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LIFE SPAN OF CLASSICAL SOLUTIONS TO $u_{tt}-u_{xx}=|u|^{1+\alpha}$

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Abstract

This paper studies the life span of classical solutions to

$$u_{tt}-u_{xx}=|u|^{1+\alpha},$$
$$t=0:\ u=\varepsilon f(x),\ u_t=\varepsilon g(x),$$

where α is a positive real number, $f\in C_0^2(R)$, $g\in C_0^1(R)$, ε is a small parameter. The upper and lower bounds of the same order of magnitude for the life span are obtained respectively.

§ I. Introduction

In this paper, we consider the following Cauchy problem

$$u_{tt}-u_{xx}=|u|^{1+\alpha},\ t>0,\ x\in R, \tag{1.1}$$
$$t=0:\ u=\varepsilon f(x),\ u=\varepsilon g(x), \tag{1.2}$$

where $\alpha>0$ is a real number, $f\in C_0^2(R)$ $g\in C_0^1(R)$, $1\geqslant\varepsilon>0$ is a small parameter.

We are interested in estimating the life-span of classical solutions to (1.1)—(1.2). By definition, the life span $T(\varepsilon)$ is sup τ, for all $\tau>0$ such that there exists a classical solution to (1.1)—(1.2) on the time interval $[0,\ \tau]$. We summarize our results as follows:

Theorem 1.1. *Suppose $g(x)$ in* (1.2) satisfies

$$\int g(x)dx\neq 0. \tag{1.3}$$

Then there exists an $\varepsilon_1>0$ such that for any ε with $0<\varepsilon\leqslant\varepsilon_1$

$$\varkappa_1\varepsilon^{-\alpha/2}\leqslant T(\varepsilon)\leqslant\varkappa_2\varepsilon^{-\alpha/2}, \tag{1.4}$$

where $\varkappa_1$ and $\varkappa_2$ are positive constants independent of ε.

Theorem 1.2. *Suppose $g(x)$ in (1.2) satisfies*

$$\int g(x)dx=0 \tag{1.5}$$

and f and g are not both identically zero. Then there exists an $\varepsilon_2>0$ such that for any ε with $0<\varepsilon\leqslant\varepsilon_2$

$$\varkappa_3\varepsilon^{-\alpha(\alpha+1)/(\alpha+2)}\leqslant T(\varepsilon)\leqslant\varkappa_4\varepsilon^{-\alpha(\alpha+1)/(\alpha+2)}, \tag{1.6}$$

where x_3 and x_4 are positive constants in dependent of ε.

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The study for the equation

$$\Box u = |u|^{1+\alpha} \tag{1.7}$$

begins with the work of F. John[1]. He considered the three dimensional case. Later, T. Kato[2], R. T. Glassey[3] and T. C. Sideris[6] got some blow up results for (1.7) for any space dimensions, but they all assume some positivity condition on the initial data. Therefore, to estimate the life span of classical solutions to (1.1)—(1.2) in its correct order of magnitude for any initial data, more work is needed.

After the completion of this work, we received a preprint copy of H. Lindblad's paper[5] which studied the same problem as ours for the case of $n=1, 2, 3$ simultaneously. To compare his results and methods with ours when $n=1$, we notice that we solved the problem for any positive α while he onlyconsidered the case $\alpha=1$ although his methods seemed to work for all positive α as well. We only proved $\varepsilon^{1/2} T(\varepsilon)$ lies between two positive bounds while he actually calculated the limit $\lim_{\varepsilon\to 0}\varepsilon^{1/2}T(\varepsilon)$ in the case $\alpha=1$ and $\int g\neq 0$, the situation $\int g=0$ is similar. Our methods are somewhat similar to his but more elementary and simple. There is also a significant difference in treating with the case $\int g=0$ between two methods.

§ 2. Lower Bound of Life Span in Theorem 1.1

Theorem 2.1. *There exists a positive constant $\varkappa_1$ independent of ε such that*

$$T(\varepsilon)\geqslant \varkappa_1 \varepsilon^{-\alpha/2}. \tag{2.1}$$

Proof No loss of generality we assume f and g are not both identically zero. Bo means of D' Alembert's formula it is easy to see that the C^2 solution to (1.1)—(1.2) is equivalent to the c^0 solution of the following intergral equation

$$u(t, x) = \varepsilon u^0(t, x) + \frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau} |u(\tau, y)|^{1+\alpha} dy d\tau \tag{2.2}$$

where

$$u^0(t, x) = [f(x+t)+f(x-t)]/2 + \int_{x-t}^{x+t} g(\beta)d\beta/2.$$

For any $v\in E$, define

$$\Phi(v)(t, x) = \varepsilon u^0(t, x) + \frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau} |v(\tau, y)|^{1+\alpha} dy d\tau, \tag{2.3}$$

$$E=\{v(t, x)\,|\,v\in C^0(R^+\times R),\ t\leqslant \tau_1\varepsilon^{-\alpha/2}\ \sup|v(t, x|\leqslant \varepsilon M_1\} \tag{2.4}$$

with

$$M_1 = 2\sup|f(\beta)| + \int_{-\infty}^{+\infty}|g(\beta)d\beta \tag{2.5}$$

and

$$\varkappa_1 = M_1^{-\alpha/2}/\sqrt{1+\alpha}. \tag{2.6}$$

It is easy to see that Φ maps E into itself and is a contraction mapping. So by contraction mapping principle
Theorem 2.1 is proved.

