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LIFE-SPAN OF CLASSICAL SOLUTIONS TO ONE DIMENSIONAL NONLINEAR WAVE EQUATIONS

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Abstract

By means of a simple and direct method, the authors obtain the sharp lower bound of the life-span of classical solutions to the Cauchy problem with small initial data for one dimensional fully nonlinear wave equations $u_{tt}-u_{xx}=F(u, Du, Du_x)$.

§1. Introduction

Consider the Cauchy problem with small initial data for one dimensional fully nonlinear wave equations

$$\begin{cases} u_{tt}-u_{xx}=F(u,\ Du,\ Du_x), & (1.1)\\ t=0:\ u=\varepsilon\varphi(x),\ u_t=\varepsilon\psi(x), & (1.2)\end{cases}$$

where

$$D=(\partial/\partial t,\ \partial/\partial x), \tag{1.3}$$

$$\varphi,\ \psi\in C_0^\infty(\mathbf{R}) \tag{1.4}$$

and $\varepsilon>0$ is a small parameter.

Let

$$\hat{\lambda}=(\lambda;\ (\lambda_i),\ i=0,\ 1;\ (\lambda_{ij}),\ i,\ j=0,\ 1,\ i+j\geqslant 1). \tag{1.5}$$

Suppose that in a neighbourhood of $\hat{\lambda}=0$, say for $|\hat{\lambda}|\leqslant 1$, the nonlinear term $F=F(\hat{\lambda})$ in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda})=O(|\hat{\lambda}|^{1+\alpha}), \tag{1.6}$$

where α is an integer$\geqslant 1$.

The aim of this paper is to study the life-span of classical solutions to (1.1)-(1.2) for all integers $\alpha\geqslant 1$. By definition, the life-span $\widetilde{T}(\varepsilon)=\sup\tau$ for all $\tau>0$ such that there exists a classical solution to (1.1)-(1.2) on $0\leqslant t\leqslant\tau$.

We outline our results as follows: There exists a small positive number ε_0 such that for any $\varepsilon\in(0,\ \varepsilon_0]$, the life-span has the following lower bounds:

(i) In the general case

$$\widetilde{T}(\varepsilon)\geqslant a\varepsilon^{-\alpha/2}; \tag{1.7}$$

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(ii) If

$$\int_{-\infty}^{+\infty} \psi(x)\, dx = 0, \tag{1.8}$$

then

$$\tilde{T}(\varepsilon) \geqslant a\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)}; \tag{1.9}$$

(iii) If

$$\partial_u^\beta F(0, 0, 0) = 0, \ \forall \alpha+1 \leqslant \beta \leqslant \beta_0, \tag{1.10}$$

then

$$\tilde{T}(\varepsilon) \geqslant a\varepsilon^{-\min(\beta_0/2, \alpha)}, \tag{1.11}$$

where a is a positive constant independent of ε, and β_0 is an integer $>\alpha$. When $\beta_0 \geqslant 2\alpha$, (1.11) becomes

$$\tilde{T}(\varepsilon) \geqslant a\varepsilon^{-\alpha}. \tag{1.12}$$

It must be pointed out that for all integers $\alpha \geqslant 1$ these lower bounds are all sharp due to P. D. Lax[1], Kong Dexing[2], H. Lindblad[3] and Y. Zhou[4, 5].

In order to prove the preceding results, by differentiation it is only necessary to consider the Cauchy problem for the following general kind of quasilinear wave equations

$$\begin{cases} u_{tt} - u_{xx} = b(u, Du)u_{xx} + 2a_0(u, Du)u_{tx} + F(u, Du), & (1.13) \\ t=0: \ u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), & (1.14) \end{cases}$$

where φ, ψ still satisfy (1.4), and for $|\tilde{\lambda}| \leqslant 1$, where $\tilde{\lambda} = (\lambda; (\lambda_i), i=0, 1)$, $b(\tilde{\lambda})$, $a_0(\tilde{\lambda})$ and $F(\tilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b(\tilde{\lambda}),\ a_0(\tilde{\lambda}) = O(|\tilde{\lambda}|^\alpha), \tag{1.15}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^{1+\alpha}) \tag{1.16}$$

and

$$a(\tilde{\lambda}) = 1 + b(\hat{\lambda}) \geqslant m_0, \tag{1.17}$$

where α is an integer $\geqslant 1$, m_0 is a positive constant. Moreover, condition (1.10) implies

$$\partial_u^\beta F(0, 0) = 0, \ \forall \alpha+1 \leqslant \beta \leqslant \beta_0. \tag{1.18}$$

§ 2. Preliminaries

For any integer $N \geqslant 0$, define

$$\|u(t, x)\|_{D, N, p} = \sum_{|k| \leqslant N} \|D^k u(t, x)\|_{L^p(\mathbb{R}^n)}, \ \forall t \geqslant 0 \tag{2.1}$$

for any function $u = u(t, x)$ such that all norms appearing on the right-hand side are bounded, where $1 \leqslant p \leqslant +\infty$, $k = (k_0, k_1, \cdots, k_n)$ is a multi-index, $|k| = k_0 + k_1 + \cdots + k_n$ and

$$D^k = \partial_t^{k_0} \partial_{x_1}^{k_1} \cdots\cdots \partial_{x_n}^{k_n}. \tag{2.2}$$

In this paper we take $n=1$, however, the following lemmas 2.1–2.3 hold for any $n \geqslant 1$.

Lemma 2.1. *Suppose that $F = F(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ with*

$$F(0)=0. \tag{2.3}$$

For any given integer $N\geqslant 0$, if a vector function $w=w(t, x)$ satisfies

$$\|w(t, \cdot)\|_{D,[N/2],\infty}\leqslant \nu_0, \ \forall t\in[0, T], \tag{2.4}$$

where $[\ \ ]$ stands for the integer part of a real number and ν_0 is a positive constant, and such that all norms appearing on the right-hand side below are bounded, we have

$$\|F(w(t, \cdot))\|_{D,N,p}\leqslant C(\nu_0)\|w(t, \cdot)\|_{D,N,p}, \ \forall t\in[0, T], \tag{2.5}$$

where $1\leqslant p\leqslant +\infty$ and $C(\nu_0)$ is a positive constant depending on ν_0.

Proof Similar to the proof of the corresponding lemma in [6].

Lemma 2.2. *Let $v=v(t, x)$ and $w=w(t, x)$ be functions such that all norms appearing on the right-hand side below are bounded. Then for any multi-index k with $|k|=N>0$ we have*

$$\|D^k(vw)\|_{L^r(\mathbb{R}^n)}\leqslant C\{\|v\|_{D,[N/2],p}\|w\|_{D,N,q}+\|Dv\|_{D,N-1,q}\|w\|_{D,\tilde{N},p}\} \tag{2.6}$$

and

$$\|D^k(vw)-vD^kw\|_{L^r(\mathbb{R}^n)}\leqslant C\{\|v\|_{D,[N/2],p}\|w\|_{D,N-1,q}+\|Dv\|_{D,N-1,q}\|w\|_{D,\tilde{N},p}\}, \tag{2.7}$$

where

$$\tilde{N}=[(N-1)/2], \tag{2.8}$$

$1\leqslant p, q, r\leqslant +\infty$ with

$$1/r=1/p+1/q \tag{2.9}$$

and C is a positive constant.

