Chin. Ann. of Math.
13B: 2(1992), 196—204.

DERIVATIONS OF THE GRADED LIE ALGEBRAS OF CARTAN TYPE**

CHIU SEN (邱 森)*

Abstract

Let F be an algebraically closed field of characteristic $p>0$, 2, $L=\bigoplus_{i=-r}^{s} L_{[i]}$ a graded Lie algebra of Cartan type over F. This paper gives a description of the adjoint module by means of the mixed product and the induced module so that the computation of $H^i(L, L)$ can be reduced to that of the cohomology of $L_{[0]}$. Then the structures of the derivation algebras of L is determined.

§ 0. Introduction

The Cartan's theorem shows that nonmodular semisimple Lie algebras do not have outer derivations. Let $L=\bigoplus_{i=-r}^{s} L_{[i]}$ be a graded Lie algebra of Cartan type over F, where F is an algebraically closed field of characteristic $p>0$. In [16, Chap. 4, § 6], the derivation algebras of the restricted Cartan type Lie algebra $W(n, \mathbf{1})$, $S(n, \mathbf{1})$, $H(n, \mathbf{1})$ and $K(n, \mathbf{1})$ ($p>2$) are investigated and the results illustrate that simple modular Lie algebras may possess outer derivations and every derivation of $W(n, \mathbf{1})$ ($p>2$) is inner. In this paper, we propose a new unifying approach to investigate the derivation algebras of Cartan type Lie algebras of characteristic p and determine the structures of $H^1(L, L)$ and the derivation algebras of L, where $L=W(n, \boldsymbol{m})$ for $p>0$, $S(n, \boldsymbol{m})$ for $p>2$, $H(n, \boldsymbol{m})$ for $p>2$ and $K(n, \boldsymbol{m})$ for $p>2$. Our results show that if $p=2$, then $W(n, \mathbf{1})$ may possess outer derivations.

Our method is mainly based on the computation of $H^1(L, L)$. We first give a description of the adjoint module L by means of the mixed product (cf. [13]) and the induced module (cf. [9, § 3]). Thus the computation of $H^1(L, L)$ is reduced to that of the cohomology of $L_{[0]}$ which is the Lie algebra of a certain reductive algebraic group and we exploit certain techniques in the representation theory of reductive algebraic groups.

Manuscript received March 31, 1990.
* Department of Mathematics, East China Normal University, Shanghai 200062, China.
** Projects supported by National Natural Science Foundation of China.

§1. Preliminaries

Throughout this paper we shall assume that F is an algebraically closed field, char $F=p>0$. Let L be a finite dimensional Lie algebra over F and V an L-module. A linear mapping $\varphi: L\to V$ is called a derivation from L into V if

$$\varphi([l_1, l_2])=l_1\varphi(l_2)-l_2\varphi(l_1),\ \forall l_1, l_2\in L.$$

Thus $\varphi\in C^1(L, V)$ is a derivation if and only if $\varphi\in Z^1(L, V)$. Hence the set of all derivations $\varphi: L\to V$ has an F-vector space structure; we shall denote the vector space by $\mathrm{Der}_F(L, V)(=Z^1(L, V))$. A derivation $\varphi: L\to V$ is called inner if there is $v\in V$ such that $\varphi(x)=x.\ v$, $\forall x\in L$. The inner derivations in $\mathrm{Der}_F(L, V)$ clearly form an F-subspace, which we denote by $I\mathrm{der}_F(L, V)$. It is clear that $\mathrm{Ider}_F(L, V)=B^1(L, V)$. It implies that

$$H^1(L, V)\cong\mathrm{Der}_F(L, V)/I\mathrm{der}_F(L, V).$$

In case of $V=L$, we denote $\mathrm{Der}_F(L, L)$ and $I\mathrm{der}_F(L, L)$ by $\mathrm{Der}_F(L)$ and $I\mathrm{der}_F(L)$, respectively. By abuse of language, $\mathrm{Der}_F(L)/I\mathrm{der}_F(L)(\cong H^1(L, L))$ is referred to as the algebra of outer derivations. If L is simple, then the adjoint representation ad: $L\to\mathrm{End}_F(L)$ gives an isomorphism of L onto $I\mathrm{der}_F(L)$. It implies that the structure of $H^1(L, L)$ determines that of the derivation algebra of L.

§2. The Adjoint Modules of the Graded Cartan Type Lie Algebras

Throughout this paper we shall assume that F is an algebraically closed field, char $F=p>0$, and all Lie algebras and modules are finite-dimensional.

