (\alpha_n(x_0)-\alpha_n(x))K'\left(\frac{F(x_0)-F(x)}{h}\right)F(dx)$$

$$\triangleq \sum_{i=1}^{4} J_{in}.$$

Lemma 5. *We have for a. e.* $x_0(F)$

$$\lim_{n\to\infty} (nh/\log\log n)^{1/2} J_{in} = 0 \quad \text{a. s}_1,\ i=2,\ 3. \tag{24}$$

Proof Let $\tilde{\alpha}_n(t) = U_n(t) - t$, $0 \leqslant t \leqslant 1$ ($U_n(t)$ is as in Lemma 1), then $\alpha_n(X) = \tilde{\alpha}_n(F(x))$, $x \in R^1$.

By the continuity of F and the boundedness of K and K', an integration by parts shows

$$\begin{aligned} |J_{3n}| &= \left| m(x_0) h^{-2} \int (\tilde{\alpha}_n(F(x_0)) - \tilde{\alpha}_n(u)) K'\left(\frac{F(x_0)-u}{h}\right) d\tilde{\alpha}_n(u) \right. \\ &= \left| \frac{m(x_0)}{2} h^{-2} \int K'\left(\frac{F(x_0)-u}{h}\right) d(\tilde{\alpha}_n(F(x_0)) - \tilde{\alpha}_n(u))^2 \right| \\ &\leqslant C h^{-3} \sup_{|u-F(x_0)|<h} (\tilde{\alpha}_n(F(x_0)) - \tilde{\alpha}_n(u))^2. \end{aligned}$$

And so from Lemma 1 and the condition i) we deduce that

$$\lim_{n\to\infty} (nh/\log\log n)^{1/2} J_{3n} = 0 \quad \text{a. s.}$$

Furthermore

$$|J_{2n}| \leqslant h^{-2} \sup_{|F(x)-F(x_0)|<h} |\alpha_n(x_0) - \alpha_n(x)| \int |m(x) - m(x_0)| \left| K'\left(\frac{F(x_0)-F(x)}{h}\right) dF_n(x).\right.$$

From this and Lemma 1 we see that to complete the proof of (24), it suffices to prove that for a. e. $x_0(F)$

$$\lim_{n\to\infty} h^{-1} \int |m(x) - m(x_0)| \left| K'\left(\frac{F(x_0)-F(x)}{h}\right) \right| dF_n(x) = 0 \quad \text{a. s.} \tag{25}$$

For this, note that

$$\begin{aligned} &h^{-1} \int |m(x) - m(x_0)| \left| K'\left(\frac{F(x_0)-F(x)}{h}\right) \right| dF(x) \\ &= \int |m(F^{-1}(F(x_0) - uh) - m(x_0)| \, |K'(u)| du. \end{aligned}$$

Since $F^{-1}(F(x_0)) = x_0$ for a. e. $x_0(F)$, Theorem 2 on pages 62–63 in [9] asserts that

$$\lim_{n\to\infty} \int |m(F^{-1}(s-uh)) - m(F^{-1}(s))| \, |K'(u)| du = 0$$

for Lebesgue almost all s, say for all $s \in A$. By the continuity of F, we thus have

$$F(\{x_0: F(x_0) \in A,\ F^{-1}(F(x_0)) = x_0)) = x_0\}) = 1$$

and so for a. e. $x_0(F)$

$$\lim_{n\to\infty} h^{-1} \int |m(x) - m(x_0)| \left| K'\left(\frac{F(x_0)-F(x)}{h}\right) \right| dF(x) = 0 \quad \text{a. s.} \tag{26}$$

Now define

$$m_n^{(1)}(x) = E(Y I_{(|Y|<a_n)} \mid X = x), \quad m_n^{(2)}(x) = m(x) - m_n^{(1)}(x),$$

$$X_{in}^{(j)} = |m_n^{(j)}(X_i) - m_n^{(j)}(x_0)| \, |K'\left(\frac{F(x_0)-F(X_i)}{h}\right),\ j=1,\ 2.$$

It is easy to see that $EX_{n1}^{(2)} = o(h)$, and, from this and (26) that $EX_{n1}^{(1)} = o(h)$.

