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UHF-ALGEBRAS AND $*$-ISOMORPHISMS

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Abstract

The following problem is discussed: If $\mathscr{M}$ is an AF-algebra, A and B are two C^*-algebras, $\mathscr{K}(A)$ and $\mathscr{K}(B)$ are the unit balls of $\mathscr{M}\otimes A$ and $\mathscr{M}\otimes B$ respectively, and there is an affine isomorphism from $\mathscr{K}(A)$ onto $\mathscr{K}(B)$, when are A and B isomorphic? This problem was inspired by K-theory of C^*-algebras. If $\mathscr{M}$ is a UHF-algebra of type $\{m^p\}$, an orientation on $\mathscr{K}(A)$ and $\mathscr{K}(B)$, i. e., an action of the special group of $\mathscr{M}$, is introduced. Then it is obtained that if the affine isomorphism commutes with the group action then A and B are isomorphic.

§ 1. Introduction

One of the problems in operator algebras is to classify all the operator algebras. The problem was partially answered for Von Neumann algebras by Von Neumann, Murray, Takesaki, Connes, et al. for a decade. However, it was not given any progress for C^*-algebras until K-theory was introduced in C^*-algebras.

The idea of K-theory of C^*-algebras is that we can construct abelian groups $K_0(A)$ and $K_1(A)$ for a C^*-algebra A. The classification of C^*-algebras was reduced to that of its K-groups. The basic problem in K-theory of C^*-algebras is when K-groups are complete invariants of $*$-isomorphic classes.

In thss paper we will deal with such problem in some sense. It is wellknown that the construction of K_0-group of a C^*-algebra A is from Grothendieck's groupization of the semi-group $H(A)$ which is the equivalent classes of $\mathrm{Proj}(A\otimes\mathscr{K})$, where $\mathscr{K}$ is the C^*-algebra of compact operators on a seperable Hilbert space. From Kadison's theorem[2], we have

$$\mathrm{Ext}((A\otimes\mathscr{K})_1^+)=\mathrm{Proj}(A\otimes\mathscr{K}),$$

where Ext denotes the set of extreme points. If B is a C^*-algebra, Φ; $\mathrm{Ext}((A\otimes\mathscr{K})_1^+)\to\mathrm{Ext}((B\otimes\mathscr{K})_1^+)$ is bijective which is equivalent to that Φ is an affine isomophism between $(A\otimes\mathscr{K})_1^+$ and $(B\otimes\mathscr{K})_1^+$, the basic problem is reduced to the following

Problem 1. 1. *If Φ: $(A\otimes\mathscr{K})_1^+\to(B\otimes\mathscr{K})_1^+$ is an affine isomorphism, when can Φ induce a $*$-isomorphism from A onto B?*

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Moreover, we have following general problem:

Problem 1. 2. *If $\mathscr{M}$ is an AF-algebra, Φ: $(A\otimes\mathscr{M})_1^+\to(B\otimes\mathscr{M})_1^+$ is an affine isomorphism, when can Φ induce a *-isomorphism from A onto B?*

It was answered by Prof. Wu Liangsen[6] for Problem 1.2 in cases that $\mathscr{M}$ is an $n\times n$ matrix algebra M_n over $\mathbf{C}$ (for $n>2$) after introducing an orientation on $(A\otimes\mathscr{M})_1^+$ and $(B\otimes\mathscr{M})_1^+$. In this paper we will discuss Problem 1.2 for $\mathscr{M}$, a UHF-algebra of type $\{m^p\}$, and also define an orientation. Therefore, the general question mentioned above is reduced to discussing the orientation for an AF-algebra and how the affine isomorphism preserves the orientation. The idea was first used by Kadison in [2], this will be discussed in a subsequent paper.

§ 2. Preliminaries

Definition 2. 1. *A C^*-algebra $\mathscr{M}$ is called a UHF-algebra if there exists a sequens C^*-subalgebras $\{A_n\}$ of $\mathscr{M}$ such that*

(1) $1\subset A_1\subset A_2\subset\cdots\subset A_n\subset\cdots$,

(2) $\bigcup\limits_{n=1}^{\infty} A_n$ *is dense in $\mathscr{M}$ with norm,*

(3) $A_n\cong M_{n_n}(\mathbf{C})$, *for every $n\in\mathbf{Z}^+$.*

It is wellknown that the UHF-algebra is the inductive limit of finite dimensional matrix algebras over $\mathbf{C}$ with the same unit and also the infinite tensor product of finite dimensional matrix algebras over $\mathbf{C}$, which is as the following theorem.