We are ready to give a brief description of the graded Lie algebras of Cartan type. Let $A(n)$ be the set of n-tuples of nonnegative integers, $\varepsilon_i=(\delta_{1i}, \cdots, \delta_{ni})\in A(n)$. For $\alpha=(\alpha_1, \cdots, \alpha_n)\in A(n)$, set

$$|\alpha| = \sum_{i=1}^{n}\alpha_i.$$

Let $\mathfrak{A}(n)$ be the divided power algebra with basis $\{x^{(\alpha)}\mid\alpha\in A(n)\}$ and mulitplication

$$x^{(\alpha)}x^{(\beta)}=C_{\alpha}^{\alpha+\beta}x^{(\alpha+\beta)},\ \alpha, \beta\in A(n),$$

where

$$C_{\beta}^{\alpha}=\prod_{i=1}^{n}C_{\beta_i}^{\alpha_i},\ \text{for } \alpha=(\alpha_1, \cdots, \alpha_n),\ \beta=(\beta_1, \cdots, \beta_n)\in A(n).$$

If $\boldsymbol{m}=(m_1, \cdots, m_n)$ is an n-tuple of positive integers and $A(n, \boldsymbol{m})=\{\alpha\in A(n)\mid\alpha_i<p^{m_i}, i=1, \cdots, n\}$, then $\mathfrak{A}=\mathfrak{A}(n, \boldsymbol{m})=\langle x^{(\alpha)}\mid\alpha\in A(n, \boldsymbol{m})\rangle$ is a subalgebra of $\mathfrak{A}(n)$. Define derivations D_i, $i=1, \cdots, n$, by

$$D_i x^{(\alpha)}=x^{(\alpha-\varepsilon_i)}$$

(We set $x^{(\alpha)}=0$, if $\alpha\notin A(n)$). Then

$$W=W(n, \boldsymbol{m}):=\left\{\sum_{j=1}^{n} a_j D_j \mid a_j\in\mathfrak{A}(n, \boldsymbol{m})\right\}$$

is a derivation algebra of $\mathfrak{A}(n, \boldsymbol{m})$. The bracket operation of $W(n, \boldsymbol{m})$ is

$$[\sum_i a_i D_i, \sum_i b_i D_i]=\sum_i\sum_j (a_j D_j(b_i)-b_j D_j(a_i))D_i. \tag{2.1}$$

Set $\mathfrak{A}_{[i]}=\langle x^{(\alpha)} \mid |\alpha|=i\rangle$ and $W_{[i]}=\langle x^{(\alpha)}D_j \mid x^{(\alpha)}\in\mathfrak{A}_{[i+1]},\ j=1, \cdots, n\rangle$. Then $W=\bigoplus_{i\geq -1} W_{[i]}$ is a graded Lie algebra of depth 1.

The subspace $S(n, \boldsymbol{m})$ $(n>3)$ spanned by

$$D_{i,j}(f):=D_j(f)D_i-D_i(f)D_j,\ f\in\mathfrak{A}(n, \boldsymbol{m}),\ i, j=1, \cdots, n, \tag{2.2}$$

is a Lie subalgebra of $W(n, \boldsymbol{m})$. Write $\pi=\{p^{m_1}-1, \cdots, p^{m_n}-1\}$. If $n=2r$, let

$$\sigma(i):=\begin{cases} 1,\ 1\leq i\leq r,\\ -1,\ r<i\leq n,\end{cases}$$

$$i'=i+(i)r,\ i=1, \cdots, n.$$

The subspace $H(n, m)$ spanned by

$$D_H(x^{(\alpha)}):=\Sigma\sigma(i')D_{i'}(x^{(\alpha)})D_i,\ \alpha\in A(n, \boldsymbol{m}),\ \alpha\neq\pi, \tag{2.3}$$

is a Lie subalgebra of $S(n, \boldsymbol{m})$.

If L is anyone of $W(n, \boldsymbol{m})$, $S(n, \boldsymbol{m})$ and $H(n, \boldsymbol{m})$, then $L=\bigoplus_{i\geq -1} L_{[i]}=\bigoplus_{i\geq -1} L\cap W_{[i]}$ is a graded Lie algebra of depth 1 and under the linear map $x^{(\varepsilon_i)}D_j\mapsto E_{ij}$, $L_{[0]}$ is isomorphic to $\mathrm{gl}(n)$, $\mathrm{sl}(n)$ and $\mathrm{sp}(n)$, respectively, where E_{ij} is the matrix whose (k, l)-component is $\delta_{ik}\delta_{jl}$.

We proceed to construct a class of graded modules of $L=W(n, \boldsymbol{m})$, $S(n, \boldsymbol{m})$ or $H(n, \boldsymbol{m})$. Let ρ_0 be a representation of $L_{[0]}$ in the module V_0 and $\widetilde{V}_0=\mathfrak{A}\otimes V_0$. If $D=\Sigma a_i D_i\in L$, then $\widetilde{D}_i=\Sigma D_i(a_j)\otimes E_{ij}\in\mathfrak{A}\otimes L_{[0]}$. Let $\widetilde{D}=\Sigma g_i\otimes l_i$, where $g_i\in\mathfrak{A}$, $l_i\in L_{[0]}$. Define a linear transformation $\widetilde{\rho}_0(D)$ of $\widetilde{V}_0$ by

$$\widetilde{\rho}_0(D)(f\otimes v)=D(f)\otimes v+\Sigma g_i f\otimes\rho_0(l_i)v,\ f\in\mathfrak{A},\ v\in V_0. \tag{2.4}$$

According to [13, Theorem 1.2], $\widetilde{\rho}_0$ is a representation of L in $\widetilde{V}_0$ and $\widetilde{V}_0=\bigoplus_{i\geq 0}\langle x^{(\alpha)}\otimes V_0 \mid |\alpha|=i\rangle$ is a graded L-module such that the base space of $\widetilde{V}_0$ is $1\otimes v_0\cong v_0$. The module $\widetilde{V}_0$ is called the mixed product of $\mathfrak{A}$ and V_0. We shall give the adjoint modules $W(n, \boldsymbol{m})$, $S(n, \boldsymbol{m})$ and $H(n, \boldsymbol{m})$ in the forms of mixed products.

