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PASSAGE, BLOCKADE, SINK AND SOURCE** OF A PLANAR DYNAMICAL SYSTEM

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Abstract

The author defines passage, blockade, sink and source of a plane multiply-connected region G with respect to a smooth dynamical system. Three theorems on the non-existence or existence of closed orbits, and the existence of a passage are proved. Also a conjecture about the number of passages in G is stated.

The Poincare's annulus theorem for the proof of the existence of limit cycles is well-known to mathematicians working in the field of qualitative theory of ODE. As to the index theorem, also due to H. Poincaré (for the proof, see [1], Chap. 7, Theorem 9.2), it has never been paid attention to, until recently, the author[2] gave a generalization as follows:

Theorem 1. *Let G be an n-multiply-connected plane region with smooth outer boundary L_1 and inner boundaries $L_2, L_3, \cdots, L_n$. With respect to a certain smooth dynamical system, there are only a finite number of critical points lying in the interior of G. Let σ, ν be the number of points where the trajectories are internally, externally tangent to the boundary of G. We call these points inner, outer contact points, respectively*[1]*. Then the sum of the indices of critical points within G is*

$$\Sigma = 2 - n + \frac{\sigma - \nu}{2}.$$

Using Theorem 1 we can give a theorem on the non-existence of closed orbits within an n-multiply connected region G.

Theorem 2. *Let G be the same as in Theorem 1, assume there are σ_i inner contact points and ν_i outer contact points on L_i (w. r. t. G) with $\nu_1 > \sigma_1$, $\sigma_i > \nu_i$ for $i = 2, \cdots, n$, and $\sum_{i=1}^{n}(\sigma_i - \nu_i) \leq 2(n-2)$, and in G there can be only critical points with non-positive indices. Then there is no closed orbit lying completely within G.*

Proof Assume there exists a closed orbit C lying completely within G. It is impossible that none of the $L_2, \cdots, L_n$ lies within C, since C must contain in its

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1). Notice that the inner (outer) contact points of G are outer (inner) contacts points of $L_2, \cdots, L_n$.

interior some critical point with positive index. Now, assume C contains in its interior some of the $L_i(i=2, \cdots, n)$ and may contain also some critical points within G. Since the sum of indices of critical points within L_i is $1+\frac{\nu_i-\sigma_i}{2}\leqslant 0$ but the sum of indices of critical points within C is 1, the existence of C is a also impossible.

Remark 1. In case $n=2$ the conditions can be relaxed to: $\sigma_i\neq\nu_i(i=1, 2)$, but $\sigma_1-\nu_1=\nu_2-\sigma_2$, and within G there are no critical points.

Remark 2. Since in Theorem 2 $\sigma_i\neq\nu_i$ but in the annulus theorem we have $\sigma_i=\nu_i=0$ for $i=1, 2$, Theorem 2 is not a generalization of the annulus theorem, it is a theorem of another kind for the non-existence of limit cycles.

Now, we can consider two different kinds of problems:

I. Under the conditions of Theorem 2, aside from the non-existence of limit cycles, what kind of other conclusions can be deduced?

II. If in the notations of Theorem 2, we have $\sigma_i=\nu_i$ for $i=1, 2, \cdots, n$, among which some or all are not zero, can we deduce the existence of limit cycles?

In considering the first problem, for the sake of simplicity, we take first G to be an annulus with boundary $L_1\supset L_2$, such that on each L_i there are only two outer contact points but no inner contact points, i. e., $\nu_1=2$, $\sigma_2=0$, $\nu_2=0$, $\sigma_2=2.$, and there are no critical point in G. Let l_1 and l_2 be the trajectories which contact L_2 externally.

Assume first that l_1 comes from the exterior of L_1.

1) If l_1 goes also to the exterior of L_1, then l_2 will have the same behavior as l_1. For, if l_2 comes from the interior of L_2, then the ω-limit set of l_2 will be in the interior of G, this is impossible. So l_2 comes from the exterior of L_1. Now, l_2 cannot go into the interior of L_2 through the boundary of the shaded region in Fig. 1, nor can it remain in G; so it must go out G and to the exterior of L_1.