Proof By chain rule, we have

$$D^k(vw)=\sum_{|k_1|+|k_2|=N} C_{k_1k_2}D^{k_1}vD^{k_2}w, \tag{2.10}$$

where $C_{k_1k_2}$ are constants.

If $|k_1|\leqslant|k_2|$, then $|k_1|\leqslant[N/2]$ and by Hölder's inequality we get

$$\|D^{k_1}vD^{k_2}w\|_{L^r(\mathbb{R}^n)}\leqslant\|D^{k_1}v\|_{L^p(\mathbb{R}^n)}\|D^{k_2}w\|_{L^q(\mathbb{R}^n)}\leqslant\|v\|_{D,[N/2],p}\|w\|_{D,N,q}. \tag{2.11}$$

If $|k_1|>|k_2|$, then $|k_1|>0$, $|k_2|\leqslant\tilde{N}$, where $\tilde{N}$ is defined by (2.8). Hence we have

$$\|D^{k_1}vD^{k_2}w\|_{L^r(\mathbb{R}^n)}\leqslant\|D^{k_1}v\|_{L^q(\mathbb{R}^n)}\|D^{k_2}w\|_{L^p(\mathbb{R}^n)}\leqslant\|Dv\|_{D,N-1,q}\|w\|_{D,\tilde{N},p}. \tag{2.12}$$

The combination of (2.10)-(2.12) leads to (2.6). Similarly we can obtain (2.7).

Lemma 2.2. *Suppose that $G=G(w)$ is a sufficiently smooth function of $w=(w_1, \cdots, w_M)$ satisfying that if*

$$|w|\leqslant\nu_0, \tag{2.13}$$

then

$$G(w)=O(|w|^\alpha), \tag{2.14}$$

where α is an integer $\geqslant 1$. Then for any integer $N>0$, if a vector function $w=(w_1, \cdots, w_M)(t, x)$ satisfies (2.4) and such that all norms appearing on the right-hand side below are bounded, we have

$$\|DG(w)\|_{D,N-1,p}\leqslant C\|w\|^{\alpha-1}_{D,[N/2],\infty}\|Dw\|_{D,N-1,p}, \tag{2.15}$$

where $1\leqslant p\leqslant+\infty$ *and* C *is a positive constant.*

Proof Noticing that

$$\widetilde{N-1}=[N/2]-1, \tag{2.16}$$

by chain rule we can get (2.15) in a way similar to the proof of Lemmas 2.1 and 2.2.

Lemma 2.4. *Let* $w^{(0)}=w^{(0)}(t,x)$ *be the solution to the Cauchy problem*

$$\begin{cases} w^{(0)}_{tt}-w^{(0)}_{xx}=0, & (2.17)\\ t=0:\ w^{(0)}=\varphi(x),\ w^{(0)}_t=\psi(x), & (2.18)\end{cases}$$

where φ, ψ *satisfy* (1.4). *Then for any* p *with* $1\leqslant p\leqslant+\infty$, *we have*

$$\|w^{(0)}(t,\cdot)\|_{L^p(\mathbf{R})}\leqslant\|\varphi\|_{L^p(\mathbf{R})}+C(1+t)^{1/p}\|\psi\|_{L^1(\mathbf{R})},\ \forall t\geqslant 0, \tag{2.19}$$

where C *is a positive constant. Moreover, if* (1.8) *holds, we have*

$$\|w^{(0)}(t,\cdot)\|_{L^p(\mathbf{R})}\leqslant\|\varphi\|_{L^p(\mathbf{R})}+\|\Psi\|_{L^p(\mathbf{R})},\ \forall t\geqslant 0, \tag{2.20}$$

where $\Psi(x)\in C_0^\infty(R)$ *such that*

$$\Psi'(x)=\psi(x),\ \forall x\in\mathbf{R}. \tag{2.21}$$

Proof By the well-known D'Alembert's formula, we have

$$w^{(0)}(t,x)=[\varphi(x-t)+\varphi(x+t)]/2+\frac{1}{2}\int_{x-t}^{x+t}\psi(\xi)\,d\xi. \tag{2.22}$$

Then it is easy to see that

$$\|w^{(0)}(t,\cdot)\|_{L^\infty(\mathbf{R})}\leqslant\|\varphi\|_{L^\infty(\mathbf{R})}+\frac{1}{2}\|\psi\|_{L^1(\mathbf{R})} \tag{2.23}$$

and

$$\|w^{(0)}(t,\cdot)\|_{L^1(\mathbf{R})}\leqslant\|\varphi\|_{L^1(\mathbf{R})}+t\|\psi\|_{L^1(\mathbf{R})}. \tag{2.24}$$

Hence (2.19) comes from (2.23)—(2.24) by interpolation.

Furthermore, if (1.9) holds, (2.22) can be rewritten as

$$w^{(0)}(t,x)=[\varphi(x-t)+\varphi(x+t)]/2+[\Psi(x+t)-\Psi(x-t)]/2 \tag{2.25}$$

and we get (2.20) immediately.

Lemma 2.5. *Let* $w(t,x)$ *be the solution to the Cauchy problem*

$$\begin{cases} w_{tt}-w_{xx}=F(t,x), & (2.26)\\ t=0:\ w=0,\ w_t=0. & (2.27)\end{cases}$$

Then for any p *with* $1\leqslant p\leqslant+\infty$ *we have*

$$\|w(t,\cdot)\|_{L^p(\mathbf{R})}\leqslant C\int_0^t(1+t-\tau)^{1/p}\|F(\tau,\cdot)\|_{L^1(\mathbf{R})}d\tau,\ \forall t\geqslant 0, \tag{2.28}$$

where C *is a positive constant, and for any* p *with* $1\leqslant p\leqslant+\infty$,

$$\|Dw(t,\cdot)\|_{L^p(\mathbf{R})}\leqslant\int_0^t\|F(\tau,\cdot)\|_{L^p(\mathbf{R})}d\tau,\ \forall t\geqslant 0. \tag{2.29}$$

Proof By Duhamal principle, we have

$$w(t,x)=\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}F(\tau,\xi)\,d\xi\,d\tau. \tag{2.30}$$

Hence (2.28) and (2.29) follow from (2.19) and a direct estimation respectively.