Let $\boldsymbol{h}(L_{[0]})$ be the standard Cartan subalgebra of $L_{[0]}=\mathrm{gl}(n)$, $\mathrm{sl}(n)$ or $\mathrm{sp}(n)$ and Λ_i, $i=1, \cdots, n$, the linear functions on $\boldsymbol{h}(\mathrm{gl}(n))=\langle E_{11}, \cdots, E_{nn}\rangle$ such that

$$\Lambda_i(E_{jj})=\delta_{ij}. \tag{2.5}$$

The restriction of Λ_i on every $\boldsymbol{h}(L_{[0]})$ will also be denoted by Λ_i. Let

$$\lambda_0=0,\ \lambda_i=\sum_{j=1}^{i}\Lambda_j,\ i=1, \cdots, n. \tag{2.6}$$

Then λ_i($i=1, \cdots, n$, for $L_{[0]}=\mathrm{gl}(n)$; $i=1, \cdots, n-1$, for $L_{[0]}=\mathrm{sl}(n)$; $i=1, \cdots, r$, for $L_{[0]}=\mathrm{sp}(n)$) are the fundamental weights of $L_{[0]}$. Every weight of $\mathrm{gl}(n)$, $\mathrm{sl}(n)$ or sp(n) is a linear combination of the fundamental weights. We have

$$\Lambda_i=\lambda_i-\lambda_{i-1},\ i=1,\ \cdots,\ n.$$

If $\lambda\in(\boldsymbol{h}(\mathrm{gl}(n)))^*$, denote by $V_0(\lambda)$ the irreducible module of gl(n) with highest weight λ. Let N_0 be an n-dimensional linear space with basis $\{e_1,\ \cdots,\ e_n\}$. On N_0, we have the natural representation ν_0 of gl(n):

$$\nu_0(E_{ij})_{ek}=\delta_{jk}e_i.$$

It is obvious that

$$\wedge^i N_0=N_0\wedge\cdots\wedge N_0(\text{ factors})=V_0(\lambda_i),\ i=1,\ \cdots,\ n.$$

By [13, Llemma 2.2], we have

$$W(n,\ \boldsymbol{m})\cong\widetilde{V}_0^*(\lambda_1), \tag{2.7}$$

that is, the adjoint module $W(n,\ \boldsymbol{m})$ is isomorphic to the mixed product of $\mathfrak{A}$ and the dual module $V_0^*(\lambda_1)$ of $V_0(\lambda_1)$. Let $I=E_{11}+\cdots+E_{nn}\in\mathrm{gl}(n)\cong W_{[0]}$. Then the action of I on $V_0^*(\lambda_1)$ is the scalar multiplication by $p-1$. As sl(n)-modules we have

$$V_0^*(\lambda_1)=V_0(\lambda_{n-1}). \tag{2.8}$$

Note that the gl(n)-module $V_0(\lambda_i)$ can be also regarded as an sl(n)-module which is sl(n)-irreducible and $V_0(\lambda_i)=\mathfrak{A}\otimes V_0(\lambda_i)$ is also a mixed product of $S(n,\ \boldsymbol{m})$. As sl(n)-modules, the adjoint module $S(n,\ \boldsymbol{m})$ is isomorphic to the minimum submodule of $W(n,\ \boldsymbol{m})\cong\widetilde{V}_0(\lambda_{n-1})$ (by [14, Theorem 2.2]). Since [15, Theorem 2.2] is also true for $p>0$, we have an exact sequence

$$0\to S(n,\ \boldsymbol{m})\to\widetilde{V}_0(\lambda_{n-1})\to F^{n+1}\to 0. \tag{2.9}$$

Next, we shall consider the case $H(n,\ \boldsymbol{m})$ $(n=2r)$ for $p>2$. Let $\mathfrak{A}':=\bigoplus_{\alpha\neq\pi}\langle x^{(\alpha)}\rangle$. Since the base space (homogeneous space of degree 0) $H_{[-1]}=W_{[-1]}$ of the adjoint module $H(n,\ \boldsymbol{m})$ is the natural $H_{[0]}(\cong\mathrm{sp}(n))$-module $V_0(\lambda_1)$ and [15, Theorem 2.3] is also true for $p>2$, by [14, Theorem 2.2], $H(n,\ \boldsymbol{m})$ is isomorphic to the minimum submodule of $\widetilde{V}_0(\lambda_1)$ and there are three exact sequences

$$0\to\mathfrak{A}'\to\mathfrak{A}(\cong\widetilde{V}_0(\lambda_0))\to F\to 0, \tag{2.10}$$

$$0\to F\to\mathfrak{A}'\to H(n,\ \boldsymbol{m})\to 0, \tag{2.11}$$

$$0\to H(n,\ \boldsymbol{m})\to\widetilde{V}_0(\lambda_1)\to\widetilde{V}_0(\lambda_1)/H(n,\ \boldsymbol{m}). \tag{2.12}$$