We see thus that in Fig. 1, l_1 and l_2 together bound a passage, such that almost all trajectories lying between l_1 and l_2 go from the exterior of L_1 into G, and go out G through L_2, then go into G again through the other part of L_2, and finally go out G through L_1.

2) If l_1 goes out G but into the interior of L_2 as shown in Fig. 2, then either the α-limit set or the ω-limit set of l_2 will remain in G, which is impossible.

Assume next that l_1 comes from the interior of L_2. There are now two possibilities as shown by the two dotted curves in Fig. 3. Then either the ω-limit set of l_1 will remain in G, or a certain half trajectory of l_2 will remain in G, all these are impossible, too.

We have proved therefore

Theorem 3. *When G is an annulus like Fig. 1, where there are two outer contact points on L_1 and other two on L_2, and no critical points lie in G, the two trajectories l_1 and l_2 bound a passage through G.*

Remark 1. We do not call the way between l_2 and l_3 (or between l_1 and l_4) a passage, because trajectories therein do not meet L_2.

Remark 2. If in Theorem 3 outer contact points on L_1 and L_2 are all replaced by inner contact points, we can get the same conclusion. But now the boundaries of the passage are made from l_3 and l_4 (see Fig. 4). Moreover, in the limiting case (see Fig. 5), the width of the passage can be reduced to zero at a point S, but we still call this a passage.

Together with the new idea "passage" introduced in Theorem 3, we will introduce other three ideas: "blockade" in the case when limit cycle appears in the annulus, "source" and "sink" in the case $\sigma_i=\nu_i=0$ but no limit cycle appears. These are shown in Figs. 7, 8, and 9, respectively. Notice that all these concepts are concerned with a 0, 1 or 2-dimensional set with respect to the trajectories in G. Here the blockade may be a 1-dimensional set, and the sink or source may be a critical point.

In conformity with Theorem 2, we can generalize Theorem 3 in two different cases:

1. G is a 3-multiply connected region with boundaries L_1, L_2 and L_3, on which $\nu_1=4$, $\sigma_1=0$, $\sigma_2=2$, $\nu_2=0$, $\sigma_3=2$, $\nu_3=0$, and there is just one saddle point in G.

2. G is still an annulus with boundaries L_1 and L_2, on which $\nu_1=4$, $\sigma_1=0$, $\sigma_2=4$, $\nu_2=0$ (or $\nu_1=0$, $\sigma_1=4$, $\nu_2=4$, $\sigma_2=0$), and there is no critical point in G.

Figs. 10, 11, 12 and 13 belong to the first case, Figs. 14 and 15 belong to the second case. It seems troublesome to prove that there are at least two passages in each figure, so we give here only a conjecture refering to the general case:

Conjecture. For an n-multiplyconnected region G with outer boundary L_1 and inner boundaries $L_2, \cdots, L_n$ such that $\nu_1=2m$, $\sigma_1=0$; $\nu_i=0$, $\sigma_i\neq 0$ and $\sum_{i=2}^{n}\sigma_i=2m$ (hence the sum of indices of critical points in G is $2-n$), moreover, we assume that there are just $n-2$ critical points (they are all saddles) lying in G. Then there exist at least m (when $n=m+1$) or $2m-n+2$ (when $n<m+1$) different passages.

Remark 3. Fig. 16 shows that when $m=2$ and $n=3$, if there are two saddle points and one focus in G (although the sum of their indices is -1), there may not exist passage, but instead, limit cycle may appear (although not completely in G). So the last condition in the Conjecture can not be weakened.

Remark 4. Fig. 13 shows that the two passages may all pass through L_2 but do not meet L_3; Fig. 15 shows that the two passages may all be of width zero.

Remark 5. If in Fig. 12, $l_2=l_3$, $l_1=l_4$, then the passage bounded by l_1 and l_2 must be counted as a double passage.