Theorem 2. 2.[3] *For $\{p_n\}\subset\mathbf{Z}$, there exists a UHF-algebra $\mathscr{M}$ of type $\{p_n\}$ iff $p_n\,|\,p_{n+1}$, $n=1, 2, \cdots$ Furtheremore, we have*

(1) $\mathscr{M}=\bigotimes\limits_{n=1}^{\infty} M_{m_n}(\mathbf{C})$, $m_1=p_1$, $m_n=p_{n-1}^{-1}p_n$, *for $n>1$, where the tensor norm is taken the minimal C^*-norm.*

(2) $\mathscr{M}=$ *the norm closure of* $\varinjlim\{M_{p_n}, \varphi_{p_{n+1}p_n}\}$, *where $\varphi_{p_{n+1}p_n}$ is a *-embedding from M_{p_n} into $M_{p_{n+1}}$ such that $\varphi_{p_{n+1}}(x)=x\otimes 1_{m_n}$, where 1_{m_n} is the unit of M_{m_n}, $m_{n+1}=p_{n+1}/p_n$.*

By construction of inductive limit, there exists a sequence of *-embeddings φ_{p_n}: $M_{p_n}\to\mathscr{M}$ such that

(1) $$\varphi_{p_n}=\varphi_{p_{n+1}}\varphi_{p_{n+1}p_n},$$

(2) $$\mathscr{M}=\bigcup_{n=1}^{\infty}\varphi_{p_n}(M_{p_n}).$$

Definition 2. 3. *Let $SU(M_{p_n})$ be the special unitary group of M_{p_n}, and*

$$SU(\mathscr{M})=\bigcup_{n=1}^{\infty}\varphi(SU(M_{p_n})),$$

where $\mathscr{M}$ is a UHF-algebra of type $\{p_n\}$. We call $SU(\mathscr{M})$ the special unitary group of the algebra.

Since M_{p_n} is generated by $SU(M_{p_n})$, $\mathscr{M}$ is generated by $SU(\mathscr{M})$ also.

Definition 2. 4. *Let $\{p_i\}\subset\mathbf{Z}$, $p_i|p_{i+1}$, $p_i\to\infty$. Define*

$$\chi\{p_i\}(x)=\sup\{n:\ x^n|p_i,\ \text{for some } i\},$$

where x is a prime number.

Theorem 2. 5.[1, Theorem 1. 1] *UHF-algebras of types $\{p_n\}$ and $\{r_n\}$ are isomorphic if and only if $\chi\{p_n\}=\chi\{r_n\}$.*

Proposition 2. 6. *If $\mathscr{M}^m$ is a UHF-algebra of type $\{m^n\}$, then for every $n>1$ there exists a *-isomorphism such that*

$$\mathscr{M}^m\otimes M_{m^n}\cong\mathscr{M}^m.$$

Proof By Theorem 2.2,

$$\mathscr{M}^m=\lim_{\rightarrow}\{M_{m^k},\ \varphi_{m^{k+1}m^k}\}.$$

Since inductive limit operation commutes with tensor product, we have

$$\mathscr{M}^m\otimes M_{m^n}=\lim_{\rightarrow}\{M_{m^k}\otimes M_{m^n},\ \varphi_{m^{k+1}m^k}\otimes \mathrm{id}m_{m^n}\}$$
$$=\lim_{\rightarrow}\{M_{m^{k+n}},\ \varphi_{m^{k+n+1}m^{k+n}}\}.$$

Hence, $\mathscr{M}^m\otimes M_{m^n}$ is a UHF-algebra of type $\{m^{k+n}\}$. Since $\chi\{m^k\}=\chi\{m^{k+n}\}$ from 2.5 we have assertion of the theorem.

In fact, we can construct the *-isomorphism as follows:

$$\varphi(\varphi_{m^k}(x)\otimes y)=\varphi_{m^{k+n}}(x\otimes y),$$

for $x\in M_{m^k}$, $y\in M_{m^n}$, $k=1, 2, \cdots$. It is easy to prove that φ can be extended to a *-isomorphism from $\mathscr{M}^m\otimes M_{m^n}$ onto $\mathscr{M}^m$.

The following is a property of tensor product of C^*-algebras which is used in this paper (Ref. § 3).