We conclude this section by considering the case of $K(n,\ \boldsymbol{m})$ $(p>2)$. Adopting the notation of [16, Chap. 4, § 5] and putting $n=2r+1$, $\|\alpha\|:=|\alpha|+\alpha_n-2$, $\mathfrak{A}(n,\ \boldsymbol{m})_{[i]}=\langle x^{(\alpha)}\mid\|\alpha\|=i\rangle$ and $\|\pi\|=s$, then we have

$$K(n,\ \boldsymbol{m})\cong\begin{cases}\mathfrak{A}(n,\ \boldsymbol{m}),\ n+3\not\equiv 0 & \mathrm{mod}(p),\\ \langle x^{(\alpha)}\mid\alpha<\pi\rangle,\ n+3\equiv 0 & \mathrm{mod}(p),\end{cases} \tag{2.13}$$

where

$$\langle x^{(\alpha)},\ x^{(\beta)}\rangle=\sum_{j=1}^{2r}\sigma(j)x^{(\alpha-\varepsilon_j)}x^{(\beta-\varepsilon_{j'})}+\left(\|\beta\|\binom{\alpha+\beta-\varepsilon_{j'}}{\beta}-\|\alpha\|\binom{\alpha+\beta-\varepsilon_n}{\alpha}\right)x^{(\alpha+\beta-\varepsilon_n)} \tag{2.14}$$

is Lie product of $\mathfrak{A}(n,\ \boldsymbol{m})$ and

$$\sigma(j)=\begin{cases} 1, & j\leqslant r, \\ -1, & j\geqslant r+1, \end{cases} \qquad j'=\begin{cases} j+r, & j\leqslant r, \\ j-1, & j\geqslant r+1. \end{cases}$$

Let $K=K(n, \boldsymbol{m})$, $K_{[j]}=\mathfrak{A}(n, \boldsymbol{m})_{[j]}$ and $K_i=\bigoplus_{j\geqslant i} K_{[j]}$. The $K=\bigoplus_{i=-2}^{s} K_{[i]}$. Let V be a K_0-module, and $U(K)$ and $U(K_0)$ the universal enveloping algebra of K and K_0, respectively. We shall define the subalgebra $\theta(K, K_0)$ of $U(K)$ and consider the induced module $U(K)\otimes_{\theta(K,K_0)}V$. It is obvious that the elements

$$z_i:=(x^{(\varepsilon_i)})^{p^{m_{i'}}},\ i=1, \cdots, 2r,$$
$$z_n:=1^{p^{m_n}}$$

belong to the center of $U(K)$. Let $\theta(K, K_0)$ be the subalgebra of $U(K)$ which is generated by $U(K_0)$ and $\{z_1, \cdots, z_n\}$. Then it is isomorphic to $F[z_1, \cdots, z_n]\otimes_F U(K_0)$. The action of $U(K_0)$ on V can be extended to $\theta(K, K_0)$ by letting the polynomial algebra $F[z_1, \cdots, z_n]$ operate via its canonical supplemantation. Henceforth all K_0-modules will be considered $\theta(K, K_0)$-modules in this fashion. Let ζ denote the natural representations of K_0 in K/K_0. Then there exist unique homomorphism σ: $U(K_0)\to F$ of F-algebras such that $\sigma(x)=\mathrm{tr}(\zeta(x))$. We iutroduce a twisted action. on V by setting $x.\ v:=xv+\sigma(x)v$. The new K_0-module will be called V_σ. By [9, Theorem 3.3], there is a natural isomorphism of $U(K)$-modules

$$U(K)\otimes_{\theta(K,\ K)}V_\sigma\cong \mathrm{Hom}_{\theta(K,\ K_0)}(U(K), V).$$

Let V be the $\langle x^{(\varepsilon_i+\varepsilon_j)}|1\leqslant i, j\leqslant 2r\rangle$-module (i. e., sp$(2r)$-module) with the highest weight λ such that the action of $x^{(\varepsilon_n)}$ on V is the scalar multiplication by c. We denote the $K_{[0]}$-module V by $V(\lambda, c)$ and extend the operations on $V(\lambda, c)$ to K_0 by letting K_1 act trivially. By [8, Proposition 4.3], there is an injective homomorphism ψ: $\mathfrak{A}(n, \boldsymbol{m})\to\mathrm{Hom}_{\theta(K,\ K_0)}(U(K), V(0, -2))$ of K-modules. Since $\dim_F\mathfrak{A}(n, \boldsymbol{m})=\dim_F \mathrm{Hom}_{\theta(K,\ K_0)}(U(K), V(0, -2))$, we have

$$\mathfrak{A}(n, \boldsymbol{m})\cong U(K)\otimes_{\theta(K,\ K_0)}V(0, -2)_\sigma. \tag{2.15}$$