§ 3. Lower Bounds (1.7) and (1.9) of the Life-Span of Classical Solutions to (1.13)-(1.14)

By the Sobolev embedding theorem, there exists $E_0>0$ so small that

$$\|f\|_{L^\infty(\mathbf{R})}\leqslant 1,\ \forall f\in H^1(\mathbf{R}),\ \|f\|_{H^1(\mathbf{R})}\leqslant E_0. \tag{3.1}$$

For any given integer $S\geqslant 4$, any given positive real numbers $E(\leqslant E_0)$ and T, introduce the following set of functions:

$$X_{S,E,T}=\{v(t,x)\mid D_{S,T}(v)\leqslant E,\ \partial_t^l v(0,x)=u_l^{(0)}(x)\ (l=0,1,\cdots,S)\}, \tag{3.2}$$

where

$$D_{S,T}(v)=\sup_{0\leqslant t\leqslant T}\|v(t,\cdot)\|_{L^\infty(\mathbf{R})}+\sup_{0\leqslant t\leqslant T}g^{-1}(t)\|v(t,\cdot)\|_{L^{1+\alpha}(\mathbf{R})}$$
$$+\sup_{0\leqslant t\leqslant T}\|Dv(t,\cdot)\|_{D,S,2} \tag{3.3}$$

in which

$$g(t)=\begin{cases}(1+t)^{1/(1+\alpha)}, & \text{if } \int\psi\neq 0;\\ 1, & \text{if } \int\psi=0.\end{cases} \tag{3.4}$$

Moreover, $u_0^{(0)}=\varepsilon\varphi(x)$, $u_1^{(0)}=\varepsilon\psi(x)$ and $u_l^{(0)}(x)$ $(l=2,\cdots,S)$ are the values of $\partial_t^l u(t,x)$ at $t=0$ formally determined from the equation (1.13) and the initial data (1.14). Obviously, $u_l^{(0)}$ $(l=0,1,\cdots,S)$ are all sufficiently smooth functions with compact support.

It is easy to prove the following

Lemma 3.1. *Endowed with the metric*

$$\rho(\bar{v},\bar{\bar{v}})=D_{S,T}(\bar{v}-\bar{\bar{v}}),\ \forall\bar{v},\ \bar{\bar{v}}\in X_{S,E,T}, \tag{3.5}$$

$X_{S,E,T}$ is a nonempty complete metric space, provided that $\varepsilon>0$ is suitably small.

Noticing that $S\geqslant 4$ and the definition of $X_{S,E,T}$, by the Sobolev embedding theorem

$$H^1(\mathbf{R})\subset L^\infty(\mathbf{R}) \text{ with continuous embedding} \tag{3.6}$$

and interpolation, we have for any $v\in X_{S,E,T}$

$$\|v(t,\cdot)\|_{D,[S/2]+2,\infty}\leqslant CE,\ \forall t\in[0,T] \tag{3.7}$$

and for any p with $2\leqslant p\leqslant+\infty$,

$$\|Dv(t,\cdot)\|_{L^p(\mathbf{R})}\leqslant CE,\ \forall t\in[0,T], \tag{3.8}$$

where C is a positive constant.

Let $\tilde{X}_{S,E,T}$ be the subset of $X_{S,E,T}$ composed of all elements in $X_{S,E,T}$ with compact support in the variable x for any fixed $t\in[0,T]$.

Theorem 3.1. *Under assumptions* (1.4) *and* (1.15)-(1.18), *for any given integer $S\geqslant 4$, there exist positive constants ε_0 and C_0 with $C_0\varepsilon_0\leqslant E_0$ such that for any $\varepsilon\in(0,\varepsilon_0]$, there exists a positive number $T=T(\varepsilon)$ such that the Cauchy problem* (1.13) -(1.14) *admits on $[0,T(\varepsilon)]$ a unique classical solution $u\in\tilde{X}_{S,C_0\varepsilon,T(\varepsilon)}$, where $T(\varepsilon)$*

can be chosen as follows:

$$T(\varepsilon)=\begin{cases} a\varepsilon^{-\alpha/2}-1, & \text{if } \int \psi \neq 0, \\ a\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)}-1, & \text{if } \int \psi = 0, \end{cases} \tag{3.9}$$

where a is a positive constant.

Moreover, with eventual modification on a set with zero measure in the variable t, we have

$$u\in C([0, T(\varepsilon)]; H^{S+1}(\mathbf{R})), \tag{3.10}$$

$$u_t\in C([0, T(\varepsilon)]; H^{S}(\mathbf{R})), \tag{3.11}$$

$$u_{tt}\in C([0, T(\varepsilon)]; H^{S-1}(\mathbf{R})). \tag{3.12}$$

In order to prove Theorem 3.1, we define a map

$$M: v\to u=Mv \tag{3.13}$$

by solving the following Cauchy problem for linear wave equations for any $v\in \widetilde{X}_{S,E,T}$:

$$\begin{cases} u_{tt}-u_{xx}=\hat{F}(v, Dv, Du_x)\triangleq b(v, Dv)u_{xx}+2a_0(v, Dv)u_{tx}+F(v, Dv), & (3.14) \\ t=0:\ u=\varepsilon\varphi(x),\ u_t=\varepsilon\psi(x). & (3.15) \end{cases}$$

It is not difficult to get the following two lemmas.

Lemma 3.2. *For any $v\in \widetilde{X}_{S,E,T}$ we have, with eventual modification on a set with zero measure in t,*

$$u=Mv\in C([0, T]; H^{S+1}(\mathbf{R})), \tag{3.16}$$

$$u_t\in C([0, T]; H^{S}(\mathbf{R})), \tag{3.17}$$

$$u_{tt}\in L^\infty(0, T; H^{S-1}(\mathbf{R})). \tag{3.18}$$

Moreover, for any fixed $t\in[0, T]$, $u=u(t, x)$ has compact support in the variable x.

Lemma 3.3. *For $u=u(t, x)=Mv$, $\partial_t^l u(0, x)$ $(l=0, 1, \cdots, S+1)$ are independent of $v\in \widetilde{X}_{S,E,T}$ and*

$$\partial_t^l u(0, x)=u_l^{(0)}(x)\ (l=0, 1, \cdots, S). \tag{3.19}$$

Furthermore,

$$\|u(0, \cdot)\|_{D,S+1,p}\leqslant C\varepsilon, \tag{3.20}$$

where $1\leqslant p\leqslant +\infty$, C is a positive constant and $\|u(0, \cdot)\|_{D,S+1,p}$ denotes the value of $\|u(t, \cdot)\|_{D,S+1,p}$ at $t=0$.

Lemma 3.4. *Under the assumptions of Theorem 3.1, for any $v\in \widetilde{X}_{S,E,T}$, $u=Mv$ satisfies*

$$D_{S,T}(u)\leqslant C_1[\varepsilon+(R+\sqrt{R})(E+D_{S,T}(u))], \tag{3.21}$$

where C_1 is a positive constant and

$$R=R(E, T)=\begin{cases} E^\alpha(1+T)^2, & \text{if } \int \psi\neq 0, \\ E^\alpha(1+T)^{(2+\alpha)/(1+\alpha)}, & \text{if } \int\psi=0. \end{cases} \tag{3.22}$$

Proof We first estimate $\|u(t, \cdot)\|_{L^\infty(\mathbf{R})}$.

We can write

$$u(t, x) = \varepsilon w^{(0)}(t, x) + w(t, x), \tag{3.23}$$

where $w^{(0)}(t, x)$ is the solution to (2.17)-(2.18), while $w(t, x)$ is the solution to (2.26)-(2.27) with $F = \hat{F}(v, Dv, Du_x)$.

