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RESOLUTION OF SINGULARITY AND OSCILLATORY INTEGRAL (I)

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Abstract

Oscillatory integral such as $\int_{B^2} e^{itf(x)}g(x)dx$ is studied and asymptotic behavior of the oscillatory integral is obtained. The phase $f(x)$ is analytic and has only isolated critical point in B^2, while $g(x)$ is smooth and compactly supported in B^2.

1. Introduction.

In this paper we treat oscillatory integral with phase having only isolated critical point. The asymptotic behavior of oscillatory integral involves the local structure of the variety of the critical point set of the phase. In multi-dimensional case, if the phase is weighted-homogeneous, Duistermaat obtained good results by using dilation transformation, for example, see [2]. Along another way, by Classification theorem, Mather, Anol'd and others obtained sharp singular index for the oscillatory integral when the phase belongs to certain function class, for details, see [1]. In one-dimensional case, Van Der-Corput's Lemma is a useful tool in obtaining the asymptotic behavior of oscillatory integral, for details, see [4, 5]. In this paper, we use the phase stationary method together with the local structure of the neighbourhood of the critical point of $f(x)$ to obtain the asymptotic behavior of the oscillatory integral, and then convince the reader by a variant of Van Der-Corpput's Lemma that the singular index for the asymptotic behavior of the oscillatory integral cannot be improved for general phase.

2. Some preliminaries.

We use in this paper extensively following two theorems which are well known in Several Complex Analysis and Algebraic Geometry:

WPT (Weierstrass Preparation Theorem). **Fix $f(z)$, which is analytic and equals zero at the origin. If**

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$$\lim_{z_1\to 0}\frac{f(z_1,0)}{z_1^k}\neq 0,$$

then we have

$$f(z_1, z')=(z_1^k+a_{k-1}(z')z_1^{k-1}+\cdots+a_0(z'))W(z)$$

for any $z=(z_1, z')\in D=\{|z|<d\}$, for some small positive d, where $W(z)$ and $P(z_1, z')=z_1^k+a_{k-1}(z')z_1^{k-1}+\cdots+a_0(z')$ are unique, analytic in D, $W(z)\neq 0$ over D, $a_j(0)=0$, $j=0, 1, \cdots, k-1$. The coefficients in the Taylor expansion of a_j are finite sum of those in the Taylor expansion of $f(z)$.

Usually, $P(z)$ is called a Weierstrass Polynomial, simply, a WP.

Definition 1. *A WP $P(z_1, z')$ is called irreducible if it cannot be decomposed into the product of two WP's.*

RSC^2 (Resolution of Singularity in C^2). If $P(z_1, z_2)=z_1^k+a_{k-1}(z')z_1^{k-1}+\cdots+a_0(z')$ is irreducible, then under coordinate transformation $C^1\ni z_2\to z_2^k$, its roots are all analytic in the new variable in some neighbourhood of the origin.

3. Normalization of roots and indices.

In this section we give the definitions of indices which are needed for the analysis of the asymptotic behavior of the oscillatory integral. Assume that the origin O is the only critical point of $f(x)$ in B^2. After coordinate transformation we can choose the x axis in (x, y) plane such that

$$\lim_{x\to 0}\frac{f(x,0)}{x^{k+1}}\neq 0,$$

$$k+1=\min_{v\in S^1}\left\{k(v):\lim_{s\to 0}\frac{f(sv)}{s^{k(v)}}\neq 0\right\}. \tag{3.1}$$

Then by the WPT, we can decompose D_xf as follows:

$$D_xf(x, y)=P(x, y)W(x, y), \tag{3.2}$$

for any $(x, y)\in D=\{|x|<d,\ |y|<d\}$, for some positive d, where $P(x, y)=x^k+a_{k-1}(y)x^{k-1}+\cdots+a_0(y)$ is a WP, $W(x, y)\neq 0$ in D. Furthermore, we can decompose $P(x, y)$ into the product of irreducible components as follows:

$$P(x, y)=P_1(x, y)\cdots P_s(x, y),\ \text{with degree } (P_m, x)=k_m,\ k_1+\cdots+k_s=k. \tag{3.3}$$