Proposition 2. 7. *Let A and B be unital C^*-algebras. π_A, π_B are faithful representations of A and B on $\mathfrak{H}_A$ and $\mathfrak{H}_B$ respectively. Then*

$$\|u\|_{\min}=\|(\pi_A\otimes\pi_B)\|,u\in A\otimes B,$$

where the right hand side is the operator norm over $\mathfrak{H}_A\otimes\mathfrak{H}_B$, and

$$A\otimes_{\min}B\cong\pi_A(A)\otimes\pi_B(B),$$

where the right hand side is the C^*-algebra generated by operator tensor product.

Proof The first assertion is the Theorem 2.3.9 in [3], the second follows from the structure of tensor product of C^*-algebras.

§ 3. Main Result

Definition 3. 1. *Let $\mathscr{M}^m$ be a UHF-algebra of type $\{m^p\}$, A a unital C^*-algebra,*

$$\mathscr{K}^m(A)=(\mathscr{M}^m\otimes_{\min}A)_1^+.$$

We call $\mathcal{K}^m(A)$ the m-hyperfinite building block of C-algebra A.*

Definition 3. 2. *Let $SU(\mathcal{M}^m)$ be the special unitary group of UHF-algebra $\mathcal{M}^m$ of type $\{m^p\}$. We define a group action on $\mathcal{K}^m(A)$:*

$$\alpha^m: SU(\mathcal{M}^m)\to Aff(\mathcal{K}^m(A)),$$

such that

$$\alpha^m(u)(x)=(u\otimes 1_A)x(u^*\otimes 1_A),\ u\in SU(\mathcal{M}^m),\ x\in\mathcal{K}^m(A),$$

where 1_A is the unit of C-algebra A, $Aff(\mathcal{K}^m(A))$ is the affine automorphism group of $\mathcal{K}^m(A)$.*

Remark 3. 3. The definitions 3.1 and 3.2 are the generalization of n-finite building block $\mathcal{K}_n(A)$ and α_n in [6]. This action gives an orientation on $\mathcal{K}^m(A)$.

Definition 3. 4. *Let A and B be C*-algebras, Φ: $A\to B$ a linear map. We call Φ a Jordan homomorphism iff*

(1) $\Phi(a^*)=\Phi(a)^*$,

(2) $\Phi(\{a, b\})=\{\Phi(a), \Phi(b)\}$,

where $\{a, b\}=ab+ba$, $a\in A$, $b\in A$.

Proposition 3. 5. *Let $\mathcal{M}$ be a UHF-algebra, A and B unital C*-algebras. If Φ: $\mathcal{M}\otimes_{\min}A\to\mathcal{M}\otimes_{\min}B$ is a Jordan isomorphism such that $\Phi\circ\alpha=\alpha\circ\Phi$, let $A=1_{\mathcal{M}}\otimes A$, $B=1_{\mathcal{M}}\otimes B$, then $\Phi(A)=B$ and Φ is a Jordan isomorphism from A to B.*

Proof Since $\mathcal{M}$ is simple[1, Theorem 5.1], every nonzero representation is faithful. Let ω be a factor state (i. e. a product state) on $\mathcal{M}$[4], which induces a factor representation $(\pi_\omega, \mathfrak{H}_\omega)$, i. e., $\pi_\omega(\mathcal{M})''$ is a factor in $\mathcal{B}(\mathfrak{H}_\omega)$, $(\pi_A, \mathfrak{H}_A)$ and $(\pi_B, {}_B\mathfrak{H})$ the universal representations of A and B respectively. Since π_ω, π_A, π_B are faithful, we have

$$\mathcal{M}\otimes_{\min}A\overset{\pi_\omega\otimes\pi_A}{\cong}\pi_\omega(\mathcal{M})\otimes\pi_A(A),$$

$$\mathcal{M}\otimes_{\min}B\overset{\pi_\omega\otimes\pi_B}{\cong}\pi_\omega(\mathcal{M})\otimes\pi_B(B),$$

which follow from Proposition 2.7.

Let

$$\pi_\omega(\alpha)(u)=(\pi_\omega\otimes\pi_A(\text{or }\pi_B))\circ\alpha(\pi_\omega^{-1}(u))\circ(\pi_\omega\otimes\pi_A(\text{or }\pi_B))^{-1},$$

$u\in SU(\mathcal{M})$. Then

$$\pi_\omega(\alpha)(u)(x)=(u\otimes 1_{\pi_A(\text{or }\pi_B)})x(u^*\otimes 1_{\pi_A(\text{or }\pi_B)}),$$

$x\in\pi_\omega(\mathcal{M})\otimes\pi_A(A)$ (or $\pi_B(B)$). Define $\Psi=(\pi_\omega\otimes\pi_B)\circ\Phi\circ(\pi_\omega\otimes\pi_A)^{-1}$. We have following observations.