§ 3. Upper Bound of Life Span in Theorem 1.1: Case $\int g>0$

Theorem 3.1. *Suppose $g(x)$ in (1.2) satisfies*

$$\int_{-\infty}^{+\infty} g(x)dx>0. \tag{3.1}$$

Then there exists a positive constant c_1 indep endent of ε such that

$$T(\varepsilon)\leqslant c_1\varepsilon^{-\alpha/2}. \tag{3.2}$$

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *The classical solution to the Goursat problem of (1.1) with boundary conditions*

$$\begin{cases} x=t; & u=1, \\ x=-t; & u=1 \end{cases} \tag{3.3}$$

must blow up in finite time.

Proof We shall deduce a contradiction by assuming thatou is a global solution. Let

$$v(t)=\int_{-t}^{t} u(t, x)dx. \tag{3.4}$$

It is easy to see that

$$v(0)=0, \tag{3.5}$$

$$v'(0)=2 \tag{3.6}$$

and by (1.1) we get

$$v''(t)=\int_{-t}^{t} |u|^{1+\alpha}+2\frac{d}{dt}(u(t, t)+u(t,-t)). \tag{3.7}$$

Noticing (3.3), (3.6) and (3.7), we have

$$v'(t)\geqslant 2 \quad \forall t\geqslant 0. \tag{3.8}$$

Then by (3.5) we obtain

$$v(t)\geqslant 2t \quad \forall t\geqslant 0. \tag{3.9}$$

According to Hölder's inequality, it follows from (3.4) that

$$v(t)\leqslant\left(\int_{-t}^{t} |u|^{1+\alpha})^{1/(1+\alpha)}\right)(2t)^{\alpha/(1+\alpha)}. \tag{3.10}$$

So (3.7) together wth (3.3) yields

$$v''(t)\geqslant v(t)^{1+\alpha}/(2t)^{\alpha}. \tag{3.11}$$

It is easy to deduce from (3.11) and (3.8)—(3.9) that

$$\frac{d}{dt}(v'^2(t)-2v^{2+\alpha}(t)/(\alpha+2)(2t)^{\alpha})>0. \tag{3.12}$$

Noticing that $v(t)/2t\to 1$ when $t\to 0$, we have

$$\lim_{t\to 0}(v'^2(t)-2v^{2+\alpha}(t)/(\alpha+2)(2t)^{\alpha})=4. \tag{3.13}$$

Hence it is easy to see that

$$v'(t) > \sqrt{2}\, v^{1+\alpha/2}(t)/\sqrt{\alpha+2}(2t)^{\alpha/}\cdot \forall t\geq 0. \tag{3.14}$$

In case $0<\alpha\leq 1$, we get from (3.14)

$$v(t) > v(1)/[1-\nu v(1)^{\alpha/2}(t^{1-\alpha/2}-1)]^{\alpha/2},\ \forall t\geq 1, \tag{3.15}$$

where

$$\nu=\sqrt{2}\,\alpha/2^{\alpha/2}(2-\alpha)\,\sqrt{2+\alpha}. \tag{3.16}$$

Thus v must blow up in finite time.

In case $\alpha>1$, we have from (3.9)

$$v(t)^{(\alpha-1)/2}\geq (2t)^{(\alpha-1)/2}. \tag{3.17}$$

Then we get from (3.14)

$$v'(t) > \sqrt{2}\, v^{2/2}(t)/\sqrt{2+\alpha}(2t)^{1/2}. \tag{3.18}$$

which has been reduced to the case $\alpha=1$, so v also blows up in finite time.

This contradiction proves Lemma 3.2.

Lemma 3.3. *Let $\tau(\varkappa)$ be the life span of classical solutions to the following Goursat problem*

$$\begin{cases} w_{tt}-w_{xx}=|w|^{1+\alpha}, \\ x=t:\ w=\varkappa, \\ x=-t:\ w=\varkappa, \end{cases} \tag{3.19}$$

where $\varkappa$ is a positive constant.

Then we have

$$\tau(\varkappa)=T_0\varkappa^{-\alpha/2}, \tag{3.20}$$

where T_0 is the life span of classical solution to Goursat problem (1.1), (3.3) *which is independent of $\varkappa$.*

Proof Let

$$u(t,x)=w(t/\varkappa^{\alpha/2},\ x/\varkappa^{\alpha/2})/\varkappa. \tag{3.21}$$

It is easy to see that u satisfies (1.1), (3.3). The conclusion of Lemma 3.3 is then obvious.

Lemma 3.4. *Let $\Omega=\{(\xi,\eta)\,|\,\xi\geq 0,\ \eta\geq 0,\ \xi+\eta<\alpha_0\}$ and $u(\xi,\eta)$, $w(\xi,\eta)\in C(\Omega)$ satisfy respectively*

$$u(\xi,\eta) > \delta+\int_0^\xi\int_0^\eta |u(\xi',\eta')|^{1+\alpha}d\xi'd\eta'/2,\ \forall(\xi,\eta)\in\Omega, \tag{3.22}$$

$$w(\xi,\eta) = \delta+\int_0^\xi\int_0^\eta |w(\xi',\eta')|^{1+\alpha}d\xi'd\eta'/2,\ \forall(\xi,\eta)\in\Omega, \tag{3.23}$$

where δ is a positive constant.