Claim 1. *The total degree of P, P_m are k, k_m respectively, $m=1, \cdots, s$, where the total degree of a function f equals*

$$\min_{v\in S^1}\left\{k(v):\lim_{s\to 0}\frac{f(sv)}{s^{k(v)}}\neq 0\right\}.$$

Proof The statement for P is valid since (3.1) holds. For the others, after a permutation we can assume that the statement for $P_m (m=1, 2, \cdots, s'\leqslant s)$ is false and then we deduce a contradiction. Expanding P_m into Taylor series as follows:

$$P_m(x, y)=x^{k_m}+\sum_{i+j<k_m}c_{m,ij}x^iy^j+\sum_{i+j\geqslant k_m}c_{m,ij}x^iy^j,\ |x|<d,\ |y|<d,\ d>0,$$

we see that $P^{\circ}=P_1\cdots P_s$ has terms $\prod_1^{s'}(\sum_{i+j<k_m} c_{m,ij}x^iy^j)x^{k_{s'+1}+\cdots+k_s}$ which contains a term different from the others in P:

$$c_{i_1j_1}c_{i_2j_2}\cdots c_{i_{s'}j_{s'}}x^{i_1+\cdots+i_{s'}+k_{s'+1}+\cdots+k_s}y^{j_1+\cdots+j_{s'}},$$

where (i_m, j_m) satisfies: $i_m+j_m=\min\{(i+j): c_{m,ij}\neq 0\}$

$$i_m=\min\{i: i+j=\min(i'+j'), c_{m,i'j'}\neq 0\}.$$

Thus we get a contradiction since P has total degree k.

Below we define various indices necessary for the analysis of oscillatory integral. We assume that we have done well the decomposition (3.3) since it is a pure algebraic problem how to decompose a given WP into the product of irreducible components.

At first, we can write also the WP $P_m(x, y)$ as follows:

$$P_m(x, y)=(x-r_{m_1}(y))\cdots(x-r_{m,k_m}(y)), \quad |x|<d, \ |y|<d, \ m=1, 2, \cdots, s. \tag{3.4}$$

By the RSC[2], we know that each $r'_{mi}(y)=r_{mi}(y^{k_m})$ is analytic in y, $|y|<d$, for some $d>0$. Expanding $r'_{mi}(y)$ into the Taylor series, we have

$$r'_{mi}(y)=\sum_{k>k_m} c_{mi,k}y^k, \quad |y|<d. \tag{3.5}$$

Pratically we can determine these coefficients $c_{mi,s}$ from equation

$$P_{mi}(\sum_{k>k_m} c_{mi,k}y^{k_m})=0.$$

From (3.5) we obtain

$$\operatorname{Re} r'_{mi}(y)=c_{mi,M_{mi}}y^{M_{mi}}+\text{higher terms},$$

$$\operatorname{Im} r'_{mi}(y)=c'_{mi,M'_{mi}}y^{M'_{mi}}+\text{higher terms}, \ c_{mi,M_{mi}}, \ c'_{mi,M'_{mi}}\neq 0, \text{ real}. \tag{3.6}$$

Since the higher terms in $\operatorname{Re} r'_{mi}(y)$ will bring us some troubles, we erase them by doing a smooth transformation as follows:

$$w=y\left(1+\frac{1}{c_{mi,M_{mi}}}(\sum_{k>M_{mi}} c_{mi,k}y^k)\right)^{1/M_{mi}}, \quad |y|<d, \text{ for some } d>0. \tag{3.7}$$

Now in the new variable w, the root $r'_{mi}(w)$ satisfies: $\operatorname{Re} r'_{mi}(w)=c_{mi,M_{mi}}w^{M_{mi}}$.

Definition 2. *The width of f with respect to root r_{mi} is*

$$d_{mi}=k_m/(M_{mi}, k_m),$$

where (p, q) denotes the common factor of two numbers p, q.