Therefore, if $n+3\not\equiv 0 \bmod(p)$, then

$$K(n, \boldsymbol{m})\cong U(K)\otimes_\theta(K, K_0), V(0, -2)_\sigma; \tag{2.16}$$

if $n+3\equiv 0 \bmod(p)$, then there is an exact sequence

$$0\to K(n, \boldsymbol{m})\to U(K)\otimes_\theta(K, K_0)V(0, -2)_\sigma\to F\to 0. \tag{2.17}$$

§ 3. The Derivation Algebras of $W(n, \boldsymbol{m})$

Suppose that $p>0$ and $W(n, \boldsymbol{m})$ is simple. Put $W=W(n, \boldsymbol{m})$ and $W_i=\bigoplus_{j\geqslant i} W_{[j]}$. Since $V^*(\lambda_1)$ is a nontrivial irreducible gl(n)-module with a highest weight; by [6, Lemma 2.1 and Lemma 3.1,] we have

$$H^1(W, W)\cong H^1(W, \tilde V_0^*(\lambda_1))\cong H^1(W_0, V_0^*(\lambda_1)). \tag{3.1}$$

Since the action of I on $V_0^*(\lambda_1)$ is the scalar multiplication by $p-1$, by [1, Lemma 3.2, (3.2) and (3.4)], we have

$$H^1(W_0, V_0^*(\lambda_1)) \cong H^1(W_1, V_0^*(\lambda_1))^{gl(n)} \cong Hom_{gl(n)}(W_1/[W_1, W_1], V_0^*(\lambda_1))$$
$$\cong \mathrm{Hom}_{gl(n)}(Y_1, V_0^*(\lambda_1)) \oplus \sum_{\mu>0} \mathrm{Hom}_{gl(n)}(Y_{p^\mu-1}, V_0^*(\lambda_1)) \quad (3.2)$$

where Y_i is the contribution to $W_1/[W_1, W_1]$ coming from $W_{[i]}$ and $Y_{p^\mu-1}$ is isomorphic to a direct sum of $\#\{j\,|\,m_j>\mu\}$-copies of $V_0^*(\lambda_1)$. Then there are $\#\{j\,|\,m_j>\mu\}$-isomorphisms, denoted by $\psi_j^{(\mu)}$ for $j=1, \cdots, n$ and $m_j>\mu$, which form a basis of $\mathrm{Hom}_{gl(n)}(Y_{p^\mu-1}, V_0^*(\lambda_1))$. Hence

$$\bigoplus_{\mu>0} \mathrm{Hom}_{gl(n)}(Y_{p^\mu-1}, V_0^*(\lambda_1)) = \langle \psi_j^{(\mu)}\,|\,j=1, \cdots, n, m_j>\mu>0\rangle. \quad (3.3)$$

If $p>2$, then

$$\mathrm{Hom}_{gl(n)}(Y_1, V_0^*(\lambda_1)) = 0.$$

Since the action of I on Y_1 is the scalar multiplication by 1. If $p=2$, then

$$\mathrm{Hom}_{gl(n)}(Y_1, V_0^*(\lambda_1)) \cong \mathrm{Hom}_{sl(n)}(Y_1, V_0(\lambda_{n-1})) \cong \mathrm{Hom}_{gl(n)}(W_{[1]}, V_0(\lambda_{n-1}))$$
$$\cong \mathrm{Hom}_{gl(n)}(\mathfrak{A}_{[2]} \otimes W_{[-1]}, V_0(\lambda_{n-1})).$$

As $sl(n)$-modules we have $\mathfrak{A}_{[2]} \cong V_0(2\lambda_1)$ and $W_{[-1]} \cong V_0(\lambda_{n-1})$. In prime characteristic, we know that the tensor product $\mathfrak{A}_{[2]} \otimes W_{[-1]}$ has a good filtration with two factors. These two factors are the Weyl modules $\overline{V}_{2\lambda_1+\lambda_{n-1}}$ and $\overline{V}_{\lambda_1}$ with highest weights $2\lambda_1+\lambda_{n-1}$ and λ_1, respectively. We claim that

$$\mathrm{Hom}_{gl(n)}(\overline{V}_{2\lambda_1+\lambda_{n-1}}, \overline{V}_{\lambda_{n-1}}) = 0,$$

where $\overline{V}_{\lambda_{n-1}}$ is the Weyl module with highest weight λ_{n-1}. Let the maximal torus T be the subgroup of $G=GL(n)$ consisting of diagonal matrices, $X(T)$ the lattice of all weights of T and $X(T)^+$ the set of dominant weights in $X(T)$. Then any weight $\mu \in X(T)$ is of the form

$$\mu(g) = (g_{11})^{\mu_1} \cdots (g_{nn})^{\mu_n},$$

where $g=(g_{ij}) \in T$ and $\mu_1, \cdots, \mu_n$ are integers, so we can identify μ with the vector $\mu=(\mu_1, \cdots, \mu_n) \in \mathbf{Z}^n$. By definition

$$X(T)^+ = \{\mu \in (T)\,|\,\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \geqslant 0\}.$$