Furthermore, for every $\varepsilon>0$

$$P\left\{\frac{1}{nh}\left|\sum_{i=1}^{n}(X_{ni}^{(2)}-EX_{ni}^{(2)})\right|>\varepsilon\right\}\leqslant E(X_{n1}^{(2)})^2/nh^2\varepsilon^2$$
$$\leqslant EY^2I_{(|Y|\geqslant a_n)}/nh^2\varepsilon^2$$

and the conditions i) and iii) imply that $\sum_{n=3}^{\infty}EY^2I_{(|Y|\geqslant a_n)}/nh^2<\infty$. Therefore

$$\lim_{n\to\infty}\frac{1}{nh}\sum_{i=1}^{n}X_{ni}^{(2)}=0 \quad \text{a. s.} \tag{27}$$

On the other hand, since $X_{ni}^{(1)}\leqslant Ca_n$, $\operatorname{Var}X_{ni}^{(1)}\leqslant C$ for $1\leqslant i\leqslant n$, with the condition i) and the assumption on a_n we get by Bernstein's inequality

$$P\left\{\frac{1}{nh}\left|\sum_{i=1}^{n}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right|>\varepsilon\right\}$$
$$\leqslant 2\exp\{-(nh\varepsilon)^2/2(n\operatorname{Var}X_{n1}^{(1)}+Ca_nnh\varepsilon)\}$$
$$\leqslant 2\exp\{-Cnh^2/(1+a_nh)\}=O(n^{-2}),$$

yielding for a. e. $x_0(F)$

$$\lim_{n\to\infty}\frac{1}{nh}\sum_{i=1}^{n}X_{ni}^{(1)}=0 \quad \text{a. s.} \tag{28}$$

Now (25) follows from (27) and (28).

Lemma 6. *We have*

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}J_{1n}=0 \quad \text{a. s.} \tag{29}$$

Proof Let M be the constant in the right hand side of (12), and put

$$\beta_n(x)=(\alpha_n(x_0)-\alpha_n(x))I_{\sup_{|F(x)-F(x_0)|<h}|\alpha_n(x_0)-\alpha_n(x)|\leqslant 2M(n^{-1}h\log\log n)^{1/2}}.$$

Then (29) follows from Lemma 1 if

$$\lim_{n\to\infty}(nh/\log\log n)^{1/2}J_{1n}=0 \quad \text{a. s.} \tag{30}$$

with

$$J_{1n}=h^{-2}\iint(y-m(x))\beta_n(x)K'\left(\frac{F(x_0)-F(x)}{h}\right)H_n(dx,dy)$$
$$=(nh^2)^{-1}\sum_{i=1}^{n}(Y_i-m(X_i))\beta_n(X_i)K'\left(\frac{F(x_0)-F(X_i)}{h}\right)$$

is achieved. To demonstrate this, put

$$Y_{ni}^{(1)}=Y_iI_{(|Y_i|<a_n)},\ Y_{ni}^{(2)}=Y_i-Y_{ni}^{(1)},\ J_{1n}^{(1)}=J_{1n}-J_{1n}^{(2)} \text{ with}$$
$$J_{1n}^{(2)}=(nh^2)^{-1}\sum_{i=1}^{n}(Y_{ni}^{(2)}-m_n^{(2)}(X_i))\beta_n(X_i)K'\left(\frac{F(x_0)-F(X_i)}{h}\right).$$

Observe that conditionally on $\Delta_n=\sigma(X_1,\cdots,X_n)$, $Y_1,\cdots,Y_n$ are indegendent, and

$$E(J_{1n}^{(1)}|\Delta_n)=E(J_{1n}^{(2)}|\Delta_n)=0,$$
$$E(J_{1n}^{(2)2}|\Delta_n)=(nh^2)^{-1}\sum_{i=1}^{n}E((Y_{ni}^{(2)}-m_n^{(2)}(X_i))^2|X_i)\left(\beta_n(X_i)K'\left(\frac{F(x_0)-F(x_i)}{h}\right)\right)^2$$

while, since $K'\left(\frac{F(x_0)-F(x)}{h}\right)=0$ whenever $|F(x_0)-F(x)|\geqslant h$,