Remark 6. Fig. 17 shows that in case $m=2$, $n=2$ but $\nu_1=4$, $\sigma_1=0$, $\nu_2=0$, $\sigma_2=2$ $\left(\text{hence } \Sigma=\frac{2-4}{2}=-1\text{, and we assume there is just one saddle point lying in } G\right)$, we can have only one passage. Therefore, when $\sum_{i=2}^{n}\sigma_i\neq\nu_1$, the conclusion of the former Conjecture is false.

Now, let us study Problem II. We investigate first Figs. 18, 19, 20 and 21, where G is an annulus, one of its boundary has no contact points, the other one has an outer as well as an inner contact points. In Fig. 18 we have a sink, where the outer contact trajectory l_1 of L_2 comes from the exterior of L_1. In Fig. 19 we have a blockade, where the outer contact trajectory l_1 of L_2 comes from the interior of L_2.

Similar phenomena happen in Figs. 20 and 21 (in Fig. 20 we have a source). These figures show that when trajectories all go into G through L_1 (or L_2) the existence of a blockade or a sink (or a source) depends on the α-limit set of l_1 or the ω-limit set of l_1'.

Secondly, let us investigate Fig. 22, where we have two inner and two outer contact points on L_1 but no contact point on L_2. We see that the existence of a limit cycle is ensured by the fact that the two inner contact trajectories of L_1 come from the exterior of L_1 and that L_2 is a repeller.

By using these facts we get a new method for the proof of the existence of limit cycles.

Example 1. In order to prove the existence of a limit cycle of the wellknown van der Pol equation

$$\dot{x}=y,\ \dot{y}=x+\mu(1-x^2)y,\ \mu>0, \tag{1}$$

we take an annulus G with inner boundary L_2: $x^2+y^2=1$ and outer boundary L_1: $x^2+y^2=C>>1$ (Fig. 23). Since for $V=x^2+y^2$

$$\dot{V}=2xy+2y[-x+\mu(1-x^2)y]=2\mu y^2(1-x^2)\begin{cases}\geqslant 0 \text{ when } |x|<1,\\ \leqslant 0 \text{ when } |x|>1,\end{cases} \tag{2}$$

we see that on L_2 trajectories all go from inside to outside (although they tangent to L_2 at (1, 0) and (−1, 0)), while on L_1 there are two inner contact points $A(1, \sqrt{C-1})$ and $B(-1, -\sqrt{C-1})$ and two outer contact points $D(1, -\sqrt{C-1})$ and $E(-1, \sqrt{C-1})$. Because the phase portrait of (1) is symmetric with respect to the origin $O(0, 0)$, in order to prove the existence of a limit cycle in G, we need only to prove that the trajectory l_1 passing through B comes from the exterior of L_1 but not from the interior of L_2. This is not difficult to prove when $C<<1$.

Moreover, once this has been proved for L_1 being $x^2+y^2=C_1$, we see at once that between $x^2+y^2=C_1$ and $x^2+y^2=C_2>C_1$ there exists no limit cycle (by considering the outer contacting circle in order to get a contradiction).

Example 2. In [3] the existence and uniqueness of limit cycle of the system

$$\dot{x}=x(1+y)(c-b-y^2),\ \dot{y}=x(1+y)(b+y^2)-ay \tag{3}$$

was studied, where $a>0$, $c>b\geqslant 0$. Aside from $O(0, 0)$ there is only one critical point $P(a\sqrt{c-b}/c(1+\sqrt{c-b}), \sqrt{c-b})$ lying in the first quadrant. Notice that $x=0$ is a trajectory and on the positive x-axis we have $\dot{y}\geqslant 0$.