By (2.23), we have

$$\|w^{(0)}(t, \cdot)\|_{L^\infty(\mathbf{R})} \leqslant C, \tag{3.24}$$

here and hereafter, C denotes a positive constant. On the other hand, by (2.28) we have

$$\|w(t, \cdot)\|_{L^\infty(\mathbf{R})} \leqslant C \int_0^t \|\hat{F}(v, Dv, Du_x)(\tau, \cdot)\|_{L^1(\mathbf{R})} d\tau. \tag{3.25}$$

By Lemma 2.1 and Hölder's inequality, noting (3.6), (3.8) and the definition of $X_{S,E,T}$, we have

$$\begin{aligned} &\|(b(v, Dv)u_{xx} + 2a_0(v, Dv)u_{tx})(\tau, \cdot)\|_{L^1(\mathbf{R})} \\ &\leqslant C\|(v, Dv)\|^{\alpha}_{L^{1+\alpha}(\mathbf{R})} \|Du_x\|_{L^{1+\alpha}(\mathbf{R})} \\ &\leqslant CE^\alpha g^\alpha(\tau) \|Du_x\|^{1-2/(1+\alpha)}_{L^\infty(\mathbf{R})} \|Du_x\|^{2/(1+\alpha)}_{L^2(\mathbf{R})} \\ &\leqslant CE^\alpha g^\alpha(\tau) \|Du_x\|_{H^1(\mathbf{R})} \leqslant CE^\alpha g^\alpha(\tau) D_{S,T}(u), \end{aligned} \tag{3.26}$$

$$\|F(v, Dv)(\tau, \cdot)\|_{L^1(\mathbf{R})} \leqslant C\|(v, Dv)\|^{1+\alpha}_{L^{1+\alpha}(\mathbf{R})} \leqslant CE^{1+\alpha} g^{1+\alpha}(\tau). \tag{3.27}$$

Hence, noting (3.4) it is easy to see that

$$\|w(t, \cdot)\|_{L^\infty(\mathbf{R})} \leqslant CE^\alpha (1+t)^k (E + D_{S,T}(u)), \tag{3.28}$$

where

$$k = \begin{cases} 2, & \text{if } \int \psi \neq 0, \\ 1, & \text{if } \int \psi = 0. \end{cases} \tag{3.29}$$

Noticing (3.22), the combination of (3.24) and (3.28) yields

$$\sup_{0 \leqslant t \leqslant T} \|u(t, \cdot)\|_{L^\infty(\mathbf{R})} \leqslant C\{\varepsilon + R(E, T)(E + D_{S,T}(u))\}. \tag{3.30}$$

We then estimate $\|u(t, \cdot)\|_{L^{1+\alpha}(\mathbf{R})}$.

By (2.19)-(2.20) and noting (3.4), we have

$$\|w^{(0)}(t, \cdot)\|_{L^{1+\alpha}(\mathbf{R})} \leqslant Cg(t). \tag{3.31}$$

Moreover, still by (2.28), using (3.26)-(3.27) and noting (3.4) and (3.22) we get

$$\begin{aligned} \|w(t, \cdot)\|_{L^{1+\alpha}(\mathbf{R})} &\leqslant C \int_0^t (1+t-\tau)^{1/(1+\alpha)} \|\hat{F}(v, Dv, Du_x)(\tau, \cdot)\|_{L^1(\mathbf{R})} d\tau \\ &\leqslant CE^\alpha (1+t)^{(2+\alpha)/(1+\alpha)} g^{1+\alpha}(t)(E + D_{S,T}(u)) \\ &\leqslant Cg(t) R(E, T)(E + D_{S,T}(u)). \end{aligned} \tag{3.32}$$

Hence, by (3.31) and (3.32) it turns out that

$$\sup_{0 \leqslant t \leqslant T} g^{-1}(t) \|u(t, \cdot)\|_{L^{1+\alpha}(\mathbf{R})} \leqslant C\{\varepsilon + R(E, T)(E + D_{S,T}(u))\}. \tag{3.33}$$

Finally, we estimate $\|Du(t, \cdot)\|_{D,S,2}$.

For any double-index $k = (k_1, k_2)$ with $0 \leqslant |k| \leqslant S$, by applying D^k to both sides of (3.14), we get the following energy integral formula

$$\|D^k u_t(t,\cdot)\|^2_{L^2(\mathbf{R})}+\int_{\mathbf{R}} a(v,Dv)(t,\cdot)(D^k u_x(t,\cdot))^2dx$$
$$=\|D^k u_t(0,\cdot)\|^2_{L^2(\mathbf{R})}+\int_{\mathbf{R}} a(v,Dv)(0,\cdot)(D^k u_x(0,\cdot))^2dx$$
$$+\int_0^t\int_{\mathbf{R}}\frac{\partial b(v,Dv)(\tau,\cdot)}{\partial\tau}(D^k u_x(\tau,\cdot))^2dx\,d\tau$$
$$-2\int_0^t\int_{\mathbf{R}}\frac{\partial b(v,Dv)(\tau,\cdot)}{\partial x}D^k u_x(\tau,\cdot)D^k u_\tau(\tau,\cdot)dx\,d\tau$$
$$-2\int_0^t\int_{\mathbf{R}}\frac{\partial a_0(v,Dv)(\tau,\cdot)}{\partial x}(D^k u_\tau(\tau,\cdot))^2dx\,d\tau$$
$$+2\int_0^t\int_{\mathbf{R}}G_k(\tau,\cdot)D^k u_\tau(\tau,\cdot)dx\,d\tau+2\int_0^t\int_{\mathbf{R}}g_k(\tau,\cdot)D^k u_\tau(\tau,\cdot)dx\,d\tau$$
$$=\|D^k u_t(0,\cdot)\|^2_{L^2(\mathbf{R})}+\int_{\mathbf{R}} a(v,Dv)(0,\cdot)(D^k u_x(0,\cdot))^2dx$$
$$+\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V},\tag{3.34}$$

in which the function $a(\cdot)$ is defined by (1.17),

$$G_k=D^k(b(v,Dv)u_{xx})-b(v,Dv)D^k u_{xx}$$
$$+2[D^k(a_0(v,Dv)u_{tx})-a_0(v,Dv)D^k u_{tx}],\tag{3.35}$$
$$g_k=D^kF(v,Dv).\tag{3.36}$$

Noting (1.15) and (3.7) it is easily seen that

$$|\mathrm{I}|,|\mathrm{II}|,|\mathrm{III}|\leqslant CE^\alpha(1+t)D^2_{S,T}(u)\leqslant CR(E,T)D^2_{S,T}(u).\tag{3.37}$$

By Lemmas 2.1–2.3 and noting (3.6), we get

$$\|G_k(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant CE^\alpha D_{S,T}(u)\tag{3.38}$$

and for $|k|>0$,

$$\|g_k(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant CE^{1+\alpha}.\tag{3.39}$$