Then, we have

$$u(\xi,\eta) > w(\xi,\eta),\ \forall(\xi,\eta)\in\Omega. \tag{3.24}$$

Proof We deduce a contradiction by assuming that (3.24) does not hold for all $(\xi,\eta)\in\Omega$. In this aase

$$\Lambda=\{(\xi,\eta)\,|\,(\xi,\eta)\in\Omega,\ u(\xi,\eta)\leq w(\xi,\eta)\} \tag{3.25}$$

is a nonempty closed set in Ω. The refore there is a point $(\xi_0,\eta_0)\in\Lambda$ which is closest to the origin. By (3.22), (3.23), it is obvious that $\xi_0\neq 0$ and $\eta_0\neq 0$; thus for all (ξ, η) such that $0 \leqslant \xi < \xi_0$, $0 \leqslant \eta < \eta_0$ (3.24) holds.

Noticing that u, v are both positive, we obtain

$$\int_0^{\xi_0}\int_0^{\eta_0} |u(\xi', \eta')|^{1+\alpha} d\xi' d\eta' > \int_0^{\xi_0}\int_0^{\eta_0} |w(\xi', \eta')|^{1+\alpha} d\xi' \eta'. \tag{3.26}$$

This together with (3.22) (3.23) implies

$$u(\xi_0, \eta_0) > w(\xi_0, \eta_0), \tag{3.27}$$

which contradicts the fact that $(\xi_0, \eta_0) \in A$. The lemma is proved.

We introduce the following notations:

$$K^+(t, x) = \{(\tau, y) \mid |y-x| \leqslant \tau - t,\ \max(0, t) \leqslant \tau\}, \tag{3.28}$$

$$K^-(t, x) = \{(\tau, y) \mid |y-x| \leqslant t - \tau,\ 0 \leqslant \tau \leqslant t\}, \tag{3.29}$$

which are nothing but the forward and backward light cones passing through (t, x) restricted on the upper plane respectively.

Proof of Theorem 3.1 We suppose that the supports of both f and g lie in the interval $(-\rho, \rho)$. Let

$$M = \frac{1}{2}\int_{\rho}^{\rho} -g(x)dx/2. \tag{3.30}$$

By assumption (3.1) $M > 0$.

For all $(t, x) \in K^+(\rho, 0)$, D'Alembert's formula yields

$$u(t, x) = \varepsilon M + \iint_{K^-(t,x)} |u(\tau, y)|^{1+\alpha} d\tau dy/2. \tag{3.31}$$

Then it is easy to see that

$$u(t, x) > \varepsilon M + \iint_{K^+(\rho,0) \cap K^-(t,x)} |u(\tau, y)|^{1+\alpha} d\tau\, dy/2. \tag{3.32}$$

Let $w(t, x)$ be a function defined on $K^+(\rho, 0)$ such that

$$w(t, x) = \varepsilon M + \iint_{K^-(t,x) \cap K^+(\rho,0)} |w(\tau, y)|^{1+\alpha} d\tau\, dy/2. \tag{3.33}$$

By means of a change of coordinates

$$\begin{cases} \xi = (t - x - \rho)/\sqrt{2}, \\ \eta = (t + x - \rho)/\sqrt{2}. \end{cases} \tag{3.34}$$

We can apply Lemma 3.4 to yield, whenever u and w both exist,

$$u(t, x) > w(t, x). \tag{3.35}$$

On the other hand, it is easy to see by the change of coordinate (3.34) that w is the solution to the following Goursat problem

$$\begin{cases} w_{tt} - w_{xx} = |w|^{1+\alpha}, & t \geqslant \rho, \\ x = t - \rho: & w = \varepsilon M, \\ x = -t + \rho: & w = \varepsilon M. \end{cases} \tag{3.36}$$

Then Lemma 3.3 together with a shifting of time shows that w blows up at time $\rho + T_0 M^{-\alpha/2} \varepsilon^{-\alpha/2}$. Therefore it follows from (3.35) that

$$T(\varepsilon)\leqslant(\rho+T_0M^{-\alpha/2})\varepsilon^{-\alpha/2}.$$

Thus Theorem 3.1 is proved.

§ 4. Upper Bound of Life Span In Theorem 1.1: Case $\int g<0$

Theorem 4.1. *Suppose g in (1.2) satisfies*

$$\int_{-\infty}^{+\infty} g(x)\,dx\leqslant 0. \tag{4.1}$$

Then there exists an $\varepsilon_1>0$ such that for any ε with $0<\varepsilon\leqslant\varepsilon_1$

$$T(\varepsilon)\leqslant c_2\varepsilon^{-\alpha/2}, \tag{4.2}$$

where c_2 is a positive constant independent of ε.

To prove Theorem 4.1, we need several lemmas:

Lemma 4.2. *Cauchy problem (1.1), (1.2) admits a classical solution $u(t, x)$ on the domain*

$$D=\{(t, x)\mid(t, x)\notin K^+(\rho, 0),\ 0\leqslant t\leqslant b\varepsilon^{-\alpha}\} \tag{4.3}$$

with

$$b=M_1^{-\alpha}/2(1+\alpha)\rho. \tag{4.4}$$

Moreover $|u(t, x)|\leqslant\varepsilon M_1,\ \forall(t, x)\in D,$

where M_1 is defined by (2.5).