Now, let us take $V=x+y$, then any straight line $V=C>0$ forms with the two coordinate axes a triangle lying in the first quadrant. Moreover, $\dot{V}=0$ is a hyperbola:

$$x=ay/c(1+y) \tag{4}$$

as shown in Fig. 24. The condition $2(c-b)/c>1/(1+\sqrt{c-b})$ ensures that P is an unstable focus or node. So we have now an annulus between the triangle ΔOAB and the small circle around P. From the direction of the trajectories, by the convention of [4], we know that A and B must be taken as outer contact points while O must be taken as an inner contact point. There is another inner contact point E, it is the intersection point of (4) and $x+y=C$. In order to prove limit cycle exists around P, it is sufficient to prove that the trajectory passing through E comes from the fourth quadrant (but not from P). This is not difficult when $C\gg 1$ (hence E is sufficicently close to the line $x=a/c$.).

Example 3. In [5] the system

$$\dot{x}=x(A_0+A_1x+A_2x^2+A_3x^2)-x^2y,\ \dot{y}=-y+x^2y \tag{5}$$

was studied, where $A_0>0$, $A_1<0$, $A_2>0$ and $A_3<0$. Critical points in the first quadrant are $O(0, 0)$, $S(1, y^*)$ and $R(x_+, 0)$, where

$$y^*=A_0+A_1+A_2+A_3,\ x_+=-A_2/A_3.$$

Since $A_3x^3+A_2x^2+A_1x+A_0=0$ under some former assumptions in [5] has only one (positive) real root x_+, the straight line $V=x+y=C$ will not intersect the quartic curve

$$\dot{V}=\dot{x}+\dot{y}=x(A_0+A_1x+A_2x^2+A_3x^3)-y=0$$

when $C>>1$. Now, $O(0, 0)$ and $R(x_+, 0)$ are saddles, and the condition $2A_3+A_2-A_0>0$ ensures that $S(1, y^*)$ is an unstable focus or node. So from Fig. 25 the existence of a limit cycle around P is obvious. This proof is simpler than that in [5].

Remark 7. Since in many mathematical models coming from ecology, chemical reactions, immune response, etc., we have planar polynomial Kolmogorov differential systems (i. e., $x=0$ and $y=0$ are integral lines), in which we see that the same terms with different signs are situated in different equations it is, a very natural way to use $x=0$, $y=0$ and $x+y=C\gg 1$ to form the outer boundary of the annulus (while the inner boundary is taken to be a small circle around the unique critical point within the triangle) and then use our method to prove the existence of limit cycles.

Similar figures can be discussed when G is an annulus with $\sigma_1=\nu_1=1$, $\sigma_2=\nu_2=1$ (Figs. 26, 27, 28) or when G is a 3-multiply connected region with $\nu_1=\sigma_1=0$, $\nu_2=\sigma_2=1$, $\nu_3=0$, $\sigma_3=2$ (Figs. 29, 30).

Our method can also be used to study the problem of the existence of trajectory connecting two critical points in a certain bounded region.

Example 4.[6] Assume in Fig. 31 within the region G there are only two critical points A and B, where A is a repeller, and no limit cycle exists. The question is: Is there a trajectory connecting A and B?

By Theorem 1, the sum of the indices of critical points within G is $\Sigma=1+\frac{-2}{2}=0$. Since the index of A is $+1$, the index of B is -1. We may assume that B is a saddle, otherwise, there would be more separatrices passing through B. Now, if l_1 or l_2 comes from A, the problem is solved. If there is a semi-stable limit cycle around A, then the α-limit set of l_2 can be this semi-stable limit cycle, but this is impossible by the assumption. So we may assume that both l_1 and l_2 come from outside of G. But then we will have a stable limit cycle around A, again a contradiction. So at least one of l_1 or l_2 must come from A.

Example 5. We have in Fig. 32 a region G with $\nu_1=4$, $\sigma_1=2$ on the boundary L_1, and two critical points A and B in G, where A is a repeller, B is a saddle. Different from Fig. 31, we have now neither limit cycle nor trajectory connecting A and B[7].