Moreover, noting (1.16) and (3.7) we have

$$\|g_0(\tau,\cdot)\|_{L^2(\mathbf{R})}=\|F(v,Dv)(\tau,\cdot)\|_{L^2(\mathbf{R})}$$
$$\leqslant C\|(v,Dv)(\tau,\cdot)\|^{1+\alpha}_{L^{2(1+\alpha)}(\mathbf{R})}.\tag{3.40}$$

By Hölder's inequality and the definition of $X_{S,E,T}$, we have

$$\|v(\tau,\cdot)\|_{L^{2(1+\alpha)}(\mathbf{R})}\leqslant\|v\|^{1/2}_{L^\infty(\mathbf{R})}\|v\|^{1/2}_{L^{1+\alpha}(\mathbf{R})}\leqslant CE(g(\tau))^{1/2},\tag{3.41}$$

then noting (3.8) it turns out that

$$\|g_0(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant CE^{1+\alpha}(g(\tau))^{(1+\alpha)/2}.\tag{3.42}$$

By means of (3.38)-(3.39) and (3.42), noting (3.4) and (3.22) we obtain

$$|\mathrm{IV}|\leqslant CE^\alpha(1+t)D^2_{S,T}(u)\leqslant CR(E,T)D^2_{S,T}(u),\tag{3.43}$$
$$|\mathrm{V}|\leqslant CE^{1+\alpha}(g(t))^{(1+\alpha)/2}(1+t)D_{S,T}(u)\leqslant CR(E,T)ED_{S,T}(u).\tag{3.44}$$

By (3.37) and (3.43)—(3.44), and noting (1.17) and (3.20), it follows from (3.34) that

$$\sup_{0\leqslant t\leqslant T}\|Du(t,\cdot)\|_{D,S,2}\leqslant C\{\varepsilon+\sqrt{R(E,T)}(E+D_{S,T}(u))\}.\tag{3.45}$$

The combination of (3.30), (3.33) and (3.45) yields (3.21).

Lemma 3.5. *Let $\bar v,\bar{\bar v}\in\tilde X_{S,E,T}$. If $\bar u=M\bar v$ and $\bar{\bar u}=M\bar{\bar v}$ also satisfy $\bar u,\bar{\bar u}\in\tilde X_{S,E,T}$, then*

$$D_{S-1,T}(\bar{u}-\bar{\bar{u}})\leqslant C_2(R+\sqrt{R})\left(D_{S-1,T}(\bar{u}-\bar{\bar{u}})+D_{S-1,T}(\bar{v}-\bar{\bar{v}})\right), \tag{3.46}$$

where C_2 is a positive constant and $R=R(E, T)$ is still defined by (3.22).

Proof Let $u^*=\bar{u}-\bar{\bar{u}}$, $v^*=\bar{v}-\bar{\bar{v}}$. By the definition of the map M, we have

$$\begin{cases} u^*_{tt}-a(\bar{v}, D\bar{v})u^*_{xx}-2a_0(\bar{v}, D\bar{v})u^*_{tx}=F^*, & (3.47)\\ t=0:\ u^*=u^*_t=0, & (3.48)\end{cases}$$

where the function $a(\cdot)$ is defined by (1.17) and

$$F^*=(b(\bar{v}, D\bar{v})-b(\bar{\bar{v}}, D\bar{\bar{v}}))\bar{\bar{u}}_{xx}+2(a_0(\bar{v}, D\bar{v})-a_0(\bar{\bar{v}}, D\bar{\bar{v}}))u^*_{t,x}$$
$$+F(\bar{v}, D\bar{v})-F(\bar{\bar{v}}, D\bar{\bar{v}}). \tag{3.49}$$

First, in a way similar to the proof of (3.30) and (3.33) we can get respectively

$$\sup_{0\leqslant t\leqslant T}\|u^*(t,\cdot)\|_{L^\infty(\mathbf{R})}\leqslant CR(E, T)\left(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)\right), \tag{3.50}$$

and

$$\sup_{0\leqslant t\leqslant T} g^{-1}(t)\|u^*(t,\cdot)\|_{L^{1+\alpha}(\mathbf{R})}\leqslant CR(E, T)\left(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)\right). \tag{3.51}$$

Then we estimate $\|Du^*(t,\cdot)\|_{D,S-1,2}$.

For any double-index $k=(k_0, k_1)$ with $|k|\leqslant S-1$, we have

$$\|D^k u^*_t(t,\cdot)\|^2_{L^2(\mathbf{R})}+\int_{\mathbf{R}} a(\bar{v}, D\bar{v})(t,\cdot)(D^k u^*_x(t,\cdot))^2dx$$
$$=\int_0^t\int_{\mathbf{R}}\frac{\partial b(\bar{v}, D\bar{v})(\tau,\cdot)}{\partial\tau}(D^k u^*_x(\tau,\cdot))^2dx\,d\tau$$
$$-2\int_0^t\int_{\mathbf{R}}\frac{\partial b(\bar{v}, D\bar{v})(\tau,\cdot)}{\partial x}D^k u^*_x(\tau,\cdot)D^k u^*_\tau(\tau,\cdot)\,dx\,d\tau$$
$$-2\int_0^t\int_{\mathbf{R}}\frac{\partial a_0(\bar{v}, D\bar{v})(\tau,\cdot)}{\partial x}(D^k u^*_\tau(\tau,\cdot))^2dx\,d\tau$$
$$+2\int_0^t\int_{\mathbf{R}}\bar{G}_k(\tau,\cdot)D^k u^*_\tau(\tau,\cdot)dx\,d\tau+2\int_0^t\int_{\mathbf{R}}\bar{g}_k(\tau,\cdot)D^k u^*_\tau(\tau,\cdot)dx\,d\tau$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}, \tag{3.52}$$

where

$$\bar{G}_k=D^k(b(\bar{v}, D\bar{v})u^*_{xx})-b(\bar{v}, D\bar{v})D^k u^*_{xx}+2(D^k(a_0(\bar{v}, D\bar{v})u^*_{tx})$$
$$-a_0(\bar{v}, D\bar{v})D^k u^*_{tx}), \tag{3.53}$$

$$\bar{g}_k=D^k F^*. \tag{3.54}$$

As in the proof of Lemma 3.4, we can get

$$|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}|, |\mathrm{IV}|\leqslant CE^\alpha(1+t)D^2_{S-1,T}(u^*)\leqslant CR(E, T)D^2_{S-1,T}(u^*). \tag{3.55}$$

In a similar way we get that for $|k|>0$

$$\|\bar{g}_k(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant CE^\alpha D_{S-1,T}(v^*) \tag{3.56}$$

and

$$\|\bar{g}_0(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant CE^\alpha(g(\tau))^{(1+\alpha)/2}D_{S-1,T}(v^*), \tag{3.57}$$

then

$$|\mathrm{V}|\leqslant CE^\alpha(g(t))^{(1+\alpha)/2}(1+t)D_{S-1,T}(u^*)D_{S-1,T}(v^*)$$
$$\leqslant CR(E, T)D_{S-1,T}(u^*)D_{S-1,T}(v^*). \tag{3.58}$$

Thus, we obtain

$$\sup_{0\leqslant t\leqslant T}\|Du^*(t,\cdot)\|_{D,S-1,T}\leqslant C\sqrt{R(E, T)}\left(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)\right). \tag{3.59}$$

The combination of (3.50)–(3.51) and (3.59) leads to the desired conclusion

(3.46),

By means of Lemmas 3.4—3.5, just as in [7] it is easy to prove that there exists $C_0>0$ such that the map M possesses a unique fixed point in $X_{S,C_0\varepsilon,T(\varepsilon)}$, provided that ε is suitably small and $T(\varepsilon)$ is given by (3.9). This finishes the proof of Theorem 3.1, then the lower bounds (1.7) and (1.9) for the life-span are verified.