Clearly $X(T)^+$ is in 1–1 correspondence with the set of partitions with at most n parts. Then $2\lambda_1+\lambda_{n-1}$ and λ_{n-1} can be identified with the partitions $\mu=(3, 1, 1, \cdots, 1, 0)$ and $\mu'=(1, 1, \cdots, 1, 0)$ respectively. Since $\sum \mu_i (=n+1) \neq \sum \mu_i' (=n-1)$ in $\mathbf{Z}$, by [2, § 3.8, Remark 2)], we obtain the asserted result. It implies that

$$\mathrm{Hom}_{gl(n)}(Y_1, V_0^*(\lambda_1)) = \mathrm{Hom}_{gl(n)}(\overline{V}_{\lambda_1}, \overline{V}_{\lambda_{n-1}}) = \mathrm{Hom}_{gl(n)}(V_0(\lambda_1)), V_0(\lambda_{n-1}))$$
$$\cong \begin{cases} F, \text{ if } n=2, \\ 0, \text{ if } n>2. \end{cases} \quad (3.4)$$

By (3.2), (3.3) and (3.4), we have

Theorem 3.1. *Suppose that $p>0$ and $W(n, \boldsymbol{m})$ is simple. Then*

$$H^1(W(n, \boldsymbol{m}), W(n, \boldsymbol{m})) \cong \begin{cases} \langle \psi_j^{(\mu)}\,|\,j=1, \cdots, n, m_j>\mu>0\rangle \oplus F, \text{ if } p=2 \text{ and } n=2, \\ \langle \psi_j^{(\mu)}\,|\,j=1, \cdots, n, m_j>\mu>0\rangle, \text{ otherwise.} \end{cases}$$

Remark 3.1. Now we shall determine the derivation algebra of $W(2, \mathbf{1})$ for $p=2$. In the same way, we can also deal with the other case. Since the gl(2)-modules $\langle x^{(\varepsilon_1+\varepsilon_2)}D_2, x^{(\varepsilon_1+\varepsilon_2)}D_1\rangle (\subseteq W_{[1]})$ and $\langle D_2, D_1\rangle$ are isomorphic to $V_D(\lambda_1)$ and $V_0^*(\lambda_1)$, respectively, we can easily see that

$$\varphi(x^{(\alpha)}D_j)=\begin{cases} D_j, & \text{if } \alpha=\varepsilon_1+\varepsilon_2 \text{ and } j=1, 2, \\ 0, & \text{otherwise}\end{cases}$$

is a 1-cocycle on W_0. Using the method in the proof of [7, Lemma 2.1], φ can be extended to a 1-cocycle $\tilde{\varphi}$ on W such that

$$\tilde{\varphi}(x^{(\alpha)}D_j)=x^{(\alpha-\varepsilon_1-\varepsilon_2)}D_j, \text{ for } \alpha\in A(n, \mathbf{m}), j=1, 2,$$

is an outer derivation of $W(2, \mathbf{1})$ and

$$H^1(W(2, \mathbf{1}), W(2, \mathbf{1}))\cong\langle\tilde{\varphi}\rangle.$$

§ 4. The Derivation Algebras of $S(n, \mathbf{m})$,

Let char. $F=p>0$, $S=S(n, \mathbf{m})$ and $S_i=\bigoplus_{j\geq i} S_{[j]}$. First, we shall compute $H^1(S(n, \mathbf{m}), \tilde{V}_0(\lambda_{n-1}))$. By [5, Lemma 2.1 and Lemma 2.2] or [8, Corollary 4.5, we have

$$H^1(S, \tilde{V}_0(\lambda_{n-1}))\cong H^1(S, S_{[-1]}, \tilde{V}_0(\lambda_{n-1}))\cong H^1(S_0, V_0(\lambda_{n-1})). \tag{4.1}$$

Using the cohomology five-term sequence, we have

$$0\to H^1(\mathrm{sl}(n), V_0(\lambda_{n-1})^{S_1})\to H^1(S_0, V_0(\lambda_{n-1}))$$
$$\to H^1(S_1, V_0(\lambda_{n-1}))^{\mathrm{sl}(n)}\to H^2(\mathrm{sl}(n), V_0(\lambda_{n-1})^{S_1}). \tag{4.2}$$

As S_1 acts trivially on $V_0(\lambda_{n-1})$,

$$H^1(S_1, V_0(\lambda_{n-1}))^{\mathrm{sl}(n)}\cong \mathrm{Hom}_{\mathrm{sl}(n)}(S_1/[S_1, S_1], V_0(\lambda_{n-1})).$$