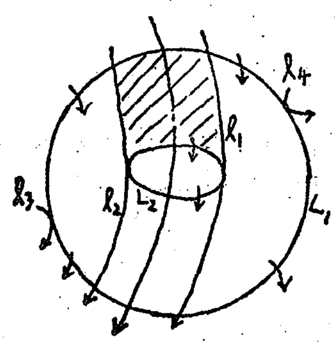


Fig. 1

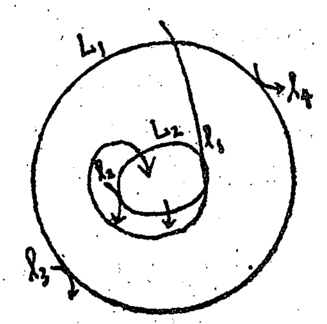


Fig. 2

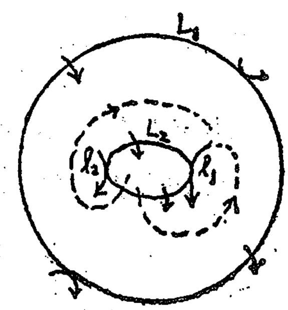


Fig. 3

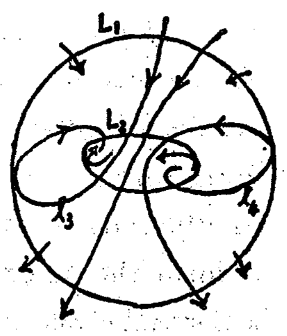


Fig. 4

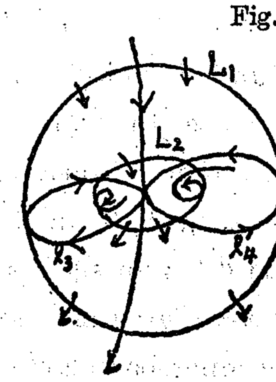


Fig. 5

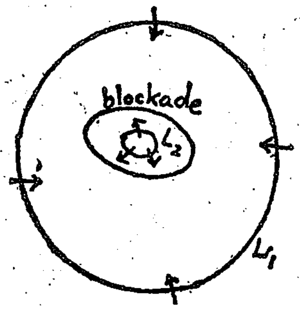


Fig. 7

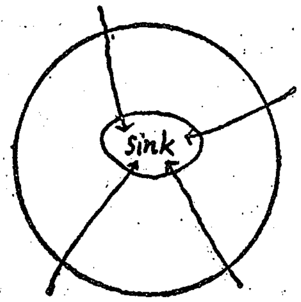


Fig. 8

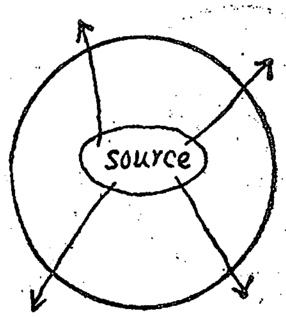


Fig. 9

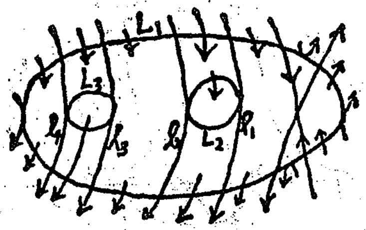
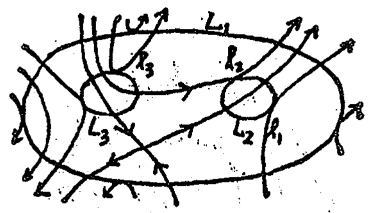


Fig. 10



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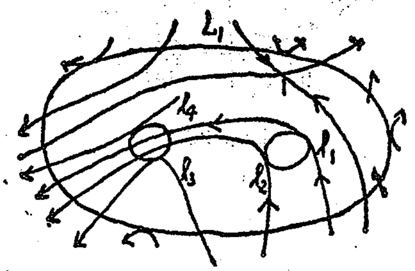


