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ON THE NECESSARY AND SUFFICI ENT CONDITIONS OF THE CONTINUITY OF M-P INVERSES A_x^+

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Abstract

Some necessary and sufficient conditions of the continuity of A_x^+ are given in [1] In this paper, some advances on the base and the condition of the continuity of A_x^+ for Semi-Fredholm operators A_x are presented.

In this paper, we assume that X is a topological space, H a Hilbert space and $B(H)$ the set of the bounded operators on H.

The following Theorem 1 in [1] will be useful in this paper.

Theorem. *Let $A_x \in B(H)$ be of closed range $R(A_x)$ and $A_x: X \to B(H)$ be continuous. The A_x^+ is continuous if and only if*

(1) $\|A_x^+\|$ *is locally bounded,*

(2) $P_{R(A_x)}$ *and* $P_{N(A_x)}$ *are continuous,*

where A_x^+ are the M-P inverses of A_x, $P(.)$ is the projection from H to $(.)$.

Now, we give the first theorem of this paper

Theorem 1. *Let A_x be of closed range $R(A_x)$ and $A_x: X \to B(H)$ be continuous. Then A_x^+ is contionuous if and if only both $P_{R(A_x)}$ and $P_{N(A_x)}$ are continuous.*

Proof Let

$$\Delta_x^1 = P_{R(A_x)} - P_{R(A_0)},$$
$$\Delta_x^2 = P_{N_x} - P_{N_0},$$
$$\Delta_x^3 = A_x - A_0, \text{ for every } x \in X,$$

where $A_0 = A_{x_0}. N_x = N(A_x), N_0 = N_{x0}$

By the assumption, we have that for any $\varepsilon > 0$ there is a neighburhood U_0 of X_0 such that

$$\|\Delta_x^1\| < \varepsilon, \ \|\Delta_x^2\| < \varepsilon, \ \|\Delta_x^3\| < \varepsilon, \text{ for every } x \in U_0.$$

From

$$A_x^+ P_{R(A_0)^\perp} = A_x^+ (P_{R(A_x)^\perp} + \Delta_x^1) = A_x^+ \Delta_x^1$$

and

$$P_N A_x^+ = (P_{N_0} + \Delta_x^2) A_x^+ = \Delta_x^2 A_0^+,$$

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it follows that

$$A_x^+ - A_0^+ = A_x^+ P_{R(A_0)} + A_x^+ P_{R(A_0)^\perp} - P_{N_x^\perp} A_0^+ - P_{N_x} A_0^+$$
$$= A_x^+ (A_0 - A_x) A_0^+ + A_x^+ P_{R(A_0)^\perp} - P_{N_x} A_0^+$$
$$= A_x^+ \Delta_x^3 A_0^+ + A_x^+ \Delta_x^1 - \Delta_x^2 A_x^+.$$

Then we have

$$\|A_x^+\| - \|A_0^+\| < \varepsilon \|A_0^+\| \|A_x^+\| + \|A_x^+\| \varepsilon + \|A_0^+\| \varepsilon.$$

Taking $\varepsilon = \min\{1, 1/2 \times (1 + \|A_x\|^{-1})\}$, we obtain the inequality $\|A_x^+\| \leqslant 4\|A_0^+\|$, for every $x \in U_0$.

This gives the boundness of $\|A_x^+\|$.

Now, the theorem follows immediadely from Theorem 1 of [1].

The following corollary of [1] is important, we need it.

Corollary. *Let H be a Hilbert space, H_0 be a closed subspace of H, X be a topological space, $A_x: X \to B(H)$ be continuous and $R(A_x) = H_0$. Then A_x^+ is continuous.*

Now we can give the main result of this paper.

Theorem 2. *Let A_x be of closed range $R(A_x)$ and $A_x: X \to B(H)$ be continuous. Then the following conditions are equivalent:*

(1) *A_x^+ is continuous;*

(2) *$P_{R(A_x)}$ is continuous;*

(3) *$P_{N(A_x)}$ is continuous.*

Proof (2)$\Rightarrow$(1): Let us consider the following operator family $A_x': H \times H \to H \times H$,

$$A_x'(\mu, \nu) = (A_x \mu \oplus P_{R(A_x)^\perp} \nu, 0), \text{ for every } (\mu, \nu) \in H \times H.$$

It is clear that $R(A_x') = H \times \{0\}$. Because of the continuity of A_x and $P_{R(A_x)}$, A_x' is continuous. With the corollary to Theorem 1 of [1], we get the contin utity of $(A_x')^+$.