$$\left|\beta_n(X_i)K'\left(\frac{F(x_0)-F(X_i)}{h}\right)\right|\leqslant C(n^{-1}h\log\log n)^{1/2}$$

we have for every $\varepsilon>0$

$$\sum_{n=3}^{\infty} P\{|(nh/\log\log n)^{\frac{1}{2}} J_{1n}^{(2)}|>\varepsilon\}$$

$$\leqslant \varepsilon^{-2} \sum_{n=3}^{\infty} (nh/\log\log n) E(E(J_{1n}^{(2)2} | \Delta_n))$$

$$\leqslant C \sum_{n=3}^{\infty} (nh/\log\log n)(n^{-1}h\log\log n) EY^2 I_{(|Y|\geqslant a_n)}/nh^4$$

$$\leqslant C \sum_{n=3}^{\infty} EY^2 I_{(|Y|\geqslant a_n)}/nh^2<\infty$$

so that

$$\lim_{n\to\infty} (nh/\log\log n)^{1/2} J_{1n}^{(2)}=0 \quad \text{a. s.}$$

To deal with $J_{1n}^{(1)}$, since

$$|Z_{ni}| \triangleq \left| (Y_{ni}^{(1)}-m_n^{(1)}(X_i))\beta_n(X_i)\, K\left(\frac{F(x_0)-F(X_i)}{h}\right)\right| \leqslant Ca_n(n^{-1}h\log\log n)^{1/2} \triangleq b_n.$$

We get for every $t>0$

$$P\{(nh/\log\log n)^{1/2} J_{1n}^{(1)}>\varepsilon \mid \Delta_n\}$$

$$\leqslant \exp\{-t\varepsilon(ah^3\log\log n)^{1/2}\} \prod_{i=1}^{n} E(\exp(tZ_{ni}) \mid \Delta_n)$$

$$\leqslant \exp\{-t\varepsilon(nh^3\log\log n)^{1/2}\} \prod_{i=1}^{n}\left(1+\frac{1}{2}Ct^2E[(Y_{ni}^{(1)}-m_n^{(1)}(X_i))^2 \mid X_i] e^{tb_n n^{-1}} h\log\log n\right)$$

an then

$$P\{(nh/\log\log n)^{1/2} J_{1n}^{(1)}>\varepsilon\}$$

$$\leqslant \exp\{-t\varepsilon(nh^3\log\log n)^{1/2}\}\left[1+\frac{1}{2}Ct^2(n^{-1}h\log\log n)\right.$$

$$\left.\times e^{tb_n} E(Y_{n1}^{(1)}-m_n^{(1)}(X_1))^2\right]^n$$

$$\leqslant \exp\left\{-t\varepsilon(nh^3\log\log n)^{1/2}+\frac{1}{2}Ct^2(h\log\log n)e^{tb_n}\right\} \triangleq P_n.$$

Noting that there exists bounded sequence t_n satisyng

$$t_n/a_n h\to 0, \quad nha_n^{-1}t_n/\log n\to\infty,$$

We thus have with $t=b_u^{-1}t_n$

$$P_n \leqslant \exp\left\{-\frac{\varepsilon}{2} b_n^{-1}t_n(nh^3\log\log n)^{1/2}\right\}=\exp\{-Cnha_n^{-1}t_n\}=O(n^{-2})$$

and so by Borel-Cantelli's lemma

$$\limsup_{n\to\infty} (nh/\log\log n)^{1/2} J_{1n}^{(1)} \leqslant \varepsilon \quad \text{a. s.} \tag{31}$$

Similarly

$$\liminf_{n\to\infty} (nh/\log\log n)^{1/2} J_{1n}^{(1)} \geqslant -\varepsilon \quad \text{a. s.} \tag{32}$$

Therefore (31) and (32) and letting $\varepsilon\to 0$ yield

$$\lim_{n\to\infty} (nh/\log\log n)^{1/2} J_{1n}^{(1)}=0 \quad \text{a. s.}$$

and so (30) holds.