Under coordinate transformation (3.7), the phase reads as follows:

$$f_{mi}(x, w)=f(x, w(y))=\sum_n \sum_{j/k_m+j'/M_{mi}=b_n} c_{jj',mi}x^jw^{j'}, \tag{3.8}$$

where $b_1<b_2<b_3\cdots$.

Definition 3. *The partial singular index N_{mi} is defined as follows:*

if $\lim_{w\to 0}\dfrac{D_w[f_{mi}(c_{mi,M_{mi}}w^{M_{mi}/k_m}, w)]}{w^d}\neq 0$, *then* $N_{mi}=d$; *otherwise* $N_{mi}=\infty$.

We see that if N_{mi} is finite, then N_{mi} is a rational number.

Definition 4. *For the given real analytic $f(x, y)$ which has the origin as its only critical point in B^2, we say that f has type at 0*

$$T(f)(0)=\left\{\begin{matrix} k_1, M_{11}, \cdots, M_{1,k_1}, M'_{11}, \cdots, M'_{1,k_1}, d_{11}, \cdots, d_{1k_1}, N_{11}, \cdots, N_{1k_1} \\ k, \cdots \cdots \cdots \cdots \cdots \cdots \\ k_s, M_{s1}, \cdots, M_{sk_s}, M'_{s1}, \cdots, M'_{sk_s}, d_{s1}, \cdots, d_{sk_s}, N_{s1}, \cdots, N_{sk_s} \end{matrix}\right\}.$$

Definiton 5. *The singular index of $f(x)$ satisfying conditions as in Definition 4 is*

$$d=\frac{1}{k+1}+\min_{\substack{i=1,\cdots,k_m\\ m=1,\cdots,s}}\left\{\max\left(\frac{1}{d_{mi}N_{mi}+1}, \frac{k_m}{(k+1)M'_{mi}}\right)\right\}. \tag{3.9}$$

4. Statement of result.

Now we can formulate our main result as follows:

Theorem. *Assume that function $f(x, y)$ is real analytic in the unit ball $B^2\subset R^2$, the critical point set of f $S(f)=\{0\}$, and f is of type $T(f)(0)$ at 0. Then for any small positive e, there exists a constant $C_e>0$ such that*

$$\left|\iint_{B^2} e^{itf(x,y)}g(x, y)\,dx\,dy\right|\leqslant\frac{C_e}{t^{d-e}}, \text{ as } t\to\infty.$$

The following sections 5—10 are devoted to the proof of Theorem.

5. Since $S(f)=\{0\}$, for any neighbourhood $U=\{|x|<d, |y|<d\}$ of the origin 0, we can find a constant $C(U)$ such that

$$|\nabla f(x, y)|>C(U)>0, \text{ for any } (x, y)\in U^c, \text{ the complement of } U. \tag{5.1}$$

Let $E(x)$ be a smooth function with compact support lying in $[-1, 1]$, and $E(x)=1$ for $-1/2<x<1/2$. Define $g_1(x, y)=g(x, y)E((x^2+y^2)/d^2)$, $g_2=g-g_1$. Then $g_2=0$ for $(x, y)\in U^{1/4}=\{|x|<d/4, |y|<d/4\}$. So by the Phase Stationary Method, we obtain

$$\left|\int e^{itf(x,y)}g_2(x, y)\,dx\,dy\right|\sim\frac{C_M}{t^M}, \text{ as } t\to\infty, \text{ for any integer } M, \tag{5.2}$$

where C_M is a constant depending on M, independent of t.

6. Now we are going to treat integral $\int e^{itf(x,y)}g_1(x, y)\,dx\,dy$, where $U=\{|x|<d, |y|<d\}$, d is sufficiently small so that all arguments in section 4 hold. Specially we know that $D_x f(x, y)=P_1\cdots P_s W$, f is of type at its critical point O: $T(f)(0)$.

For any given small positive e, we define sets as follows:

$$U_{mi}=\{(x, y)\in U: |x-\mathrm{Re}r_{mi}(y)|<t^{-s}, |\mathrm{Im}r_{mi}(y)|<t^{-s}\}, \tag{6.1}$$

where $s=(1-e)/(k+1)$; $i=1, \cdots, k_m$, $m=1, \cdots, s$.