Observation 1. Ψ is a π_ω (α)-invarint Jordan isomorphism.

Clearly, Ψ is a Jordan isomorphism, the invarince of Ψ is from the following.

$$\begin{aligned}\Psi\circ\pi_\omega(\alpha)(u)&=(\pi_\omega\otimes\pi_B)\circ\Phi\circ(\pi_\omega\otimes\pi_A)^{-1}\circ(\pi_\omega\otimes\pi_A)\circ\alpha(\pi_\omega^{-1}(u))\circ(\pi_\omega\otimes\pi_A)^{-1}\\&=(\pi_\omega\otimes\pi_B)\circ\alpha(\pi_\omega^{-1}(u))\circ\Phi\circ(\pi_\omega\otimes\pi_A)^{-1}=\pi_\omega(\alpha)(u)\circ\Psi.\end{aligned}$$

Observation 2. Let $A=1_{\mathcal{M}}\otimes A$, $\mathcal{L}=\mathcal{M}\otimes 1_A$, $B=1_{\mathcal{M}}\otimes B$, etc. Then $\Psi(\pi_A(A))=\pi_B(B)$.

Indeed, for every $x \in A$, $u \in SU(\mathscr{M})$,

$$x = (u \otimes 1_A)(1_{\mathscr{M}} \otimes x)(u^* \otimes 1_A).$$

Then

$$\pi_A(x) = (\pi_\omega \otimes 1_{\pi_A(A)})(1_{\pi_\omega(\mathscr{M})} \otimes \pi_A(x))(\pi_\omega(u^*) \otimes 1_{\pi_Y(A)}).$$

Following the invarince of Ψ, we have

$$\Psi(\pi_A(x)) = (\pi_\omega(u) \otimes 1_{\pi_B(B)}) \Psi(\pi_A(x))(\pi_\omega(u^*) \otimes 1_{\pi_B(B)}).$$

Therefore

$$\Psi(\pi_A(x)) \in ((\pi_\omega(SU(\mathscr{M})) \otimes 1_{\pi_B B}))' \cap (\pi_\omega(\mathscr{M}) \otimes \pi_B(B)).$$

By Commutant theorem[3], we have

$$\Psi(\pi_A(x)) \in \pi_B(B).$$

Hence

$$\Psi(\pi_A(A)) \subseteq \pi_B(B).$$

Considering Ψ^{-1}, we get

$$\Psi^{-1}(\pi_B(B)) \subseteq \pi_A(A),$$

which completes the observation.

Observation 3. Let $\Phi' = \pi_B^{-1} \circ \Psi \circ \pi_A$. Then it is a Jordan isomorphism and $\Phi' = \Phi$.

The first conclusion follows from the above discussion. The second is verified as following

$$\begin{aligned}\Phi'(x) &= (\pi_B^{-1} \circ \Psi \circ \pi_A)(x) = (\pi_\omega \otimes \pi_B)^{-1} \circ \Psi \circ (\pi_\omega \otimes \pi_A)(1_{\mathscr{M}} \otimes x) \\ &= \Phi(1_{\mathscr{M}} \otimes x) = \Phi(x).\end{aligned}$$

Following the three observations, we complete the proof of the theorem.

Now we prove the main theorem in this paper.

Theorem 3. 6. *Let $\mathscr{M}^m$ be a UHF-algebra of type $\{m^p\}$, A, B two unital C^*-algebras, $\mathscr{K}^m(a)$ and $A^m(B)$ the hyperfinite buiding blocks of A and B and α^m the action of $SU(\mathscr{M}^m)$ on $\mathscr{K}^m(A)$ and $\mathscr{K}^m(B)$. If Φ: $\mathscr{K}^m(A) \to \mathscr{K}^m(B)$ is an α^m-invariant affine isomorphism and $\Phi(0)=0$, then A and B are *-isomorphic.*

Proof By [7, Lemma 2.2], Φ can be extended as an α^m-invariant positive Jordan isomorphism from $\mathscr{M}^m \otimes_{\min} A$ to $\mathscr{M}^m \otimes_{\min} B$. Following Theorem 2.6, we have:

(1) $$\mathscr{M}^m \otimes_{\min} M_{m^p} \overset{\varphi}{\cong} \mathscr{M}^m,$$