Proof By finite propogation speed of waves we know that u vanshes outside $K^+(-\rho, 0)$. So by D'Alembert's formula the C^2 solution to (1.1) (1.2) outside $K^+(\rho, 0)$ is equivalent to the C^0 solution of the intergral equation

$$u(t, x)=\varepsilon\, u^0(t, x)+\frac{1}{2}\iint_{K^-(t,x)\cap K^+(-\rho,0)}|u(\tau, y)|^{1+\alpha}d\tau dy. \tag{4.5}$$

Define

$$\phi_1(v)(t, x)=\varepsilon u^0(t, x)+\iint_{K^-(t,x)\cap K^+(-\rho,0)}|v(\tau, y)|^{1+\alpha}d\tau\, dy/2 \tag{4.6}$$

for any $v\in E_1$, where

$$E_1=\{v(t, x)\mid v(t, x)\in C(D),\ \sup|v(t, x)|\leqslant\varepsilon M_1\}. \tag{4.7}$$

As in the proof of Theorem 2.1, we can prove ϕ_1 is a contraction mapping from E_1 into itself. Thus, by contraction mapping principle, Lemma 4.2 is proved.

Lemma 4.3. *The Goursat problem*

$$\begin{cases} u_{tt}-u_{xx}=|u|^{1+\alpha},\\ x=t: u=-c,\\ x=-t: u=-c \end{cases} \tag{4.8}$$

is equivalent to the following problem for ordinary differential equations:

$$\begin{cases} \tau v''(\tau)+v'(\tau)=\tau|x(\tau)|^{1+\alpha},\ \tau>0,\\ v(0)=-c. \end{cases} \tag{4.9}$$

so they have the same life Span. Here c is a constant.

Proof Let u be the solution of (4.8). By the Lorentz invariance of the equation and boundary conditions and noticing the uniquness of solutions to (4.8), we conclude that there exists a function v such that

$$u(t, x)=v((t^2-x^2)^{1/2}). \tag{4.10}$$

From (4.10) we derive

$$v(\tau)=u(\tau, 0), \tag{4.11}$$

so v is a C^2 function. Insert (4.10) to (4.8), we get (4.9).

Conversely, if v is the solution of (4.9), then u defined by (4.10) solves (4.8).

Lemma 4.4. *If $c>0$, then the solution of (4.8) must blow up in finite time.*

Proof We deduce a contradiction by assuming u is a global solution of (4.8). Define v by (4.11). By Lemma 4.3, v satisfies (4.9). Integrate (4.9), we get

$$\lambda v'(\lambda)=\int_0^\lambda \tau|v(\tau)|^{1+\alpha}d\tau. \tag{4.12}$$

Noticing $v(0)=-c<0$, we see that there exists a $\lambda_0>0$ such that

$$v(\lambda)<-c/2,\ \forall 0<\lambda<\lambda_0. \tag{4.13}$$

So we get from (4.12) that when $\lambda\geqslant\lambda_0$

$$\lambda v'(\lambda)\geqslant\lambda_0^2(c/2)^{1+\alpha}/2. \tag{4.14}$$

Then

$$v(\lambda)\geqslant v(\lambda_0)+\lambda_0^2(c/2)^{1+\alpha}(\ln\lambda-\ln\lambda_0)/2. \tag{4.15}$$

Thus v becomes positive when λ is big enough. Let λ_1 be such that

$$u(\lambda_1, 0)=v(\lambda_1)>0. \tag{4.16}$$

For $(t, x)\in K^+(\lambda_1, 0)$, it is easy to see that

$$\begin{aligned} u(t, x)&=-c+\iint\limits_{K^+(t,x)\cap K^+(0,0)}|u(\tau, y)|^{1+\alpha}d\tau\, dy \\ &\geqslant -c+\iint\limits_{K^-(\lambda_1,0)\cap K^+(0,0)}|u(\tau, x)|^{1+\alpha}d\tau\, dy+\iint\limits_{K^+(\lambda_1,0)\cap K^-(t,x)}|u(\tau, y)|^{1+\alpha}d\lambda\, dy \\ &=u_1(\lambda_1, 0)+\iint\limits_{K^+(\lambda_1,0)\cap K^-(t,x)}|u(\tau, y)|^{1+\alpha}d\tau\, dy \\ &>u_1(\lambda_1, 0)/2+\iint\limits_{K^+(\lambda_1,0)\cap K^-(t,x)}|u(\tau, y)|^{1+\alpha}d\tau\, dy. \end{aligned} \tag{4.17}$$

By Lemma 3.3 and Lemma 3.4. u must blow up in finite time, which contradicts our assumption. The lemma is then proved.

Now, we return to Cauchy problem (1.1) (1.2). Define

$$c=-g(x)dx/2. \tag{4.18}$$

We have

$$u(t, x)=-\varepsilon c+\iint\limits_{K^-(t,x)}|u(\tau, y)|^{1+\alpha}d\tau\, dy/2,\ \forall(t, x)\in K^+(\rho, 0). \tag{4.19}$$

(4.19) can be rewritten as

$$u(t, x)=-\varepsilon c+\iint\limits_{K^-(t,x)\cap K^+(\rho,0)}|u(\tau, y)|^{1+\alpha}d\tau\, dy/2+H(t, x, \varepsilon), \tag{4.20}$$

where

$$H(t, x, \varepsilon)=\frac{1}{2}\left(\iint_{R_1(t,x)}+\iint_{R_2(t,x)}+\iint_{K^-(\rho,0)}\right)|u(\tau, y)|^{1+\alpha}d\tau\, dy$$

$$=I_1(t, x, \varepsilon)+I_2(t, x, \varepsilon)+I_3(\varepsilon) \tag{4.21}$$

with

$$R_1(t, x)=\{(\tau, y)\,|-\rho\leqslant y-\tau\leqslant\rho,\ \rho\leqslant y+\tau\leqslant t+x\}, \tag{4.22}$$

$$R_2(t, x)=\{(\tau, y)\,|\,x-t\leqslant y-\tau\leqslant-\rho,\ -\rho\leqslant y+\tau\leqslant\rho\}. \tag{4.23}$$