Now we are to find $(A_x')^+$.

Let $B_x: H \times H \to H \times H$ be the following operator family:

$$B_x(\mu, \nu) = (A_x^+ \mu, P_{R(A_x)^\perp} \mu) \tag{1}$$

for every $(\mu, \nu) \in H \times H$.

It is clear that

$$N(B_x) = \{0\} \times H = H \times H \ominus R(A_x'),$$

and $R(B_x) = N(A_x)^\perp \times R(A_x)^\perp$.

We have

$$A_x' B_x(\mu, \nu) = A_x'(A_x^+ \mu, P_{R(A_x)^\perp} \mu) = (A_x A_x^+ \mu + P_{R(A_x)^\perp} \mu, 0)$$
$$= (\mu, 0) = P_{R(A_x')}(\mu, \nu)$$

and

$$B_x A_x'(\mu, \nu) = B_x(A_x \mu + P_{R(A_x)^\perp} \nu, 0)$$
$$= (A_x^+ A_x \mu, P_{R(A_x)^\perp} \nu)$$
$$= P_{N(A_x)^\perp} \mu, P_{R(A_x)^\perp} \nu).$$

Clearly $B_x A'_x$ is self-adjoint and idempotent.

In fact, $B_x=(A'_x)^+$. For this it suffices to show

$$N(A'_x)^{\perp}=N(A_x)^{\perp}\times R(A_x)^{\perp} \tag{2}$$

(next to do this).

Given $(x, y)\in N(A'_x)^{\perp}$, let

$$x=x_1+x_2,\ y=y_1+y_2,$$

where $x_1\in N(A_x), x_2\in N(A_x)^{\perp}$, $y_1\in R(A_x)$, $y_2\in R(A_x)^{\perp}$.

For each $x_0\in N(A_x)$, since $(x_0, 0)\in N(A'_x)$, we have

$$\langle(x_1+x_2, y_1+y_2), (x_0, 0)\rangle=0,$$

where $\langle, \rangle$ denotes the inner product of $H\times H$. It follows that $\langle x_1, x_0\rangle=0$. So $x_1=0$. Similarly $y_1=0$.

Hence we have

$$N(A'_x)^{\perp}\subset N(A_x)^{\perp}\times R(A_x)^{\perp}.$$

Thus (2) is proved.

Consequently, we have that A_x^+ is continuous from (1) and the continuity of $(A'_x)^+$.

(3)$\Rightarrow$(1): Since $P_{N(A_x)}$ is continuous, so is $P_{R(A_x^*)}=P_{N(A_x)^{\perp}})$, So, from the first part of the proof we get the continuity of $(A_x^*)^{\perp}$. Since $(A_x^+)^*=(A_x^*)^+$, A_x^+ is continuous.

That (1) implies (2) and (3) follows immediately from Theorem 1 of [1].

Lemma 1. *Let D be a positive operator. If there exists a positive number β such that $(0, \beta)\subset\rho(D)$, then*

$$\|(D-\xi I)^{-1}\|<\max\{\beta_0^{-1}, (\beta-\beta_0)^{-1}\}$$

for every $\xi\in\{\xi\in\mathbf{C}: |\xi|=\beta_0\}$, where β_0 is such a number that $0<\beta_0<\beta$.