Proof of Theorem 1 From all above we conclude that (4) follows from the following

$$\limsup_{n\to\infty} \pm (nh/\log\log n)^{1/2}(T'_{1n}+J_{4n}-\tilde{m}_n(x_0))$$
$$=(\mathrm{Var}(Y|X=x_0)\int K^2(u)du)^{1/2} \quad \text{a. s.} \tag{33}$$

To establish (33), we have by integration by parts

$$J_{4n}=-m(x_0)h^{-1}\int(\alpha_n(x_0)-\alpha_n(x))dK\left(\frac{F(x_0)-F(x)}{h}\right)$$
$$=-m(x_0)h^{-1}\int K\left(\frac{F(x_0)-F(x)}{h}\right)d\alpha_n(x).$$

Then

$$T'_{1n}+J_{4n}-\tilde{m}_n(x_0)=h^{-1}\iint(y-m(x_0))K\left(\frac{F(x_0)-F(x)}{h}\right)(H_n(dx,dy)-H(dx,dy))$$

and by Theorem 2 of [3]

$$\limsup_{n\to\infty} \pm(2n\sigma_n^2(n)\log\log n)^{-1/2}(T'_{1n}+J_{4n}-\tilde{m}_n(x_0))=1 \quad \text{a. s.} \tag{34}$$

where, with $m_0(x)=E((Y-m(x_0))^2|X=x)$,

$$\sigma_n^2(n)=(nh)^{-2}\,\mathrm{Var}\left((Y-m(x_0))K\left(\frac{F(x_0)-F(x)}{h}\right)\right)$$
$$\sim\frac{1}{n^2h}\int m_0(F^{-1}(F(x_0)-ht))K^2K^2(t)dt$$
$$=\frac{1}{n^2h}\int m_0(F^{-1}(F(x_0)-ht))-m_0(x_0))K^2(t)dt$$
$$+\frac{1}{n\,h}m_0(x_0)\int K^2(u)du.$$

Note that $m_0(x_0)=\mathrm{Var}(Y|X=x_0)$, and similarly to (26)

$$\int(m_0(F^{-1}(F(x_0)-ht))-m_0(x_0))K^{\cdot}(t)dt\to 0 \quad \text{for a. e. } x_0(F),$$

(33) follows from (34).

To prove (5), set

$$f_n(x)=\frac{1}{nh}\sum_{i=1}^{n}K\left(\frac{F_n(x)-F_n(X_i)}{h}\right).$$

Then by taking $Y_i=1$ in (4), (4) ensures that for a. e. $x_0(F)$

$$\limsup_{n\to\infty} \pm(nh/\log\log n)^{1/}\left(f_n(x_0)-\int_{-1}^{1}K(u)du\right)=0 \quad \text{a. s.} \tag{35}$$

(since in this case, for $0<F(x_0)<1$ and n large enough

$$\tilde{m}_n(x_0)=h^{-1}\int K\left(\frac{F(x_0)-F(x)}{h}\right)dF(x)=\int_{(F(x_0)-1)/h}^{F(x_0)/h}K(u)du=\int_{-1}^{1}K(u)du$$

while $F(\{x_0: 0<F(x_0)<1\})=1$). Now

$$\hat{m}_n(x_0)-\tilde{m}_n(x_0)b^{-1}$$
$$=(m_n(x_0)-\tilde{m}_n(x_0))/f_n(x_0)+\tilde{m}_n(x_0)(b-f_n(x_0))/f_n(x_0)b.$$

(5) thus follows from (4) and (35).

Remark. From above proofs we can see that the comparatively strict condition

$\lim_{|t|\to 1} K(t)=0$ can be cancelled at the expense of losing the exact value of LIL. For example we have

Corollary of Theorem 1. *Under all conditions of Theorem 1 except for* $\lim_{|t|\to 1} K(t)=0$, *we have*

$$\limsup_{n\to\infty} (nh/\log\log n)^{1/2} |m_n(x_0)-\tilde{m}_n(x_0)| \leq C \quad \text{a. s.}$$

and

$$\limsup_{n\to\infty} (nh/\log\log n)^{2/2} |\hat{m}_n(x_0)-\tilde{m}_n(x_0)/b| \leq C \quad \text{a. s.}$$

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