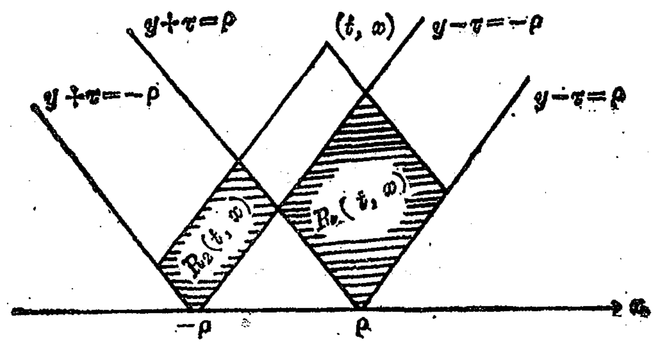

Fig. 1

According to Lemma 4.2, it is easy to see that when

$$t\leqslant b\varepsilon^{-\alpha} \tag{4.24}$$

we have

$$0\leqslant I_1(t, x, \varepsilon)\leqslant(\varepsilon M_1)^{1+\alpha}(2t-\rho)\rho/2, \tag{4.25}$$

$$0\leqslant I_2(t, x, \varepsilon)\leqslant(\varepsilon M_1)^{1+\alpha}(t-x-\rho)\rho/2, \tag{4.26}$$

$$0\leqslant I_3(\varepsilon)\leqslant\rho^2(\varepsilon M_1)^{1+\alpha}/2. \tag{4.27}$$

So we get

$$0\leqslant H(t, x, \varepsilon)\leqslant\rho(\varepsilon M_1)^{1+\alpha}(2t-\rho)/2. \tag{4.28}$$

We next estimate $\partial H(t, x, \varepsilon)/\partial t$ and $\partial H(t, x, \varepsilon)/\partial x$. In order to estimate $\partial I_1/\partial t$ and $\partial I_1/\partial x$, we make a change of coordinates

$$\begin{cases}\xi'=y-\tau-\rho,\\ \eta'=y+\tau-\rho.\end{cases} \tag{4.29}$$

It is easy to see that

$$I_1=\int_0^{2\rho}\int_0^{x+t-\rho}|u(\eta'-\xi',\ \rho+\xi'+\eta')|^{1+\alpha}d\eta' d\xi'/4. \tag{4.30}$$

Then we get

$$|\partial I_1/\partial t|,\ |\partial I_2/\partial x|\leqslant\rho(\varepsilon M_1)^{1+\alpha}/2. \tag{4.31}$$

Similar estimates hold for $\partial I_2/\partial t$ and $\partial I_2/\partial x$. Then we conclude that

$$|\partial H(t, x, \varepsilon)/\partial x|+|\partial H(t, x, \varepsilon)/\partial t|\leqslant 2\rho(\varepsilon M_1)^{1+\alpha}. \tag{4.32}$$

Now let

$$w^\varepsilon(t, x)=u(\varepsilon^{-\alpha/}t+\rho,\ \varepsilon^{-\alpha/2}x)/\varepsilon,\ \forall(t,, x)\in k^+(\rho, 0). \tag{4.33}$$

According to (6.17), it is easy to see that w^ε satisfies

$$w^\varepsilon(t, x)=-c+\widetilde{H}(t, x, \varepsilon)+\iint_{K(t,x)\cap K(0,0)}|w^\varepsilon(\tau, y)|^{1+\alpha}d\tau dy/2$$

$$\forall(t, x)\in K^+(0, 0) \tag{4.34}$$

where

$$\widetilde{H}(t, x, \varepsilon)=H(\varepsilon^{-\alpha/2}t+\rho,\varepsilon^{-\alpha/2}x, \varepsilon)/\varepsilon. \tag{4.35}$$

It follows from (4.28) and (4.23) that as long as

$$\varepsilon^{-\alpha/2}t+\rho<bt^{-\alpha} \tag{4.36}$$

we have

$$0\leqslant\widetilde{H}(t, x, \varepsilon)\leqslant\rho M_1^{1+\alpha}\varepsilon^{\alpha/2}(2t+\varepsilon^{\alpha/2}\rho)/2, \tag{4.37}$$

$$|\partial\widetilde{H}(t, x, \varepsilon)/\partial t|+|\partial\widetilde{H}(t, x, \varepsilon)/\partial x|\leqslant 2\rho M_1^{1+\alpha}\varepsilon^{\alpha/2}. \tag{4.38}$$

Let T_1 be the life span of solution of (4.8) with c be defined by(4.18). By Lemma 3.4 $T_1<+\infty$, we claim

Lemma 4.5. *There exists an $\varepsilon_1>0$ such that, for any ε with $0<\varepsilon<_1$, $w^\varepsilon(t, x$ blows up at least at time T_1+1.*

Proof We prove the lemma by contradiction. If this ε_1 does not exist, then there exist $\varepsilon_n\to 0$ (when $n\to\infty$) such that $w^n(t, x)=w^{\varepsilon n}(t, x)$ exist until time T_1+1

We choose N_0 so big that when $n>N_0$ (3.36) hold for $t=T_1+1$ and $\varepsilon=\varepsilon_n$.