(2) $$\mathscr{M}^m \otimes_{\min} M_{m^p} \otimes_{\min} A \overset{\varphi \otimes \mathrm{id}_A}{\cong} \mathscr{M}^m \otimes_{\min} A,$$

$$\mathscr{M}^m \otimes_{\min} M_{m^p} \otimes_{\min} B \overset{\phi \otimes \mathrm{id}_A}{\cong} \mathscr{M}^m \otimes_{\min} B,$$

where φ is defined by $\varphi(\varphi_{m^k}(x) \otimes y) = \varphi_{m^{k+p}}(x \otimes y)$, for all $x \in M_{m^k}$, $y \in M_{m^p}$. From (2), we define

$$\Psi = (\varphi \otimes \mathrm{id}_B)^{-1} \circ \Phi \circ (\varphi \otimes \mathrm{id}_A),$$

$$A_1 = M_{m^p} \otimes_{\min} A,$$

$$B_1 = M_{m^p} \otimes_{\min} B.$$

Clearly, Ψ is a positive Jordan isomorphism.

Let α_1 be the inner action of $SU(\mathscr{M}^m)$ on $\mathscr{M}^m\otimes A_1$ and $\mathscr{M}^m\otimes B_1$, i. e.,

$$\alpha_1:\ SU(\mathscr{M}^m)\to \mathrm{Int}(\mathscr{M}^m\otimes_{\min}A_1),$$
$$u\mapsto (u\otimes \mathrm{id}_{A_1})(\cdot)(u^*\otimes \mathrm{id}_{A_1}),$$
$$SU(\mathscr{M}^m)\to \mathrm{Int}(\mathscr{M}^m\otimes_{\min}B_1),$$
$$u\mapsto (u\otimes \mathrm{id}_{B_1})(\cdot)(u^*\otimes \mathrm{id}_{B_1}),$$

where $\mathrm{Int}(\cdot)$ denotes the inner automorphism group.

Now we have Ψ is α_1-invariant. Indeed, for every $u\in SU(\mathscr{M}^m)$, $x\in M_{m^k}$, $y\in M_{m^p}$, $a\in A$, there exists $q\geq k$ such that $\varphi_{m^q}^{-1}(u)\in M_{m^q}$ and

$$\begin{aligned}
\Psi\{\alpha_1(u)[\varphi_{m^k}(x)\otimes y\otimes a]\} &= \Psi\{[u\varphi_{m^k}(x)u^*]\otimes y\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_A)^{-1}\circ\Phi\circ(\varphi\otimes \mathrm{id}_A)\{[u\varphi_{m^k}(x)u^*]\otimes y\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\{\varphi[(u\varphi_{m^k}(x)u^*)\otimes y]\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_A)^{-1}\circ\Phi\{\varphi_{m^{q+p}}[\varphi_{m^q}^{-1}(u\varphi_{m^k}(x)u^*)\otimes y]\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\{\varphi_{m^{q+p}}[(\varphi_{m^q}^{-1}(u)\otimes 1_{M_{m^p}})\\
&\quad\times(\varphi_{m^q}^{-1}(\varphi_{m^k}(x))\otimes y)(\varphi_{m^q}^{-1}(u^*)\otimes 1_{M_{m^p}})]\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\{(\varphi_{m^{q+p}}[(\varphi_{m^q}^{-1}(u)\otimes 1_{M_{m^p}}]\varphi_{m^{q+p}}\\
&\quad\times[\varphi_{m^q}^{-1}(\varphi_{m^k}(x))\otimes y]\varphi_{m^{q+p}}[\varphi_{m^q}^{-1}(u^*)\otimes 1_{M_{m^p}}])\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\{(\varphi_{m^{q+p}}[(\varphi_{m^q}^{-1}(u)\otimes 1_{M_{m^p}}]\otimes 1_B)(\Phi\\
&\quad\times[\varphi_{m^{q+p}}(\varphi_{m^q}^{-1}((\varphi_{m^k}(x))\otimes y)\otimes a])\\
&\quad\times(\varphi_{m^{q+p}}[\varphi_{m^q}^{-1}(u^*)\otimes 1_{M_{m^p}}]\otimes 1_B)\}\\
&= \{(\varphi\otimes \mathrm{id}_B)^{-1}(\varphi_{m^{q+p}}[(\varphi_{m^q}^{-1}(u)\otimes 1_{M_{m^p}}]\otimes 1_B)\}\\
&\quad\times\{(\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi[\varphi_{m^{q+p}}(\varphi_{m^q}^{-1}((\varphi_{m^k}(x))\otimes y)\otimes a]\}\\
&\quad\times\{(\varphi\otimes \mathrm{id}_B)^{-1}(\varphi_{m^{q+p}}[\varphi_{m^q}^{-1}(u^*)\otimes 1_{M_{m^p}}]\otimes 1_B)\}\\
&= (u\otimes 1_{B_1})\{(\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\circ(\varphi\otimes \mathrm{id}_B)\\
&\quad\times[\varphi_{m^k}(x)\otimes y\otimes a]\}(u^*\otimes 1_{B_1})\\
&= \alpha_1(u)\circ\Psi[\varphi_{m^k}(x)\otimes y\otimes a],
\end{aligned}$$