Now we can do unit smooth decomposition with respect to U_{mi} as follows:

$$g_1=g_{11}+g_{12}+\cdots+g_{sk_s}+g'_{sk_s},$$

$$g_{11}=g_1E([(x-\mathrm{Re}r_{11}(y))^2+(\mathrm{Im}r_{11}(y))^2]t^{2s}\times 5),$$

$$g_{mi}=g_1\prod_{(m',i')<(m,i)}(1-E)([(x-\mathrm{Re}r_{m'i'}(y))^2+(\mathrm{Im}r_{m'i'}(y))^2]t^{2s}C_{m'i'})$$
$$\times E([(x-\mathrm{Re}r_{mi}(y))^2+(\mathrm{Im}r_{mi}(y))^2)t^{2s}C_{mi}),$$

$$\text{for } (m, i):\ (1, 1) < (m, i) \leqslant (s, k_s),$$

$$g'_{sk_s} = g_1 \prod_{(m,i)} (1-E)\left([(x-\mathrm{Re}\,r_{mi}(y))^2 + (\mathrm{Im}\,r_{mi}(y))^2] s^{2s} C_{mi}\right), \tag{6.2}$$

where (m, i) is ordered lexicographically: $(m, i) < (m', i')$ iff $m < m'$ or $m = m'$, $i < i'$; $C_{mi} = 5^i$ for $m = 1$, $i = 1, \cdots, k_1$, and $C_{mi} = 5^{k_1 + \cdots + k_{m-1} + i}$ for the others. It is obvious that supp $(g_{mi}) \subset U_{mi}$, while supp$(g'_{sk_s}) \subset U / \bigcup U_{mi}$.

According to (6.2), we have

$$\int e^{itf(x,y)} g_1(x, y)\, dx\, dy = \sum_{m,i} \int e^{itf(x,y)} g_{mi}(x, y)\, dx\, dy + \int e^{itf(x,y)} g'_{sk_s}\, dx\, dy. \tag{6.3}$$

Among the integrals in (6.3), the last is the easiest, we treat it below. Integrating by part shows that

$$\int e^{itf(x,y)} g'_{sk_s}(x, y)\, dx\, dy = -\int e^{itf(x,y)} D_x \left\{ \frac{g'_{sk_s}(x, y)}{\mathrm{it}\, D_x f(x, y)} \right\} dx\, dy. \tag{6.4}$$

From (6.2), we get

$$D_x \frac{g'_{sk_s}(x, y)}{D_x f(x, y)} = -\frac{D_x^2 f(x, y)}{(D_x f(x, y))^2} g'_{sk_s}(x, y) + \frac{D_x g'_{sk_s}(x, y)}{D_x f(x, y)};$$

$$D_x g'_{sk_s}(x, y) = D_x g_1 (1-E) \cdots (1-E) - \sum_{m,i} t^{2s} C_{mi} DE\left([(x-\mathrm{Re}\,r_{mi}(y))^2 + (\mathrm{Im}\,r_{mi}(y))^2]\right.$$

$$\left. \times t^{2s} C_{mi}\right) \times 2(x - \mathrm{Re}\,r_{mi}(y))\, g_1 (1-E) \cdots (1-E), \tag{6.5}$$

where DE is the derivative of E, and $DE(x) = 0$ for $-1/2 < x < 1/2$ or $|x| > 1$. We have also

$$\frac{D_x^2 f(x, y)}{(D_x f(x, y))^2} = \frac{1}{P_1 \cdots P_s} \frac{D_x W}{W} + \frac{1}{P_1 \cdots P_s} \sum_{m,i} \frac{1}{x - r_{mi}(y)}. \tag{6.6}$$

Since

$$|a_1 a_2 \cdots a_k| \leqslant C \sum_j |a|^k \text{ for some positive } C, \tag{6.7}$$

we get

$$\left| D_x \frac{g'_{sk_s}(x, y)}{D_x f(x, y)} \right| \leqslant C \sum_{m,i} \left\{ \frac{1}{|x - r_{mi}(y)|^{k+1}} + \frac{1}{|x - r_{mi}(y)|^k} \right.$$

$$\left. \times \left(1 + C_{mi} t^s \left| DE\left((x - \mathrm{Re}\,r_{mi}(y))^2 + (\mathrm{Im}\,r_{mi}(y))^2\right) t^{2s} C_{mi}\right) \right| \right) \right\}. \tag{6.8}$$