We regard $S_1/[S_1, S_1]$ as an $S_{[0]}(\cong \mathrm{sl}(n))$-module and investigate its weight vectors with the weight $\lambda_{n-1}=\sum_{i=1}^{n-1}\Lambda_i$. Let $D_{i,j}(x^{(\alpha)})+[S_1, S_1]$ be one of them, where $\alpha=(\alpha_1, \cdots, \alpha_n)\in\mathfrak{A}(n, \mathbf{m})$. Write $D=D_{i,j}(x^{(\alpha)})$ and note that

$$\begin{cases} \alpha_k\equiv\delta_{ki}+\delta_{kj}+\alpha_1-\delta_{1i}-\delta_{1j} \bmod (p), \ k=2, \cdots, n-1, \\ \alpha_n\equiv\delta_{ni}+\delta_{nj}+\alpha_1-1-\delta_{1i}-\delta_{1j} \bmod (p)\end{cases} \tag{4.3}$$

and

$$[x^{(k)}Dl, D]\in[S_1, S_1], \text{ for } 1\leq k<l\leq n. \tag{4.4}$$

By (4.3) and (4.4), we can show that

$$D=D_{i,n}(x^{(\varepsilon_i+p^k\varepsilon_1)}), \text{ for } i=1, \cdots, n-1, l=1, \cdots, n \text{ and } 1\leq k<m_1.$$

Since $D_{i,n}(x^{(\varepsilon_i+p^k\varepsilon_l)})=-x^{(p^k\varepsilon_l)}D_n$, i, $l=1, \cdots, n-1$, and $D_{1,n}(x^{(\varepsilon_1+p^k\varepsilon_n)})=D_{i,n}(x^{(\varepsilon_i+p^k\varepsilon_n)})$, $i=2, \cdots, n-1$, we have

$$\mathrm{Hom}_{\mathrm{sl}(n)}(S_1/[S_1, S_1], V_0(\lambda_{n-1}))\cong\langle D_{1,n}(x^{(\varepsilon_1+p^k\varepsilon_l)})\ l=1, \cdots, n;\ 1<k<m_l\rangle. \tag{4.5}$$

Suppose that $p>2$. By [10, Theorem III, 2], we have

$$H^*(sl(n), V_0(\lambda_{n-1}))=0. \tag{4.6}$$

Therefore, we have

Lemma 4.1. *Suppose that $p>2$. Then*

$$H^1(S(n, \boldsymbol{m}), \tilde{V}_0(\lambda_{n-1}) \cong \langle D_{1,n}(x^{(\varepsilon_i+p^i\varepsilon_j)}) \mid 1 \leqslant i < m_j, j=1, \cdots, n \rangle.$$

Next, we shall compute $H^1(S, S)$. Since [15, Theorem 2.2] is also true for $p>0$, we have an exact sequence

$$0 \to S \to \tilde{V}_0(\lambda_{n-1}) \to F^{n+1} \to 0.$$

Then we have the long exact sequence

$$0 \to H^0(S, F^{n+1}) \to H^1(S, S) \to H^1(S, \tilde{V}_0(\lambda_{n-1})) \to H^1(S, F^{n+1}). \tag{4.7}$$

Since $H^0(S, F^{n+1}) \cong F^{n+1}$ and $H^1(S, F^{n+1})=0$, by Lemma 4.1, we have

Theorem 4. 1. *Suppose that $p>2$. Then*

$$H^1(S(n, \boldsymbol{m}), S(n, \boldsymbol{m})) \cong F^{n+1} \oplus \langle D_{1,n}(x^{(\varepsilon_i+p^i\varepsilon_j)}) \mid 1 \leqslant i < m_j, j=1, \cdots, n \rangle.$$

Remark 4. 1. In (4.7), the linear map

$H^1(S, S) \to H^1(S, \tilde{V}_0(\lambda_{n-1})) (\cong \langle D_{1,n}(x^{(\varepsilon_i+p^i\varepsilon_j)}) \mid 1 \leqslant i < m_j, j=1, \cdots, n \rangle)$ sends the outer derivation $(\mathrm{ad}\, D_j)^{p^i}$ of $S(n, \boldsymbol{m})$ onto $D_{1,n}(x^{(\varepsilon_i+p^i\varepsilon_j)})$.

§ 5. The Derivation Algebras of $H(n, \boldsymbol{m})$

Since $H(n, \boldsymbol{m})$ $(p>2)$ possesses a nondegenerate associative form (cf, [16, Char. 4, Theorem 6.5]), $H(n, \boldsymbol{m})^* \cong H(n, \boldsymbol{m})$ (regarded as $H(n, m)$-modules). By [6, Theorem 5.1], we have

Theorem 5. 1. *Suppose that $p>2$. Then*

$$\dim_F H^1(H(n, \boldsymbol{m}), H(n, \boldsymbol{m})) = |\boldsymbol{m}|+2.$$

§ 6 The Derivation Algebras of $K(n, \boldsymbol{m})$

First, suppose that $n+3 \equiv 6 \bmod (p)$. By (2.17), we have the long exact sequence

$$0 \to F \to H^1(K(n, \boldsymbol{m}), K(n, \boldsymbol{m})) \to H^1(K(n, \boldsymbol{m}), U(K) \otimes_{\theta(K, K_0)} V(0, -2)_\sigma \to 0. \tag{6.1}$$

By [8, Theorem 4.1],

$$H^1(K, U(K) \otimes_{\theta(K, K_0)} V(0, -2)_\sigma) \cong H^1(K_0, V(0, -2)). \tag{6.2}$$

Using the cohomology five-term sequence, we have

$$\begin{aligned} &0 \to H^1(K_{[0]}, V(0, -2)) \to H^1(K_0, V(0, -2)) \\ &\to H^1(K_1, V(0, -2))^{K_{[0]}} \to H^2(K_{[0]}, V(0, -2)). \end{aligned} \tag{6.3}$$