We first prove that

$$w^n(T_1, 0) \text{ is unbound for } n. \tag{4.39}$$

We again prove this by contradiction. If there exists a c_0 such that

$$|w^n(T_1, 0)|\leqslant c_0, \tag{4.40}$$

then we get from (4.34)

$$w^n(T_1, 0)-w^n(t, x)\geqslant\widetilde{H}(T_1, 0, \varepsilon_n)-\widetilde{H}(t, x, \varepsilon_n),$$

$$\forall(t, x)\in K^-(T_1, 0)\cap K^+(0, 0). \tag{4.41}$$

From (4.37) we know that when n is big enough

$$\widetilde{H}(T_1, 0, \varepsilon_n)-\widetilde{H}(t, x, \varepsilon_n)\geqslant -1. \tag{4.42}$$

Then we get from (4.41), (4.40)

$$w^n(t, x)\leqslant c_0+1,\ \forall(t, x)\in K^-(T_1, 0)\cap K^+(0, 0). \tag{4.43}$$

From (4.34), (4.37) one obviously has

$$w^n(t, x)\geqslant -c, \tag{4.44}$$

so there exists a constant c_3 such that

$$|w^n(t, x)|\leqslant c_3\ \forall(t, x)\in K^-(T_1, 0)\cap K^+(0, 0). \tag{4.45}$$

Then, noticing (4.38) we get from (4.34) that there exists a constant c_4 such that

$$|\partial w^n(t, x)/\partial t|+|\partial w^n(t, x)/\partial x|\leqslant c_4,\ \forall(t, x)\in K^-(T_1, 0)\cap K^+(0, 0). \tag{4.46}$$

By Ascoli-Alzera theorem, there exists a subsequence of $\{w^n\}$, still denoted by $\{w^n\}$ for convenience, such that

$$w^n(t, w)\to w^0(t, x) \text{ uniformly in } K(T_1, 0)\cap K^+(0, 0). \tag{4.47}$$

By passing to the limit in (4.34), we know w^0 satisfies

$$w^0(t, x)=-c+\iint_{K^-(t,x)\cap K^+(0,0)}|w^0(\tau, y)|^{1+\alpha}d\tau\, dy/2.$$

Noticing Lemma 4.3, we know that $w^0(t, x)$ can be extended to a solution of (4.8) untill time T_1 which contradicts the fact that T_1 is the life span. Thus, (4.39) hold. Noticing (4.44), we know that $w^n(T_1, 0)$ is bounded from below. Hence, for any given $L>0$, we can find N such that

$$w^N(T_1, 0)>L+1. \tag{4.48}$$

Then we get from (4.34) that for any $(t, x)\in K^+(T_1, 0)$

$$w^n(t, x)-w^N(T_1, 0)\geqslant \widetilde{H}(t, x, \varepsilon_N)-\widetilde{H}(T_1, 0, \varepsilon_N) + \iint_{K^-(t,x)\cap K^+(T_1)} |w^N(\tau, y)|^{1+\alpha}d\tau dy/2. \tag{4.49}$$

Noticing (4.37). we can choose N so big that

$$\widetilde{H}(T_1, 0, \varepsilon_N)-\widetilde{H}(t, x, \varepsilon_N)<1. \tag{4.50}$$

Therefore we get from (4.48), (4.49) and (4.50)

$$w^N(t, x)>L+\iint_{K^-(t,x)\cap K^+(T_1,0)} |w^N(\tau, y)|^{1+\alpha}d\tau\, dy/2,\ \forall(t, x)\in K^+(T_1, 0). \tag{4.51}$$

By Lemma 3.3 and Lemma 3.4, we know that $w^N(t, x)$ at least blows up at time $T_1+L^{-\alpha/2}T_0$. We choose L so big that

$$L^{-\alpha/2}T_0<1. \tag{4.52}$$

Then we get $w^N(t, x)$ blows up before time T_1+1, this is a contradiction. So the lemma is proved.

Proof of Theorem 4.1 By Lemma 4.5, there exists an $\varepsilon_1>0$ such that for any ε with $0<\varepsilon<\varepsilon_1$

$$T(\varepsilon)\leqslant\rho+(T_1+1)\varepsilon^{-\alpha/2}. \tag{4.53}$$

So Theorem 4.1 is true with

$$c_2=\rho+T_1+1. \tag{4.54}$$

§5. Lower Bound of Life Span in Theorem 1. 2

Theorem 5.1. *Under the assumption of Theorem* 1.2, *there exists an* $\varepsilon_1>0$ *such that for any* ε *with* $0<\varepsilon\leqslant\varepsilon_0$

$$T(\varepsilon)\geqslant\varkappa_3\varepsilon^{-\alpha(\alpha+1)/(\alpha+2)}, \tag{5.1}$$

where $\varkappa_3$ *is a positive constant independent of* ε.

To prove Theorem 5.1, we first prove

Lemma 5.2. *Let* u *be the solution to* (1.1) *with initial data*

$$t=0:\ u=f_1(x)\ u_t=g_1(x), \tag{5.2}$$

where $f_1\in C_0^2(R)$, $g_1\in C_0^1(R)$. *Let* T *be the life span of* u. *If* $T<+\infty$, *then* $u(t, x)$ *is unbounded when* $t\to T$.

Proof By finite propogation speed of waves, if the supports of both f_1 and g_1 lie in $(-\rho, \rho)$, then the support of $u(t, x)$ lies in $[-(T+\rho), T+\rho]$, for $0\leqslant t<T$. If the conclusion of Lemma 4.2 is false, then $|u(t, x)|$ is bounded in $[0, T)\times R$. Again by differentiating the intergral equation obtained by D'Alembert's formula

we know that $|u_t(t, x)|$ is bounded in $[0, T)\times R$. Hence, $\int |u_t(t, x)|dx$ is also bounded. Thus, by Theorem 2.1 it can be extended beyond time T, which contradicts the fact that T is the life span. So the lemma is proved.