§ 4. Lower Bound (1.11) of the Life-Span of Classical Solutions to (1.13)-(1.14)

In this section we consider the special case that (1.18) holds. The discussion is similar to the preceding section, therefore we only point out some essentirl points here.

In this case, no matter whether $\int \psi\, dx$ equals zero or not, instead of (3.3) we take

$$D_{S,T}(v)=\sup_{0\le t\le T}\|v(t,\cdot)\|_{L^\infty(\mathbf{R})}+\sup_{0\le t\le T}(1+t)^{-1/(1+\beta_0)}\|v(t,\cdot)\|_{L^{1+\beta_0}(\mathbf{R})}$$
$$+\sup_{0\le t\le T}\|Dv(t,\cdot)\|_{D,S,2},\tag{4.1}$$

then we have

Theorem 4.1. *Under the assumptions of Theorem 3.1, if (1.18) holds, we have the same conclusion as in Theorem 3.1 provided that, instead of (3.9), $T(\varepsilon)$ is given by*

$$T(\varepsilon)=a\varepsilon^{-\min(\beta_0/2,\alpha)}-1,\tag{4.2}$$

where a is a positive constant.

In order to obtain Theorem 4.1, it is only necessary to prove the following two lemmas.

Lemma 4.1. *Under the assumptions of Theorem 4.1, for any $v\in\widetilde{X}_{S,E,T}$, $u=Mv$ satisfies*

$$D_{S,T}(u)\le C_1[\varepsilon+(R^2+R+\sqrt{R})(E+D_{S,T}(u))],\tag{4.3}$$

where C_1 is a positive constant and

$$R=R(E,T)=E^{\min(\beta_0/2,\alpha)}(1+T).\tag{4.4}$$

Proof Noting (1.18), $\hat{F}$ can be rewritten as

$$\begin{aligned}\hat{F}(v,Dv,Du_x)&=(b(v,0)u_x)_x-b_x(v,0)u_x+(b(v,Dv)-b(v,0))u_{xx}\\&\quad+2(a_0(v,0)u_x)_t-2a_{0t}(v,0)u_x+2(a_0(v,Dv)-a_0(v,0))u_{xt}\\&\quad+(F(v,Dv)-F(v,0)-F_{Dv}(v,0)Dv)+F(v,0)+F_{Dv}(v,0)Dv\\&=\sum_{i=0}^{1}\partial_iG_i(v,u_x)+\sum_{i=0}^{1}A_i(v)v_{x_i}u_x+\sum_{i,j=0}^{1}B_{ij}(v,Dv)v_{x_i}u_{xx_j}\\&\quad+\sum_{i,j=0}^{1}C_{ij}(v,Dv)v_{x_i}v_{x_j}+F(v,0),\end{aligned}\tag{4.5}$$

where $(x_0,x_1)=(t,x)$, $(\partial_0,\partial_1)=(\partial_t,\partial_x)=D$ and in a neighbourhood of the origin we have

$$G_i(\bar{\lambda}) = O(|\bar{\lambda}|^{1+\alpha}),\ i=0,\ 1,\ \bar{\lambda}=(\lambda,\ \lambda_1) \tag{4.6}$$

and $G_i(\bar{\lambda})$ is affine in λ_1,

$$A_i(\lambda) = O(|\lambda|^{\alpha-1}),\ i=0,\ 1, \tag{4.7}$$

$$B_{ij}(\tilde{\lambda}),\ C_{ij}(\tilde{\lambda}) = O(|\tilde{\lambda}|^{\alpha-1}),\ i,\ j=0,\ 1,\ \tilde{\lambda}=(\lambda,\ \lambda_0,\ \lambda_1), \tag{4.8}$$

$$F(\lambda,\ 0) = O(|v|^{1+\beta_0}). \tag{4.9}$$

Hence, the solution $u=Mv$ to (3.14)-(3.15) can be expressed as

$$u = \varepsilon w^{(0)} + \sum_{i=0}^{1} \partial_i w^{(i)} - u^{(0)} + u^{(1)} + u^{(2)}, \tag{4.10}$$

where $w^{(0)}$ is still defined by (2.17)-(2.18), while $w^{(i)}(i=0,\ 1)$, $u^{(0)}$, $u^{(1)}$ and $u^{(2)}$ satisfy respectively

$$w^{(i)}_{tt} - w^{(i)}_{xx} = G_i(v,\ u_x),\ (i=0,\ 1), \tag{4.11}$$

$$u^{(0)}_{tt} - u^{(0)}_{xx} = 0, \tag{4.12}$$

$$u^{(1)}_{tt} - u^{(1)}_{xx} = \sum_{i=0}^{1} A_i(v) v_{x_i} u_x + \sum_{i,j=0}^{1} B_{ij}(v,\ Dv) v_{x_i} u_{xx_j} + \sum_{i,j=0}^{1} C_{ij}(v,\ Dv) v_{x_i} v_{x_j} \tag{4.13}$$

and

$$u^{(2)}_{tt} - u^{(2)}_{xx} = F(v,\ 0) \tag{4.14}$$

with the zero initial data for $w^{(i)}(i=0,\ 1)$, $u^{(1)}$ and $u^{(2)}$ and the following initial data for $u^{(0)}$:

$$t=0:\ u^{(0)}=0,\ u^{(0)}_t = G_0(v,\ u_x)(0,\ x). \tag{4.15}$$

We first estimate $\|u(t,\ \cdot)\|_{L^\infty(\mathbb{R})}$.

By (2.29) and noting (4.6) and (3.6), we have, for $i=0,\ 1$,

$$\begin{aligned}\|\partial_i w^{(i)}(t,\ \cdot)\|_{L^\infty(\mathbb{R})} &\le \int_0^t \|G_i(v,\ u_x)(\tau,\ \cdot)\|_{L^\infty(\mathbb{R})} d\tau \le CE^\alpha(1+t)(E+D_{S,T}(u)) \\ &\le CR(E,\ T)(E+D_{S,T}(u)),\end{aligned} \tag{4.16}$$

here and in what follows $R(E,\ T)$ is defined by (4.4).