where the seventh equality is obtained from the α^m-invariancity and $1_{M_{m^p}}$, 1_A, 1_B, 1_{A_1}, 1_{B_1} are the identities in M_{m^p}, A, B, A_1, B_1 respectively.

From Theorem 4.2, let $A_1=1_{\mathscr{M}^m}\otimes A_1$, $B_1=1_{\mathscr{M}^m}\otimes B_1$, then $\Psi(A_1)=B_1$ and Ψ is a Jordan isomorphism. Following [6], let α_{m^p} be the inner action of $SU(m^p)$ on A_1 and B_1, then Ψ is communicative with the action α_{m^p}. Indeed, for every $u\in SU(m^p)$, $x\in M_{m^p}$, $a\in A$, we have

$$\begin{aligned}
&\Psi(\alpha_{m^p}(u)(x\otimes a))\\
&= \Psi(1_{\mathscr{M}^m}\otimes uxu^*\otimes a) = (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\circ(\varphi\otimes \mathrm{id}_A)(1_{\mathscr{M}^m}\otimes uxu^*\otimes a)\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi[\varphi(1_{\mathscr{M}^m}\otimes uxu^*)\otimes a] = (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes uxu^*)\otimes a]\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\{\varphi_{m^{k+p}}[(1_{M_{m^k}}\otimes u)(1_{M_{m^k}}\otimes x)(1_{M_{m^k}}\otimes u^*)]\otimes a\}\\
&= (\varphi\otimes \mathrm{id}_B)^{-1}\circ\Phi\{[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u)\otimes 1_A][\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes x)\otimes a]\\
&\quad[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u^*)\otimes 1_A]\}
\end{aligned}$$

$$= (\varphi\otimes\mathrm{id}_B)^{-1}\{[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u)\otimes 1_B]\Phi[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes x)\otimes b] \times [\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u^*)\otimes 1_B]\}$$
$$= \{(\varphi\otimes\mathrm{id}_B)^{-1}[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u)\otimes 1_B]\}\{(\varphi\otimes\mathrm{id}_B)^{-1}\circ\Phi[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes x)\otimes a]\} \times\{(\varphi\otimes\mathrm{id}_B)^{-1}[\varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u^*)\otimes 1_B]\}$$
$$= (1_{\mathscr{A}^m}\otimes u\otimes 1_B)\Psi(1_{\mathscr{A}^m}\otimes x\otimes a)(1_{\mathscr{A}^m}\otimes u^*\otimes 1_B) = \alpha_{m^p}(u)\circ\Psi(x\otimes a),$$

where the first equality is derived by the definition of the action α_{m^k} and $M_{m^p}\otimes A \cong 1_{\mathscr{A}^m}\otimes M_{m^p}\otimes A$, the second by the definition of Ψ, the third, fourth, fifth, and sixth by the definition of φ and the fact that $\varphi_{m^{k+p}}$ is a *-embedding, the seventh by the α^m-invariance of Φ, the eighth by the fact that $\varphi\otimes\mathrm{id}_B$ is a *-isomorphism, the ninth by

$$\varphi(1_{\mathscr{A}^m}\otimes u) = \varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u),$$
$$\varphi(1_{\mathscr{A}^m}\otimes u^*) = \varphi_{m^{k+p}}(1_{M_{m^k}}\otimes u^*)$$

and the tenth by the definition of α_{m^p} and

$$M_{m^p}\otimes A = 1_{\mathscr{A}^m}\otimes M_{m^p}\otimes A,$$
$$M_{m^p}\otimes B = 1_{\mathscr{A}^m}\otimes M_{m^p}\otimes B.$$

Let $p>1$, then $m^p>2$. By [6, Theorem 3.1], $\Psi(A)=B$ and Ψ is a *-isomorphism.

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