Substituting (6.8) into (6.4), we get

$$\left| \int e^{itf(x,y)} g'_{sk_s}(x, y)\, dx\, dy \right|$$

$$\leqslant C \sum_{m,i} \int \left\{ \frac{E(|x - r_{mi}(y)|^2 t^{2s} C_{mi})}{t\, |x - r_{mi}(y)|^{k+1}} + \frac{E(|x - r_{mi}|\, t^s)}{t\, |x - r_{mi}(y)|^k} \right.$$

$$\left. \times \left(1 + t^s \left| DE\left([(x - \mathrm{Re}\,r_{mi}(y))^2 + (\mathrm{Im}\,r_{mi}(y))^2] t^{2s} C_{mi}\right) \right| \right) \right\} dx\, dy$$

$$\leqslant \frac{C(e)}{t^e}, \text{ as } t \to \infty. \tag{6.9}$$

Integrating by parts M times shows that

$$\left| \int e^{itf(x,y)} g'_{sk_s}(x, y)\, dx\, dy \right| \leqslant C \sum_{m,i} \int_{U_{mi}} \frac{1}{c(t\, |x - r_{mi}(y)|^{k+1})^M}\, dx\, dy$$

$$\leqslant C_M / t^{eM}, \text{ as } t \to \infty, \tag{6.10}$$

where C_M is a constant depending on M and f, g, but independent of t.

7. Global estimate of the oscillatory integral $\int e^{itf(x,y)} g_{mi}(x, y)\,dx\,dy$: the first approach.

Recalling (3.4), (3.5), (3.6), we see that if $r'_{mi}(y) \neq 0$, $\operatorname{Im} r'_{mi}(y) \neq 0$, then $c'_{mi, M'_{mi}} \neq 0$ for some M'_{mi}. In this case we have

$$\left|\iint e^{itf} g_{mi}\,dx\,dy\right| \leqslant C \int E([(x-\operatorname{Re} r_{mi}(y))^2 + (\operatorname{Im} r_{mi}(y))^2] t^{2s} C_{mi})\,dx\,dy$$

$$\leqslant \frac{C}{t^c},\ d = s(1+k_m/M'_{mi}), \text{as } t\to\infty.$$

8. Global estimate for $\int e^{itf} g_{mi}\,dx\,dy$: the second approach.

At first, note that N_{mi} is defined well even though $\operatorname{Re} r'_{mi}(y) \equiv 0$, for in this case (3.7) becomes $w=y$, and the $d_{mi}=1$ in Definition 2, $c_{mi,M_{mi}}=0$ in Definition 3. Since f has only one critical point in B^2, we see that in case $\operatorname{Im} r'_{mi} \equiv 0$, $N_{mi} \neq \infty$. In this section we treat the case: $N_{mi} \neq \infty$.

If $\operatorname{Re} r'_{mi}(y) \neq 0$, we do transformation (3.7) and replace f, g by $f(x, y(w))$, $g(x, y(w))y'(w)$ respectively, denoted also by f, g. By our assumption: $N_{mi} \neq \infty$, we see that

$$D_w f(u + c_{mi,M_{mi}} w^{M_{mi}/k_m}, w) = \sum_{jM_{mi}/k_m + j' \geq 1} a_{jj'}(u) w^{jM_{mi}/k_m + j' - 1}, \tag{8.1}$$

and $N_{mi} \neq \infty$ guarantees that $\sum_{jM_{mi}/k_m + j' - 1 = k_{mi}} a_{jj'}(0) \neq 0$,

furthermore, the function $a_{jj'}(u)$ is analytic in $u^{1/d_{mi}}$.