By Lemma 3.3 and (2.23), we get

$$\|u^{(0)}(t,\ \cdot)\|_{L^\infty(\mathbb{R})} \le C\varepsilon. \tag{4.17}$$

Using (2.28), it is easy to see that

$$\begin{aligned}\|u^{(1)}(t,\ \cdot)\|_{L^\infty(\mathbb{R})} \le C\Big\{&\int_0^t \sum_{i=1}^{1} \|A_i(v) v_{x_i} u_x(\tau,\ \cdot)\|_{L^1(\mathbb{R})} d\tau \\ &+ \int_0^t \sum_{i,j=0}^{1} \|B_{ij}(v,\ Dv) v_{x_i} u_{xx_j}(\tau,\ \cdot)\|_{L^1(\mathbb{R})} d\tau \\ &+ \int_0^t \sum_{i,j=0}^{1} \|C_{ij}(v,\ Dv) v_{x_i} v_{x_j}(\tau,\ \cdot)\|_{L^1(\mathbb{R})} d\tau\Big\} \\ \le\ & CE^\alpha(1+t)(E+D_{S,T}(u)) \\ \le\ & CR(E,\ T)(E+D_{S,T}(u)).\end{aligned} \tag{4.18}$$

Again by (2.28) and the definition of $\tilde{X}_{S,E,T}$, noting (4.9) we get

$$\begin{aligned}\|u^{(2)}(t,\ \cdot)\|_{L^\infty(\mathbb{R})} &\le C\int_0^t \|F(v,\ 0)(\tau,\ \cdot)\|_{L^1(\mathbb{R})} d\tau \\ &\le C\int_0^t \|v(\tau,\ \cdot)\|^{1+\beta_0}_{L^{1+\beta_0}(\mathbb{R})} d\tau \le CE^{1+\beta_0}(1+t)^2\end{aligned}$$

$$\leqslant CER^2(E, T). \tag{4.19}$$

The combination of (4.16)-(4.19) and (3.24) yields

$$\sup_{0\leqslant t\leqslant T} \|u(t, \cdot)\|_{L^\infty(\mathbf{R})} \leqslant C\{\varepsilon + (R^2(E, T) + R(E, T))(E + D_{S,T}(u))\}. \tag{4.20}$$

We then estimate $\|u(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})}$.

Similar to (4.16), using (2.29) we obtain, for $i=0, 1$,

$$\begin{aligned}\|\partial_i w^{(i)}(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} &\leqslant \int_0^t \|G_i(v, u_\alpha)(\tau, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} d\tau \\ &\leqslant C\int_0^t \|v(\tau, \cdot)\|^\alpha_{L^\infty(\mathbf{R})} \|(v, u_\alpha)(\tau, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} d\tau \\ &\leqslant CE^\alpha (1+t)^{1+1/(1+\beta_0)}(E + D_{S,T}(u)) \\ &\leqslant C(1+t)^{1/(1+\beta_0)} R(E, T)(E + D_{S,T}(u)).\end{aligned} \tag{4.21}$$

Similarly, by (2.19) and (2.28) we can get

$$\|w^{(0)}(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})},\ \|u^{(0)}(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} \leqslant C\varepsilon(1+t)^{1/(1+\beta_0)}, \tag{4.22}$$

$$\|u^{(1)}(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} \leqslant C(1+t)^{1/(1+\beta_0)} R(E, T)(E + D_{S,T}(u)) \tag{4.23}$$

and

$$\|u^{(2)}(t, \cdot)\|_{L^{1+\beta_0}(\mathbf{R})} \leqslant C(1+t)^{1/(1+\beta_0)} ER^2(E, T). \tag{4.24}$$

Therefore, we have

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T} (1+t)^{-1/(1+\beta_0)} \|u(t, \cdot)\|_{L^{1+\beta_0}(R)} \\ &\leqslant C\{\varepsilon + (R^2(E, T) + R(E, T))(E + D_{S,T}(u))\}.\end{aligned} \tag{4.25}$$

Finally, we estimate $\|Du(t, \cdot)\|_{D,S,2}$.

In the present case, we still have (3.34)-(3.37), (3.43) and (3.39). Moreover, since we can write $F(v, Dv)$ as

$$F(v, Dv) = F(v, 0) + \widetilde{F}(v, Dv)Dv, \tag{4.26}$$

where $\widetilde{F}(v, Dv)$ is sufficiently smooth and

$$\widetilde{F}(\tilde{\lambda}) = O(|\tilde{\lambda}|^\alpha),\ \tilde{\lambda} = (\lambda, \lambda_0, \lambda_1) \tag{4.27}$$

in a neighbourhood of $\tilde{\lambda}=0$, noting (4.9) and the definition of $\widetilde{X}_{S,E,T}$ we have

$$\begin{aligned}\|F(v, 0)(\tau, \cdot)\|_{L^2(\mathbf{R})} &\leqslant \|F(v, 0)(\tau, \cdot)\|^{1/2}_{L^\infty(\mathbf{R})} \|F(v, 0)(\tau, \cdot)\|^{1/2}_{L^1(\mathbf{R})} \\ &\leqslant C\|v(\tau, \cdot)\|^{(1+\beta_0)/2}_{L^\infty(\mathbf{R})} \|v(\tau, \cdot)\|^{(1+\beta_0)/2}_{L^{1+\beta_0}(\mathbf{R})} \\ &\leqslant CE^{1+\beta_0}(1+\tau)^{1/2}\end{aligned} \tag{4.28}$$

and

$$\|\widetilde{F}(v, Dv)Dv\|_{L^2(\mathbf{R})} \leqslant \|\widetilde{F}(v, Dv)\|_{L^\infty(\mathbf{R})} \|Dv\|_{L^2(\mathbf{R})} \leqslant CE^{1+\alpha}, \tag{4.29}$$

then

$$\|g_0(\tau, \cdot)\|_{L^2(\mathbf{R})} \leqslant CE^{1+\beta_0}(1+\tau)^{1/2} + CE^{1+\alpha}. \tag{4.30}$$

From (3.39) and (4.30) it follows that

$$\begin{aligned}|V| &\leqslant CE^{1+\alpha}(1+t)D_{S,T}(u) + CE^{1+\beta_0}(1+t)^{3/2}D_{S,T}(u) \\ &\leqslant C(R(E, T) + R^2(E, T))ED_{S,T}(u).\end{aligned} \tag{4.31}$$

By (3.37), (3.43) and (4.31), it follows from (3.34) that

$$\sup_{0\leqslant t\leqslant T} \|Du(t, \cdot)\|_{D,S,2} \leqslant C\{\varepsilon + (R(E, T) + \sqrt{R(E, T)})(E + D_{S,T}(u))\}. \tag{4.32}$$

The combination of (4.20), (4.25) and (4.32) gives (4.3).