Since $\langle x^{(\varepsilon_n)} \rangle$ is an ideal of $K_{[0]}$, there is the Hochschild-Serre spectral sequence $\{E_r^{i,j}\}$, whose E_∞ term is associated with $H^*(K_{[0]}, V(0, -2))$, that is,

$$E_2^{i,j} = H^i(K_{[0]}/\langle x^{(\varepsilon_n)} \rangle, H^j(\langle x^{(\varepsilon_n)} \rangle, V(0, 2)) \Rightarrow H^k(K_{[0]}, V(0, -2))$$

for $k=i+j$. Since $H^j(\langle x^{(\varepsilon_n)} \rangle, V(0, -2))=0$, we have

$$H^k(K_{[0]}, V(0, -2))=0, \text{ for } k\geq 0. \quad (6.4)$$

By (6.3), (6.4) and direct computation, we have

$$\begin{aligned} H^1(K_0, V(0, -2)) &\cong H^1(K_1, V(0, -2))^{K_{(0)}} \\ &\cong \mathrm{Hom}_{K_{(0)}}(K_1/[K_1, K_1], V(0, -2)) \\ &\cong \langle x^{(p^i\varepsilon_j)} \mid i=1, \cdots, m_j-1, j=1, \cdots, n\rangle. \end{aligned} \quad (6.5)$$

By (6.1), (6.2) and (6.5), we have

$$H^1(K, K)\cong F\oplus\langle x^{(p^i\varepsilon_j)} \mid i=1, \cdots, m_j-1, j=1, \cdots, n\rangle. \quad (6.6)$$

Next, suppost that $n+3\not\equiv 0 \mod (p)$. By (2.16), (6.2) and (6.5), we have

$$H^1(K, K)\cong\langle x^{(p^i\varepsilon_j)} \mid i=1, \cdots, m_j-1, j=1, \cdots, n\rangle. \quad (6.7)$$

Therefore, we have

Theorem 6.1. *Suppose that* $p>2$. *Then*

$$H^1(K(n, m), K(n, m))\cong\begin{cases} F\oplus\langle x^{(p^i\varepsilon_j)} \mid i=1, \cdots, m_j-1, j=1, \cdots, n\rangle, \\ \quad \text{if } n+3\equiv 0 \mod (p), \\ \langle x^{(p^i\varepsilon_j)} \mid i=1, \cdots, m_j-1, j=1, \cdots, n\rangle, \\ \quad \text{otherwise}. \end{cases}$$

References

[1] Andersen, H. H. & Jantzen, J. C., Cohomology of induced representations for algebraic groups, *Math. Ann.* **269** (1984), 487—525.

[2] Carter, R. W. & Lusztig, G., On the modular representations of the general linear and symmetric groups, *Math. Z.*, **136** (1974), 193—242.

[3] Ju. Celousov, M., Derivations of Lie algebras of Cartan type, Izv. Vyss. Ucebn. Zaved Mathematika 7, 126—134.

[4] Chiu Sen, On the restricted cohomology of the Lie algebras of the simple algebraic groups (in Chinese), *Chin. Ann. of Math.*, **6A**: 3 (1985), 281—289.

[5] Chiu Sen, Cohomology of graded Lie algebras of Cartan type $S(n, m)$, *Chin. Ann. of Math.*, **10B**: 1 (1989), 105—114.

[6] Chiu Sen, Central extensions and $H^1(L, L^*)$ of the graded Lie algebras of Cartan type, *J. Algebra* (to appear).

[7] Chiu, S. & Yu. Shen, G., Cohomology of graded Lie algebras of Cartan type of characterstic p, *Abh. Math. Sem. Univ. Hamburg*, **57** (1987), 139—156.

[8] Farnsteiner, R., Extension functors of modular Lie algebras and Frobenius algebras (preprint).

[9] Farnsteiner, R. & Strade, H., Cohomology theory of modular Lie algebras: Shapiro's lemma and its consequences (preprint).

[10] Feldvoss, J., Homologischs aspekte der darstellungstheorie modularer Lie-algabren, der Universität Hamburg, Dissertation, 1989.

[11] Humphreys, J., Introduction to Lie algebr as and representation theory, Springer, New York, 1972.

[12] Rotman, J., An introduction to homological algebra, New York, Academic Press, 1979.

[13] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (I) mixed product of modules, *Scientia Sinica (Ser. A)*, **29**:6 (1986), 570—581.

[14] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (II)——positive and negative graded nodules, *Scientia Sinica (Ser. A)*, **29**: 10 (1986), 1009—1019.

[15] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (II)——irreducible modules, *Chiu. Ann. of Math.*, **9B**: 4 (1988), 404—417.

[16] Strane, R. & Farhsteiner, R., Modular Lie algebras and their representations, Dekker 126, New York, 1988.

[17] Wan Zhexian, Lie algebras, Science Publishing House, Beijing, 1964 (in Chinese).

[18] Wilson, R., A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic *J. Algebra*, **40** (1976), 418—465.