Chin. Ann. of Math.
13B: 3(1992), 298—314.

THE CAUCHY PROBLEM FOR GAS DYNAMIC SYSTEMS IN MULTI-DIMENSIONAL SPACE WITH WEAKLY SINGULAR DATA**

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Abstract

The Cauchy problem for gas dynamic systems with weakly discontinuous initial data is discussed. The local existence of the solution to such problem is proved. Meanwhile, it is shown that the singularities of the solution spread on all characteristic surfaces issuing from the manifold carrying initial singularities.

§ 1. Preliminary

Since A. Majda's work[5] published in 1983, there has been noticeable development in the research of initial problems for the gas dynamic systems in multidimensional space with singular data. When the initial data are discontinuous and satisfy some conditions, A. Majda[5] and S. Alinhac[2] discussed the local existence of shock fronts and rarefaction waves respectively. When the initial data are continuous, but their derivatives are discontinuous on a lower dimensional manifold of the initial surface, if certain compatibility conditions are satisfied, then there will exist a gradient wave issuing from the manifold carrying singularities of initial data; such a conclusion has also been proved mathematically (see [8]). However, in general case, this problem is more complicated because the singularities of the solution will spread on all characteristic surfaces issuing from the manifold carrying initial singularities. In this paper, we are going to discuss such Cauchy problems with singular data, and give the structure of singularities of the solution. We will mainly discuss the system of isentropic compressible flow in three dimensional space in order to show that the result in this paper is available to some quasilinear hyperbolic systems with multiple characteristics. Our conclusion in this paper is also valid for the case of non-isentropic flow. In addition, the method in this paper can also be applied to treat waves with higher order weak discontinuity. The solution which we obtained is piecewise H^s smooth, but some regularity is lost

Manuscript received April 18, 1990.
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** Projects supported by NNSF and SNSF.

in the process of finding the solution. To improve the regularity of the weakly singular solution is worth studying further, in this aspect some results have been obtained in [1].

§ 2. Introduction of Problems and Main Results

Let us consider the Cauchy problem of the gas dynamic system in three dimensional space:

$$\begin{cases} B_0\dfrac{\partial U}{\partial t}+B_1\dfrac{\partial U}{\partial x}+B_2\dfrac{\partial U}{\partial y}+B_3\dfrac{\partial U}{\partial z}=0, & (2.1)\\ U=\begin{cases} U_0^-(x,\ y,\ z),\ z<\varphi_0(x,\ y)\\ U_0^+(x,\ y,\ z),\ z>\varphi_0(x,\ y)\end{cases},\ t=0 & (2.2)\end{cases}$$

where $U=(u,\ v,\ w,\ p)^T$,

$$B_0=\begin{bmatrix}\rho &&& \\ &\rho && \\ &&\rho & \\ &&& c^{-2}\rho^{-1}\end{bmatrix},\quad B_1=\begin{bmatrix}\rho u &&& 1\\ &\rho u && \\ &&\rho u & \\ 1 &&& c^{-2}\rho^{-2}u\end{bmatrix},$$

$$B_2=\begin{bmatrix}\rho v &&& 1\\ &\rho v && \\ &&\rho v & \\ 1 &&& c^{-2}\rho^{-1}v\end{bmatrix},\quad B_3=\begin{bmatrix}\rho w &&& \\ &\rho w && \\ &&\rho w & 1\\ &&1 & c^{-2}\rho^{-1}w\end{bmatrix}$$

with $c=(p'(\rho))^{1/2}$ being the sound speed.

For simplicity, we set $\Gamma=\{z=\varphi_0(x,\ y)\}$ to be a smooth hypersurface through the origin on $t=0$, $\omega^-=\{t=0\}\cap\{z<\varphi_0(x,\ y)\}$, $\omega^+=\{t=0\}\cap\{z>\varphi_0(x,\ y)\}$. For the problem (2.1) (2.2), we assume that $U_0^+|_\Gamma=U_0^-|_\Gamma$, $U_0^\pm\in H^\lambda(\omega^\pm)$, and $D_nU_0^+|_\Gamma\neq D_nU_0^-|_\Gamma$, where n is the normal direction of Γ. Our main result in this paper is

Theorem 2.1. *For given $s>0$, there exists $\lambda>0$ such that under the above assumptions on the initial data, there exists an neighborhood Ω of the origin O, and functions $\varphi_1(t,\ x,\ y)\leqslant\varphi_2(t,\ x,\ y)\leqslant\varphi_3(t,\ x,\ y)$ (The equalities hold if and only if $t=0$.) defined in $\Omega\cap\{z=0\}$, and functions U_0, U_1, U_2 and U_3, which are defined in $\Omega^0=\{z<\varphi_1(t,\ x,\ y)\}\cap\Omega$, $\Omega^1=\{\varphi_1(t,\ x,\ y)<z<\varphi_2(t,\ x,\ y)\}\cap\Omega$, $\Omega^2=\{\varphi_2(t,\ x,\ y)<z<\varphi_3(t,\ x,\ y)\}\cap\Omega$ and $\Omega^3=\{z>\varphi_3(t,\ x,\ y)\}\cap\Omega$ respectively, satisfying*

(1) *$\varphi_j(j=1,2,3)$ and $U_k(k=0,1,2,3)$ belong to H^s in their domains of definition.*

(2) $\varphi_j|_{t=0}=\varphi_0(x,\ y)$, $U_0|_{t=0}=U_0^-(x,\ y,\ z)$, $U_3|_{t=0}=U_0^+(x,\ y,\ z)$ and $U_{j-1}|_{z=\varphi_j}=U_j|_{z=\varphi_j}(j=1,\ 2,\ 3)$.