By the WPT, we obtain

$$D_w f(u + c_{mi,M_{mi}} w^{M_{mi}/k_m}, w) = ((w^{1/d_{mi}})^{d_{mi}N_{mi}} + \cdots + b_0(u)) W_1(w, u), \tag{8.2}$$

$W_1(w, u)$ is analytic in $w^{1/d_{mi}}$, $u^{1/d_{mi}}$, and $W \neq 0$ for $|w|<d$, $|u|<d$, for some $d>0$.

Letting $w = z^{d_{mi}}$, then we have

$$D_w = \frac{1}{d_{mi} z^{d_{mi}-1}} D_z. \tag{8.3}$$

For (8.2) we can write also

$$D_w f(u + c_{mi,M_{mi}} w^{M_{mi}/k_m}, w) = (z^{d_{mi}N_{mi}} + \cdots + b_0(u)) W_1(z, u),$$

$$z^{d_{mi}N_{mi}} + \cdots + b(u) = (z - s_{mi,1}(u)) \cdots (z - s_{mi,d_{mi}N_{mi}}(u)). \tag{8.4}$$

Now we can define sets as follows:

$$V_n = \{(u, z): |z - \operatorname{Re} s_{mi,n}(u)| < t^{-\delta}\},\ \delta = (1-e)/(d_{mi}N_{mi}+1),$$

$$n = 1, \cdots, d_{mi}N_{mi}. \tag{8.5}$$

Doing smooth unit decomposition with respect to V_n, $n=1, \cdots, d_{mi}N_{mi}$, we obtain

$$g_{mi} = g_{mi,1} + \cdots + g_{mi,d_{mi}N_{mi}} + g'_{mi,d_{mi}N_{mi}};$$

$$g_{mi,n} = g_{mi} \prod_{j=1}^{n-1} (1-E)((z - \operatorname{Re} s_{mi,j}(u)) t^{(1-e)})$$

$$\times E((z - \operatorname{Re} s_{mi,n}(u)) t^{\delta}),\ n \geqslant 2; \tag{8.6}$$

when $n=1$, there is no $(1-E)$ term;

$$g'_{mi,d_{mi}N_{mi}}=g_{mi}\prod_{j=1}^{d_{mi}N_{mi}}(1-E)((z-\mathrm{Res}_{mi,j}(u))t^{s}).$$

Therefore we get decomposition

$$\int e^{itf(u+c_{ni},M_{mi}w^{M_{mi}/k_m},w)}g_{mi}\,dw\,du=\sum_n\int e^{itf}g_{mi,n}\,dw\,du+\int e^{itf}g'_{mi,d_{mi}N_{mi}}\,dw\,du. \tag{8.7}$$

Among the integrals in (8.7), the last is the most difficult, we treat the easy at first. We estimate $\int e^{itf}g_{mi,n}\,dw\,du$ by absolute integral:

$$\left|\int e^{itf}g_{mi,n}\,dw\,du\right|\leqslant C\int_{V_n\cap\{|u|<t^{-s}\}}du\,dw\leqslant Ct^{-s-\delta},\ \text{as } t\to\infty;\ n=1,\cdots,d_{mi}N_{mi}. \tag{8.8}$$

9. Estimate of $\int e^{itf}g'_{mi,d_{mi}M_{mi}}\,dw\,du$: the case not affected by $E(|x-r_{m'i}(y)|t^{s})$.

Integrating by part shows that

$$\int e^{itf}g'_{mi,d_{mi}N_{mi}}\,du\,dw=-\int e^{itf}D_w\left\{\frac{g'_{mi,d_{mi}mN_i}}{itD_wf}\right\}du\,dw. \tag{9.1}$$