Fig. 12

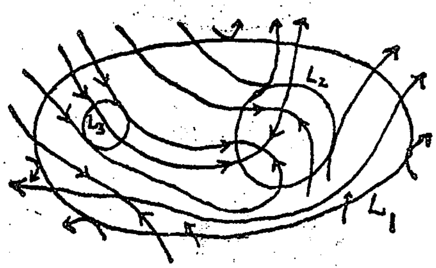


Fig. 13

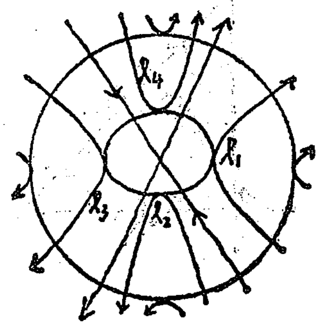


Fig. 14

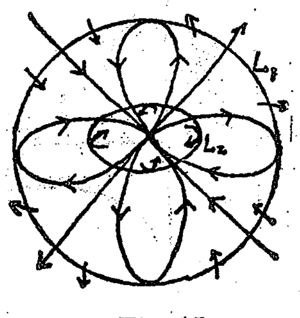


Fig. 15

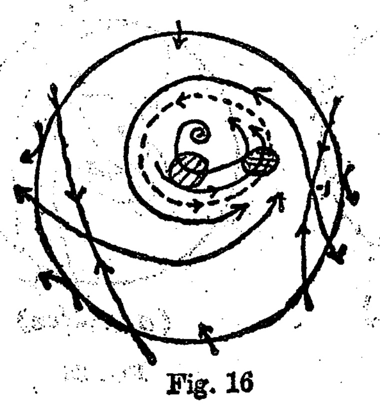


Fig. 16

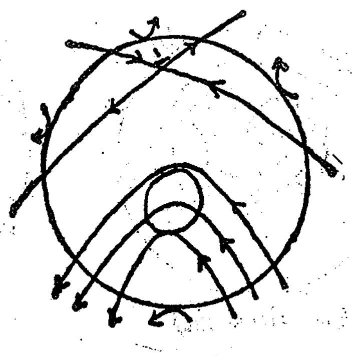


Fig. 17

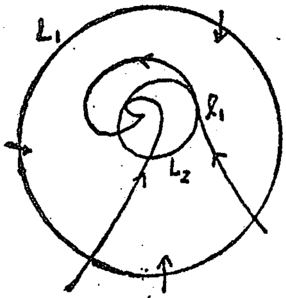
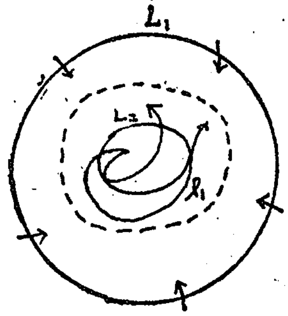


Fig. 18



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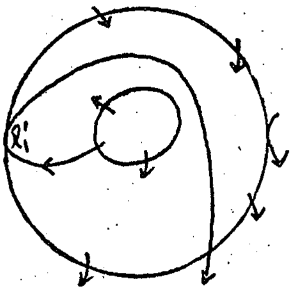


Fig. 20

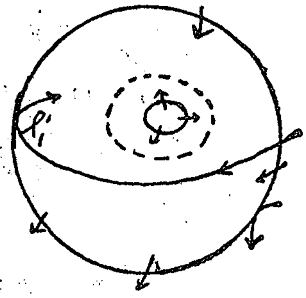


Fig. 21

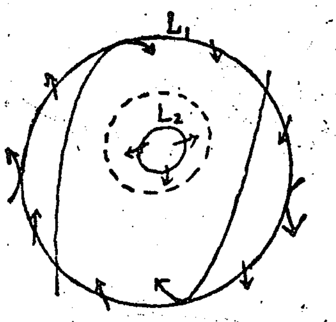
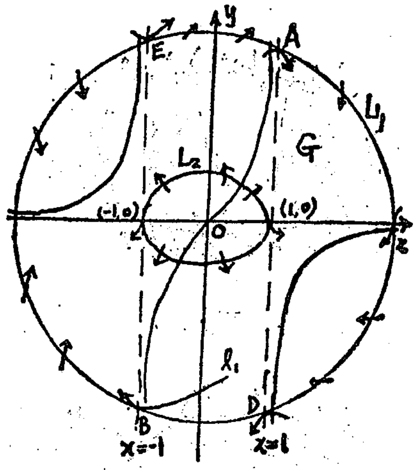