Lemma 4.2. *Let $\bar{v}$, $\bar{\bar{v}}\in\tilde{X}_{S,E,T}$. If $\bar{u}=M\bar{v}$ and $\bar{\bar{u}}=M\bar{\bar{v}}$ also satisfy $\bar{u}$, $\bar{\bar{u}}\in\tilde{X}_{S,E,T}$, then*

$$D_{S-1,T}(\bar{u}-\bar{\bar{u}})\leqslant C_2(R^2+R+\sqrt{R})\,(D_{S-1,T}(\bar{u}-\bar{\bar{u}})+D_{S-1,T}(\bar{v}-\bar{\bar{v}})),\qquad(4.33)$$

where C_2 is a positive constant and $R=R(E,T)$ is still defined by (4.4).

Proof We first estimate $\|u^*(t,\cdot)\|_{L^\infty(\mathbf{R})}$.

We have

$$\begin{aligned}b(\bar{v},D\bar{v})-b(\bar{\bar{v}},D\bar{\bar{v}})&=b_1(\tilde{v},D\tilde{v})v^*+b_2(\tilde{v},D\tilde{v})Dv^*\\&=b_1(\tilde{v},0)v^*+(b_1(\tilde{v},D\tilde{v})-b_1(\tilde{v},0))v^*\\&\quad+b_2(\tilde{v},D\tilde{v})Dv^*\\&=b_1(\tilde{v},0)v^*+b_3(\tilde{v},D\tilde{v})D\tilde{v}v^*+b_2(\tilde{v},D\tilde{v})Dv^*\end{aligned}\qquad(4.34)$$

and

$$\begin{aligned}&F(\bar{v},D\bar{v})-F(\bar{\bar{v}},D\bar{\bar{v}})\\&=F(\bar{v},0)-F(\bar{\bar{v}},0)+\sum_{i=0}^{1}(F_i(\bar{v},0)\partial_i\bar{v}-F_i(\bar{\bar{v}},0)\partial_i\bar{\bar{v}})\\&\quad+\sum_{i,j=0}^{1}(F_{ij}(\bar{v},D\bar{v})\partial_i\bar{v}\,\partial_j\bar{v}-F_{ij}(\bar{\bar{v}},D\bar{\bar{v}})\partial_i\bar{\bar{v}}\,\partial_j\bar{\bar{v}})\\&=\frac{\partial F(\tilde{v},0)}{\partial u}v^*+\sum_{i=0}^{1}\partial_i(G_i(\bar{v})-G_i(\bar{\bar{v}}))\\&\quad+\sum_{i,j=0}^{1}[F_{ij}(\bar{v},D\bar{v})\partial_i\bar{v}\,\partial_jv^*+F_{ij}(\bar{v},D\bar{v})\partial_iv^*\,\partial_j\bar{\bar{v}}\\&\quad+(F_{ij}(\bar{v},D\bar{v})-F_{ij}(\bar{\bar{v}},D\bar{\bar{v}}))\partial_i\bar{\bar{v}}\partial_j\bar{\bar{v}}]\\&=\frac{\partial F}{\partial u}(\tilde{v},0)v^*+\sum_{i=1}^{1}\partial_i(\hat{G}_i(\tilde{v})v^*)+\sum_{i,j=0}^{1}(F_{ij}(\bar{v},D\bar{v})\partial_i\bar{v}\,\partial_jv^*\\&\quad+F_{ij}(\bar{v},D\bar{v})\partial_iv^*\partial_j\bar{\bar{v}}+\hat{F}_{ij}(\tilde{v},D\tilde{v})v^*\partial_i\bar{\bar{v}}\partial_j\bar{\bar{v}})\\&\quad+\sum_{i,j,k=0}^{1}F_{ijk}(\tilde{v},D\tilde{v})\partial_i\bar{\bar{v}}\,\partial_j\bar{\bar{v}}\,\partial_kv^*,\end{aligned}\qquad(4.35)$$

where

$$\tilde{v}=(\bar{v},\bar{\bar{v}})\qquad(4.36)$$

and $G_i(v)$ is a primitive function of $F_i(v,0)$. Thus in a way similar to the proof of Lemma 4.1, we can get

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\|u^*(t,\cdot)\|_{L^\infty(\mathbf{R})}+\sup_{0\leqslant t\leqslant T}(1+t)^{-1/(1+\beta_0)}\|u^*(t,\cdot)\|_{L^{1+\beta_0}(\mathbf{R})}\\&\leqslant C(R^2(E,T)+R(E,T))\,(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)).\end{aligned}\qquad(4.37)$$

On the other hand, we still have (3.52)–(3.56), Moreover, noting that

$$\begin{aligned}&F(\bar{v},D\bar{v})-F(\bar{\bar{v}},D\bar{\bar{v}})\\&=F(\bar{v},0)-F(\bar{\bar{v}},0)+\sum_{i=0}^{1}(F_i(\bar{v},D\bar{v})\partial_i\bar{v}-F_i(\bar{\bar{v}},D\bar{\bar{v}})\partial_i\bar{\bar{v}})\\&=\frac{\partial F}{\partial u}(\tilde{v},0)v^*+\sum_{i=0}^{1}(F_i(\bar{v},D\bar{v})\partial_iv^*+(F_i(\bar{v},D\bar{v})-F_i(\bar{\bar{v}},D\bar{\bar{v}}))\partial_i\bar{\bar{v}}),\end{aligned}\qquad(4.38)$$

as in the proof of Lemma 4.1, we get

$$\|\bar{g}_0(\tau,\cdot)\|_{L^2(\mathbf{R})}=\|F^*(\tau,\cdot)\|_{L^2(\mathbf{R})}\leqslant C(E^{\beta_0}(1+\tau)^{1/2}+E^{\alpha})D_{S-1,T}(v^*),\qquad(4.39)$$

then we have

$$|V|\leqslant C(R^2(E,T)+R(E,T))D_{S-1,T}(u^*)D_{S-1,T}(v^*). \tag{4.40}$$

Thus, we obtain

$$\sup_{0\leqslant t\leqslant T}\|Du^*(t,\cdot)\|_{D,S-1,2}\leqslant C(R(E,T)+\sqrt{R(E,T)})(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)). \tag{4.41}$$

The desired conclusion (4.33) comes directly from (4.37) and (4.41).

Remark 4.1. For the Cauchy problem

$$\begin{cases} u_{tt}-u_{xx}=u_t^{1+\alpha}, \\ t=0:\ u=\varepsilon\varphi(x),\ u_t=\varepsilon\psi(x), \end{cases} \tag{4.42}$$

we have the sharp lower bound of the life-span of classical solutions as follows:

$$T(\varepsilon)\geqslant a\varepsilon^{-\alpha}\ (a>0\ \text{constant}), \tag{4.43}$$

while for the Cauchy problem

$$\begin{cases} u_{tt}-u_{xx}=u^{1+\beta_0}, \\ t=0:\ u=\varepsilon\varphi(x),\ u_t=\varepsilon\psi(x), \end{cases} \tag{4.44}$$

the sharp lower bound of the life-span is

$$T(\varepsilon)\geqslant a\varepsilon^{-\beta_0/2}\ (a>0\ \text{constant}). \tag{4.45}$$

Hence, in Theorem 4.1 hypothesis (1.18) can not be weakened.

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