Similar to (6.4)—(6.9), we have following relations:

$$D_w\left(\frac{g'_{mi,d_{mi}N_{mi}}}{D_wf}\right)dw=D_z\left(\frac{g'_{mi,d_{mi}N_{mi}}}{D_wf}\right)dz, \tag{9.2}$$

$$D_wf(u+c_{mi,M_{mi}}w^{M_{mi}/k_m},w)(z)=(z-s_{mi,1}(u))\cdots(z-s_{mi,d_{mi}N_{mi}}(u))W_1(z,u), \tag{9.3}$$

$$D_z\frac{g'_{mi,d_{mi}N_{mi}}}{D_wf}=-\frac{g'_{mi,d_{mi}N_{mi}}}{\prod_u(z-s_{mi,n}(u))}\left\{\frac{D_zW_1}{W}+\sum_n\frac{1}{z-s_{mi,n}(u)}\right\}$$
$$+\frac{1}{\prod_u(z-s_{mi,n}(u))W_1}D_zg'_{mi,d_{mi}N_{mi}}, \tag{9.4}$$

$$g'_{mi,d_{mi}N_{mi}}(u,z)=g_1(u,z)\prod_{(m',i')<(m,i)}(1-E)([(u+\mathrm{Re}r_{mi}-\mathrm{Re}r_{m'i'}(z))^2$$
$$+(\mathrm{Im}r_{m'i'}((z))^2t^{2s}C_{m'i'})E((u^2+\mathrm{Im}^2r_{mi}(z))t^{2s}C_{mi})$$
$$\times\prod_n(1-E)(z-\mathrm{Res}_{mi,n}(u))t^{s}), \tag{9.5}$$

$$D_zg'_{mi,d_{mi}N_{mi}}(u,z)$$
$$=D_zg_1(u,z)\ (1-E)\cdots E(1-E)\cdots(1-E)-\sum_n DE((z-\mathrm{Res}_{mi,n}(u))t^{s})t^{s}$$
$$\times g_1(1-E)\cdots E(1-E)\cdots(1-E)$$
$$-\sum_{(m'',i'')<(m,i)}D_z((u+\mathrm{Re}r_{mi}-\mathrm{Re}r_{m'i'})^2+\mathrm{Im}^2r_{m'i'}(z))t^{2s}C_{m'i'}$$
$$\times DE([(u+\mathrm{Re}r_{mi}-\mathrm{Re}r_{m'i'})^2+\mathrm{Im}^2r_{m'i'}(z)]t^{2s}C_{m'i'})$$
$$\times(1-E)\cdots E(1-E)\cdots(1-E)$$
$$+C_{mi}D_z(\mathrm{Im}^2r_{mi}(z))t^{2s}DE((u^2+\mathrm{Im}^2r_{mi})t^{2s}C_{mi})(1-E)\cdots(1-E)$$
$$=h_0-\sum_n h_n-\sum_{(m',i')<(m,i)}h_{m'i'}+h_{mi}. \tag{9.6}$$

Among the terms in (9.6), h_0, $h_n(n=1,\cdots,d_{mi}N_{mi})$ are easy to treat, we leave the terms $h_{m'i'}$ and h_{mi} to section 10.

Now (9.4) can be written as follows:

$$D_z \frac{g'_{mi,d_{mi}N_{mi}}}{D_w f} = \frac{-\sum h_{m'i'} + h_{mi}}{\prod_n (z - s_{mi,n}(u)) W_1} + \text{rest}, \tag{9.7}$$

where "rest" satsfies

$$|\text{"rest"}| \leqslant C \sum_n |z - \operatorname{Res}_{mi,n}(u)|^{-(d_{mi}N_{mi}+1)}$$

$$+ \sum_u \frac{1}{|z - \operatorname{Res}_{m,n}(u)|^{d_{mi}N_{mi}}} \Big(1 + \sum_u t^\delta |DE((z - \operatorname{Res}_{mi,n}) t^{\delta(1-e)})|\Big). \tag{9.8}$$

Inserting (9.8) into (9.1), we obtain

$$\left| \iint e^{itf} \text{"rest"}\, du\, dw \right| \leqslant C \int_{\substack{(\cup V_n)^c \\ |n| < t^{-\varepsilon}}} |\text{"rest"}|\, dz\, du \leqslant C t^{-\varepsilon-\delta} \tag{9.9}$$