Fig. 22



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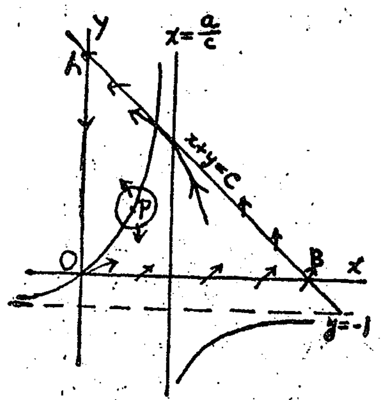

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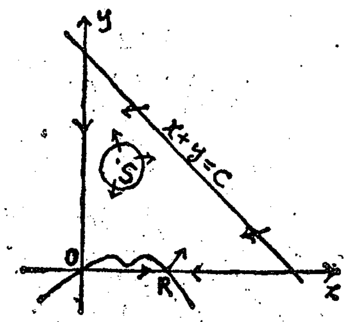
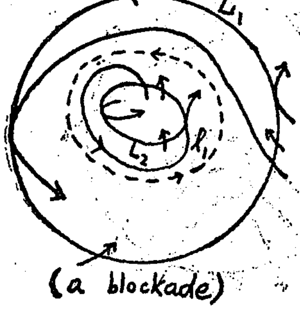


Fig. 25



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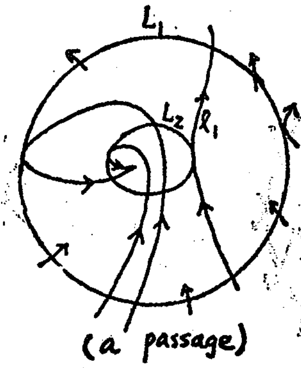

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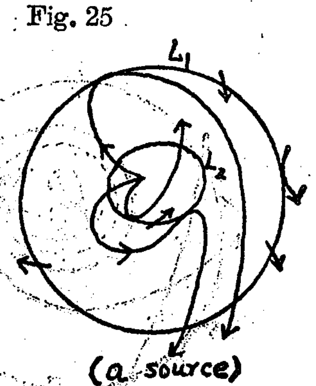

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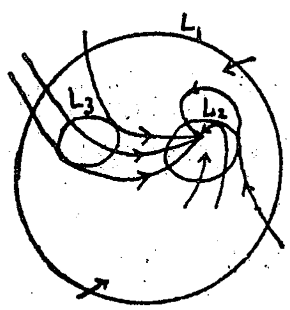

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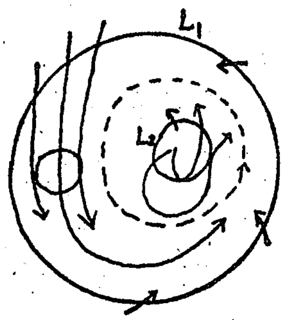

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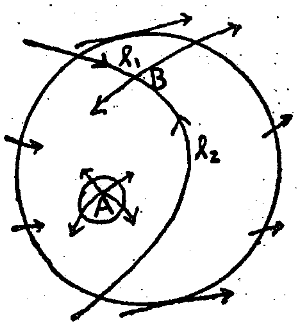

Fig. 31

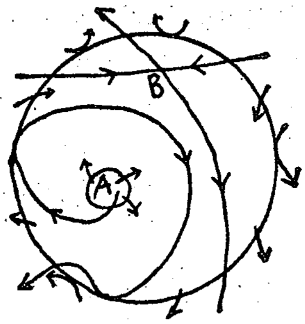

Fig. 32

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