Proof of Theorem 5.1 By D'Alembert's formula and noticing (1.5), for any $(t, x)\in K^+(\rho, 0)$, we get

$$u(t, x)=\iint_{K^+(t,x)} |u(\tau, y)|^{1+\alpha}d\tau dy/2. \tag{5.3}$$

When $t\leqslant b\varepsilon^{-\alpha}$, by Lemma 4.2 we have

$$\begin{aligned} u(t, x)&=\iint_{K^-(t,x)\cap K^+(\rho,0)} u(\tau, y)|^{1+\alpha}d\tau\, dy/2+\iint_{K^+(t,x)\cap(K^+(-\rho,0)-K^+(\rho,0))} u(\tau, y)|^{1+\alpha}d\tau\, dy/2 \\ &\leqslant \iint_{K^-(t,x)\cap K^+(\rho,0)} |u(\tau, y)|^{1+\alpha}d\tau\, dy/2+\frac{1}{2}(\varepsilon M_1)^{1+\alpha}\iint_{K^+(t,x)\cap(K^+(-\rho,0)-K^+(\rho,0))} d\tau\, dy \\ &= \iint_{K^-(t,x)\cap K^+(\rho,0)} |u(\tau, y)|^{1+\alpha}d\tau\, dy/2+\rho(\varepsilon M_1)^{1+\alpha}(t-\rho)+\rho^2(\varepsilon M_1)^{1+\alpha}/2. \end{aligned} \tag{5.4}$$

Let

$$\hat{u}(t, x)=A^{2/\alpha}u(At+\rho, Ax), \tag{5.5}$$

where

$$A=\rho^{-\alpha/(\alpha+2)}(\varepsilon M_1)^{-\alpha(\alpha+1)/(\alpha+2)}. \tag{5.6}$$

Then by (5.4), when ε is small enough, we have

$$\hat{u}(t, x)<\iint_{K^-(t,x)\cap K^+(0,0)} \hat{u}(\tau, y)|^{1+\alpha}d\tau\, dy/2+t+1,\ (t, x)\in K^+(0, 0). \tag{5.7}$$

Let v be defined on $K^+(0, 0)$ such that

$$v(t, x)=\iint_{K^-(t,x)\cap K^+(0,0)} |v(\tau, y)|^{1+\alpha}d\tau\, dy/2+t+1. \tag{5.8}$$

It is easy to see that (5.8) at least has a local classical solution on $0\leqslant t\leqslant T_2$. By a comparison argument similar to what we have applied in Lemma 3.4, we can get

$$\hat{u}(t, x)<v(t, x). \tag{5.9}$$

If we take ε so small that $AT_2+\rho\leqslant b\varepsilon^{-\alpha}$, then by Lemma 5.2 we conclude that $\hat{u}(t,x)$ can exist untill time T_2. Noticing (5.5) and (5.6) we get

$$T(\varepsilon)\geqslant\rho^{-\alpha(\alpha+2)}(\varepsilon M_1)^{-\alpha(\alpha+1)/(\alpha+2)}T_2+\rho. \tag{5.10}$$

Thus Theorem 5.1 is proved with

$$\varkappa_3=\rho^{-\alpha/(\alpha+2)}M_1^{\alpha(\alpha+1)/(\alpha+2)}T_2. \tag{5.11}$$

§ 6. Upper Bound of Life Span in Theorem 1.2

Theorem 6.1. *Under the assumption of Theorem 1.2, there exists a $\varepsilon_3>0$ such that for any ε with $0<\varepsilon<\varepsilon_3$*

$$T(\varepsilon)\leqslant\varkappa_4\varepsilon^{-\alpha(\alpha+1)/(\alpha+2)}, \tag{6.1}$$

where $\varkappa_4$ is a positive constant independent of ε.

To prove Theorem 6.1, we need

Lemma 6.2. *Let v be the solution to the following Coursat problem*

$$\begin{cases} v_{tt}-v_{xx}=|v|^{1+\alpha}, \\ x=t:\ v=t, \\ x=-t:\ v=0, \end{cases} \tag{6.2}$$

then v blows up in finite time.

Proof This lemma can be proved in a way similar to that of the proof of Lemma 3.2, but with more care.

Proof of Theorem 6.1 By D'Alembert's formula we get

$$u(t,\ x)=\varepsilon F(x+t)+\varepsilon G(x-t)+\iint_{K^-(t,x)}|u(\tau,\ y)|^{1+\alpha}d\tau\ dy/2, \tag{6.3}$$

where

$$F(s)=f(s)/2+\int_{-\infty}^{s}g(\beta)d\beta/2, \tag{6.4}$$

$$G(s)=f(s)/2-\int_{-\infty}^{s}g(\beta)d\beta/2. \tag{6.5}$$

By (1.5), the supports of both F and G lie in $(-\rho,\ \rho)$. Moreover, F and G are not both identically zero. No loss of generality, we assume G is not identically zero. So there exist constants a_0, b_0 and μ such that

$$|G(s)|>\mu,\ \forall a_0<s<b_0. \tag{6.6}$$

Obviously we have

$$-\rho<a_0<b_0<\rho. \tag{6.7}$$

Let T_3 be the life span of solution to pro blem (6.2). By Lemma 6.2, $T_3<+\infty$. Let

$$\tau(\varepsilon)=[(b_0-a_0)/2]^{-\alpha/(\alpha+2)}T_3(\mu\varepsilon/2)^{-\alpha(\alpha+1)/(\alpha+2)}+\rho. \tag{6.8}$$

We claim that if the solution exists on $[0,\ \tau(\varepsilon))\times R$, then it must blow up at that time.