10. Estimate of $\int e^{itf} g'_{mi,d_{mi}N_{mi}} dw\, du$: the case affected by $E(|x - r_{m'i'}(y)| t^\varepsilon)$. We deal with $h_{m'i'}$ and h_{mi}. We have

$$|u + \operatorname{Re} r_{mi} - \operatorname{Re} r_{m'i'}(z)| < \frac{1}{t^{(1-e)e'}}, \quad |\operatorname{Im} r_{m'i'}(z)| < t^{-\varepsilon},$$

$$\text{if } DE([(u + \operatorname{Re} r_{mi} - \operatorname{Re} r_{m'i'})^2 + \operatorname{Im}^2 r_{m'i'}] t^{2\varepsilon} C_{m'i'}) \neq 0, \tag{10.1}$$

$$|u| < t^{-\varepsilon} \text{ if } E((u^2 + \operatorname{Im}^2 r_{mi}) t^{2\varepsilon}) \neq 0. \tag{10.2}$$

From (9.6), (10.1) and (10.2), we know that

$$|h_{m'i'}(u, z)| \leqslant t^\varepsilon \{|D_z(\operatorname{Re} r_{mi} - \operatorname{Re} r_{m'i'})(z)| + |D_z \operatorname{Im} r_{m'i'}(z)|\}. \tag{10.3}$$

If $\operatorname{Re} r_{mi} - \operatorname{Re} r_{m'i'} \neq 0$, then we have

$$\operatorname{Re} r_{mi}(z) - \operatorname{Re} r_{m'i'}(z) = c'' z^N + \text{higher terms} = c'' z^N (1 + o(1)), \tag{10.4}$$

where $o(1)$ denotes a function which is continuous and takes zero if $z=0$, $N>0$ (indeed, N is rational).

From (10.1) and (10.2), we know that

$$|\operatorname{Re} r_{mi} - \operatorname{Re} r_{m'i'}(z)| \leqslant 2t^{-\varepsilon}, \tag{10.5}$$

therefore $|z| \leqslant C t^{-\varepsilon/N}$ by (10.4).

Similarly we have: if $\operatorname{Im} r_{m'i'}(z) \neq 0$ then $\operatorname{Im} r_{m'i'}(z) = d' z^{N'} +$ higher terms, and

$$|\operatorname{Im} r_{m'i'}(z)| \leqslant t^{-\varepsilon}: \ |z| \leqslant C t^{-\varepsilon/N'}. \tag{10.6}$$

At last, by the compactness of $h_{m'i'}$, from (9.6), (9.7), (10.1)—(10.6), we deduce following

$$\left| \iint e^{itf} \frac{h_{m'i'}}{\prod_n (z - s_{mi,n}(u)) W_1} du\, dz \right|$$

$$\leqslant C \int_{|u| < t^{-\varepsilon}} du \times t^{\varepsilon-\delta} \Big[\int_{|z| < C t^{-\varepsilon/N}} |z^{N-1}| dz + \int_{|z| < C t^{-\varepsilon/N'}} |z^{N'-1}| dz$$

$$\leqslant C t^{-\varepsilon-\delta}. \tag{10.7}$$

Summing up results in (5.2), (6.10), (7.1), (8.8), (9.9) and (10.7), we prove theorem.

11. Some remarks.

We state here a theorem of E. M. Stein:

Proposition. *Assume that* $a_1 > a_2 > \cdots > a_n$, $f(t) = t^{a_1} + b_2 t^{a_2} + \cdots + b_n t^{a_n}$, b_i ($i=$

2, ⋯, n) are real parameters. Then for any $0 \leqslant A < B \leqslant 1$, *we have*

$$\left|\int_A^B e^{isf(t)} g(t)\,dt\right| \leqslant Cs^{-d},\ s \to \infty,\ d = \min(1/a_1, 1/n).$$

Above estimate is sharp. For details, see Page 184—186, [4].

The singular index d in Theorm is sharp in following meaning: after getting a factor $\frac{1}{k+1}$ by integration along direction x, we want to get more from (8.1). In case $(M_{mi}, k_m) = 1$, the number of terms cw^a in (8.1) with power $a \leqslant N_{mi}$ is about kN_{mi}. Tnen above proposition prevents us from getting a larger index than $1/kN_{mi}$.

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