Proof Let E_λ denote the spectral resolution unit for D, and let

$$U_\xi(\lambda)=\begin{cases}\dfrac{1}{\lambda-\xi} & \lambda\in(-\infty, \varepsilon]\cup[\beta-\varepsilon, \infty),\\ 0, & \lambda\in(\varepsilon, \beta-\varepsilon),\end{cases}$$

where ε is such a number that

$$0<\varepsilon<\min\{\beta_0, \beta-\beta_0\}.$$

Noting that $E_{\lambda_1}=E_{\lambda_2}$ for every $\lambda_1, \lambda_2\in(0, \beta)$, we have

$$\begin{aligned} U_\xi(D)(D-\xi I)&=\int_{-\infty}^{\infty}U_\xi(\lambda)dE_\xi,\int_{-\infty}^{\infty}(\lambda-\xi)dE_\lambda\\ &=\int_{-\infty}^{\infty}U_\xi(\lambda)(\lambda-\xi)dE_\lambda\\ &=\int_{-\infty}^{\varepsilon}dE_\lambda+\int_{\beta-\varepsilon}^{\infty}dE_\lambda\end{aligned}$$

$$=\int_{-\infty}^{\varepsilon} dE_\lambda+\int_{\varepsilon}^{\beta-\varepsilon} dE_\lambda+\int_{\beta-\varepsilon}^{\infty} dE_\lambda$$
$$=I.$$

Similarly, the equality $(D-\xi I)U_\xi(D)=I$ also holds.

So $(D-\xi I)^{-1}=U_\xi(D)$.

Hence

$$\|(D-\xi I)^{-1}\|=\|U_\xi(D)\|=\|\int_{-\infty}^{\infty} U_\xi(\lambda)dE_\lambda\|$$
$$\leqslant \max\{(\beta_0-\varepsilon)^{-1}, (\beta-\beta_0-\varepsilon)^{-1}\} \quad (*)$$

for every $\xi: |\xi|=\beta_0$.

The following lemma is Lemma 1 of [1], we need it.

Lemma 2. *Let $A\in B(H)$, $R(A)$ be closed. Then, for any β satisfying $0<\beta<\|A^+\|^{-1}$, we have*

$$(0, \beta)\subset\rho(A^*A)\cap\rho(AA^*),$$

where $\rho(.)$ indicates the set of the regular point of the correspondent operator.

Theorem 3. *Let $A_x\in B(H)$ be of closed range $R(A_x)$, and $A_x: X\to B(H)$ be neicontinuous. Then A_x^+ is continuous if and only if for each $x_0\in X$ there exists a neighbourhood U_0 of x_0 and $\beta>0$ such that*

$$(0, \beta)\subset\rho(A_x^+A_x) \text{ for all } x\in U_0.$$

Proof Sufficiency: Let β_0 be such that $0<\beta_0<\beta$ and ∂s denote the curve $\{z\in \mathbf{C}, |z|=\beta_0\}$ and $D_x:=A_x^*A_x$.

Put $P_x:=\frac{1}{2\pi i}\oint_{\partial s}(D_x-\xi I)^{-1}d\xi$ for all $x\in U_0$. Then we have

$$P_x=\frac{1}{2\pi i}\oint_{\partial s}\left(\int_{-\infty}^{\infty}\frac{1}{\lambda-\xi}\, dE_{x,\lambda}\right)d\xi$$
$$=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi i}\oint_{\partial s}\frac{1}{\lambda-\xi}\, d\xi\right)dE_{x,\lambda}$$
$$=\int_{-\beta}^{\beta} dE_{x,\lambda}=P_{N_x}, \quad \text{for all } x\in U_0,$$

where $E_{x,\lambda}$ is the spectral resolution of unit for D_x.

Since

$$\|P_x-P_0\|=\|\frac{1}{2\pi i}\oint_{\partial s}\{(D_x-\xi I)^{-1}-(D_0-\xi I)^{-1}\}d\xi\|$$
$$=\|\frac{1}{2\pi i}\oint_{\partial s}(D_x-\xi I)^{-1}(D_x-D_0)(D_0-\xi I)^{-1}d\xi\|$$
$$\leqslant M\|D_x-D_0\|,$$

where $D_0=D_{x0}$, $P_0=P_{N_{x0}}$, $M=\max\{\beta_0^{-1}, (\beta-\beta_0)^{-1}\}$, it follows that P_{N_x} is continuous.