For this purpose we choose ε so small that

$$\tau(\varepsilon)\leqslant b\varepsilon^{-\alpha} \tag{6.9}$$

where b is defined by (4.4).

For $(t,\ x)\in K^+(\rho,\ 0)$, we define

$$D^+(t,\ x)=\{(\tau,\ y)\,|\,a_0<y-\tau<b_0,\ \rho<y+\tau<t+x\}. \tag{6.10}$$

For $(\tau,\ y)\in D_+(t,\ x)$, by (6.3) we have

$$u(\tau,\ y)=\varepsilon G(y-\tau)+\iint_{K^-(\tau,y)\cap K^+(-\rho,0)}|u(\tau',\ y')|^{1+\alpha}d\tau'dy'/2. \tag{6.11}$$

Then by (6.6) we get

$$|u(\tau,\ y)|>\varepsilon\mu-\iint_{K^-(\tau,y)\cap K^+(-\rho,0)}|u(\tau',\ y')|^{1+\alpha}d\tau'dy'/2. \tag{6.12}$$

Noticing (5.8) and using Lemma 4.2, we get for $t<\tau(\varepsilon)$

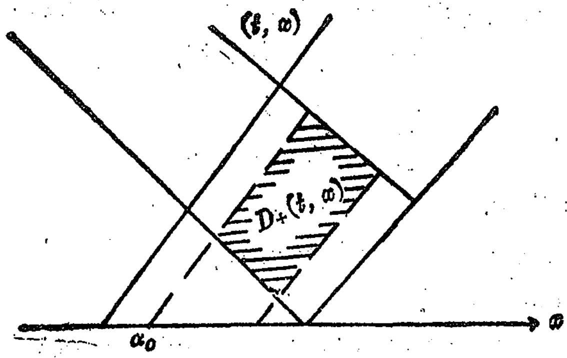

Fig. 2

$$\iint_{K^-(\tau,y)\cap K^+(-\rho,0)} |u(\tau', y')|^{1+\alpha}d\tau'dy'/2 \leqslant (sM_1)^{1+\alpha} \iint_{K^-(\tau,y)\cap K^-(-\rho,0)} d\tau'dy'/2$$
$$\leqslant (sM_1)^{1+\alpha}\rho\tau \leqslant (\tau M_1)^{1+\alpha}\rho\tau(s). \qquad (6.13)$$

So when s is small enough, we have

$$\iint_{K^-(\tau,y)\cap K^+(-\rho,0)} |u(\tau', y')|^{1+\alpha}d\tau'dy'/2 \leqslant \mu s/2. \qquad (6.14)$$

Hence, it follows from (6.12) that

$$|u(\tau, y)| > \mu s/2, \ \forall(\tau, y) \in D_+(t, x). \qquad (6.15)$$

On the other hand, for any $(t, x) \in K^+(\rho, 0)$, we have

$$u(t, x) = \iint_{K^-(t,x)} |u(\tau, y)|^{1+\alpha}d\tau dy/2. \qquad (6.16)$$

Then

$$u(t, x) > \frac{1}{2}\iint_{K^-(t,x)\cap K^-(\rho,0)} |u(\tau, y)|^{1+\alpha}d\tau\, dy + \frac{1}{2}\iint_{D_+(t,x)} |u(\tau, y)|d\tau dy. \qquad (6.17)$$

Therefore, we have for $t < \tau(s)$

$$\frac{1}{2}\iint_{D^+(t,x)} |u(\tau, y)|^{1+\alpha}d\tau\, dy \geqslant (\mu s/2)^{1+\alpha}\iint_{D_+(t,x)} d\tau\, dy/2$$
$$= (b_0 - a_0)(\mu s/2)^{1+\alpha}(t+x-\rho)/4. \qquad (6.18)$$

Thus we get from (6.17), (6.18)

$$u(t, x) > \iint_{K^-(t,x)\cap K^+(\rho,0)} |u(\tau, y)|^{1+\alpha}d\tau\, dy/2 + (b_0 - a_0)(\mu s/2)^{1+\alpha}(t+x-\rho)/4. \qquad (6.19)$$

Let w be defined on $K^+(\rho, 0)$ such that

$$w(t, x) = \iint_{K(t,x)\cap K(^-\rho,0)} |w(\tau, y)^{1+\alpha}d\tau\, dy/2 + (b_0 - a_0)(\mu s/2)^{1+\alpha}(t+x-\rho)/4. \qquad (6.20)$$

Then w is the solution to the following Goursat problem

$$\begin{cases} w_{tt} - w_{xx} = |w|^{1+\alpha}, \\ x = t-\rho, \ w = (b_0 - a_0)(\mu s/2)^{1+\alpha}/2, \\ x = -t+\rho, \ w = 0 \end{cases} \qquad (6.21)$$

Again by a scaling argument, we know that the life span of w is exactly $\tau(s)$. Then a comparison argument similar to that in Lemma 3.4 gives

$$u(t, x) > w(t, x), \ \forall t < \tau(s). \tag{6.22}$$

Therefore u blows up at time $\tau(s)$. Then the life span of u satisfies (6.1) with

$$\varkappa_4 = [(b_0 + a_0)/2]^{-\alpha/(\alpha+2)}(\mu/2)^{-\alpha(\alpha+1)/(\alpha+2)}T_3 + \rho.$$

Theorem 6.1 is thus proved.

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