Necessity: Because of the continity of A_x^*, there exists a neighbourhood U_0 of X_0 such that $\|A_x^*\|<M$ for all $x\in U_0$. By Lemma 2, $(0, M^{-1})\subset\rho(A_x^*A_x)$.

This completes the proof.

Corollary. *Suppose A_x are of colsed range $R(A_x)$ and $A_x: X \to B(H)$ is continuous. Then A_x^+ is continuous if and only if $\|A_x^+\|$ are locally bounded.*

Proof The necessity is obvious.

Sufficiency: There is a neighbourhood U_0 of X_0 such that $\|A_x^+\| < M$ for each $x \in U_0$. From Lemma 2, we have $(0, M^{-1}) \subset \rho(A_x^* A_x)$ for all $x \in U_0$. Thus A_x^+ is continuous by Theorem 3.

The above results are reduced to the following theorem:

Theorem 4. *Suppose A_x are of closed range $R(A_x)$ and $A_x: X \to B(H)$ is continuous. Then the following conditions are equivalent:*

(1) *A_x^+ is continuous;*

(2) *$P_{R(A_x)}$ is continuous;*

(3) *$P_{N(A_x)}$ is continuous;*

(4) *$\|A_x^+\|$ is continuous;*

(5) *$\|A_x^+\|$ are locally bounded;*

(6) *For any $x_0 \in X$, there is a neighbourhood U_0 of x_0 and a constant $\beta > 0$ such that*

$$(0, \beta) \subset \rho(A_x^* A_x), \text{ for all } x \in U_0;$$

(7) *For each $x_0 \in X$, there is a neighbourhood U_0 of x_0 and a constant $\beta > 0$, such that*

$$(0, \beta) \subset \rho(A_x A_x^*), \text{ for all } x \in U_0.$$

Proof The equivalences of (1), (2), (3), (5) and (6) have been proved before.

It is obvious that (1) implies (4), and that (4) implies (5).

Now we are to show that (7) implies (1):

Since A_x is continuous, A_x^* is continuous. And from the condition of (7) and the equivalence of (6) and (1), $(A_x^*)^+$ is continuous. Because $(A^+)^* = (A^*)^+$, A_x^+ is continuous.

By Lemma 2, (1) implies (7).

This concludes the proof.

Let $SF(H)$ be the set of all bounded Semi-Fredholm operators on H.

Theorem 5. *Suppose that $A_x: X \to SF(H)$ is continuous. Then A_x^+ is continuous if ond only if either* $\dim N(A_x)$ *or* $\dim N(A_x)$ *is locally finite and locally constant.*

Proof Sufficiency: We first suppose that there is an open set U_0 such that $\dim N(A_x) = \dim N(A_0) < \infty$ whenever x and x_0 are in U_0.

Let $B_x = A_x^* A_x$. Obviously $N(B_x) = R(B_x)^\perp = N(A_x)$.

So B_x are Fredholm operators. By Theorem 3 of [1], when x is in U_x, B_x: $X \to F(H)$ is continuous. So $P_{N(B_x)}$ and $P_{N(A_x)}$ are continuous. Hence A_x^+ is continuous when x is in U_0.

Similarly, when we suppose that there is an open set U_0 such that $\dim N(A_x^*)$

$=\dim N(A_0^*)<\infty$, and let $B_x=A_xA_x^*$, we then get that A_x^+ is continuous when x is in U_0.

Necessity: For any $x_0\in X$, we suppose that $\dim N(A_{x_0})<\infty$.

Let $B_x:=A_x^*A_x$. Obviously, B_x is a Fredholm operator. Because B_x is continuous, $P_{N(A_x)}$ is continuous, and since $P_{N(A_x)}=P_{N(B_x)}$ so is $P_{N(B_x)}$. Therefore B_x^+ is continuous. By Theorem 3 of [1], there is a neighbourhood U_0 of X_0 such that $\dim N(B_x)=\dim N(B_{x_0})<\infty$. Since $N(B_x)=N(A_x)$, the necessary assertion is valid when $\dim N(A_{x_0})<\infty$. Similarly, when $\dim N(A_{x_0}^*)<\infty$, let $B_x:=A_xA_x^*$; then the necessary assertion holds.

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