(3)

$$\frac{\partial\varphi_1}{\partial t}+u_1\frac{\partial\varphi_1}{\partial x}+v_1\frac{\partial\varphi_1}{\partial y}-w_1+c_1\left(1+\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\left(\frac{\partial\varphi_1}{\partial y}\right)^2\right)^{1/2}=0,\ on\ z=\varphi_1;$$

$$\frac{\partial\varphi_2}{\partial t}+u_2\frac{\partial\varphi_2}{\partial x}+v_2\frac{\partial\varphi_2}{\partial y}-w_2=0, \text{ on } z=\varphi_2;$$

$$\frac{\partial\varphi_3}{\partial t}+u_2\frac{\partial\varphi_3}{\partial x}+v_2\frac{\partial\varphi_3}{\partial y}-w_2-c_2\Big(1+\Big(\frac{\partial\varphi_3}{\partial x}\Big)^2+\Big(\frac{\partial\varphi_3}{\partial y}\Big)^2\Big)^{1/2}=0, \text{ on } z=\varphi_3$$

where c_1 and c_2 are the sound speeds of U_1 and U_2 respectively. Besides, $U_k(k=0,1,2,3)$ satisfies (2.1) *in its domain of definition.*

Remark. By the continuity of U, we know that the second relation of (3) can be also written as $\frac{\partial\varphi_2}{\partial t}+u_1\frac{\partial\varphi_2}{\partial x}+v_1\frac{\partial\varphi_2}{\partial y}-w_1=0$.

By using the property of finite speed of propagation for hyperbolic systems, we immediately obtain the functions φ_1, φ_3, U_0 and U_3 from initial data (2.2), so we only need to find the remains: φ_2, U_1 and U_2.

First, let us give the values of all derivatives of φ_2, U_1 and U_2 on Γ, such that $\varphi_j(j=1, 2, 3)$ and $U_k(k=0, 1, 2, 3)$ satisfy certain compatibility conditions on Γ. Without loss of generality, we will always assume $\varphi_0(x, y)=0$; otherwise we will easily be led to this by using a simple transformation $z'=z-\varphi_0(x, y)$.

Set $\varphi_{jm}=\left.\frac{\partial^m\varphi_j}{\partial t^m}\right|_{t=0}$ and $U_{km}=\left.\frac{\partial^m U_k}{\partial z^m}\right|_{t=0}$ $(j=1, 2, 3;\ k=0, 1, 2, 3;\ m\geqslant 0)$. Since $z=\varphi_j(t, x, y)$ is the j-th characteristic surface starting from $\{z=t=0\}$ $(j=1, 2, 3)$, we have

$$\varphi_{jt}=\lambda_j(U, \varphi_{jx}, \varphi_{jy}, -1),\ \varphi_j(0, x, y)=0, \tag{2.3}$$

where $\lambda_j(U, \xi_1, \xi_2, \xi_3)$ is the j-th eigenvalue of $B_0^{-1}(\xi_1B_1+\xi_2B_2+\xi_3B_3)$. Therefore, we establish the 0-th order compatibility condition as $\varphi_{j1}=\lambda_j(U, 0, 0, -1)$.

Differentiating (2.3) along with $z=\varphi_j(t, x, y)$, we have

$$\frac{\partial^2\varphi_j}{\partial t^2}=\frac{\partial\lambda_j}{\partial U}(\partial_t+\varphi_{jt}\partial_z)U+\frac{\partial\lambda_j}{\partial\xi_1}\frac{\partial^2\varphi_j}{\partial x\partial t}+\frac{\partial\lambda_j}{\partial\xi_2}\frac{\partial^2\varphi_j}{\partial y\partial t}.$$

The restriction of the above relation to $t=0$ is

$$\begin{aligned}\varphi_{j2}=&\frac{\partial\lambda_j}{\partial U}(\varphi_{j1}-B_0^{-1}B_3)U_{j1}-\frac{\partial\lambda_j}{\partial U}\Big(B_0^{-1}B_1\frac{\partial U_j}{\partial x}+B_0^{-1}B_2\frac{\partial U_j}{\partial y}\Big)\\&+\frac{\partial\lambda_j}{\partial\xi_1}\frac{\partial\varphi_{j1}}{\partial x}+\frac{\partial\lambda_j}{\partial\xi_2}\frac{\partial\varphi_{j1}}{\partial y}.\end{aligned}\tag{2.4}$$

Furthermore, this relation is still valid as substituting U_{j1} by $U_{j-1,1}$. Because the tangential derivatives of U_j and $U_{j-1,1}$ along the surface $\{z=\varphi_j\}$ are continuous on this surface, and their normal derivatives could have jump, we have

$$\frac{\partial\lambda_j}{\partial U}(\varphi_{j1}-B_0^{-1}B_3)(U_{j1}-U_{j-1,1})=0. \tag{2.5}$$

We call this relation the first order compatibility condition. Similarly, we can establish the higher order compatibility conditions as

$$(\varphi_{j1}-B_0^{-1}B_3)^m(U_{jm}-U_{j-1,m})=0,\ m=1, \cdots, \tag{2.6}$$

By (2.6) we have $U_{jm}-U_{j-1,m}=W_{jm}e_j(j=1 \text{ or } 3)$ and $U_{2m}-U_{1m}=W'_{2m}e'_2+W''_{2m}e''_2$, where e_1 (resp. e_3) is the first (resp. third) unit eigenvector of $B_0^{-1}B_3$, e'_2 and e''_2 are its two linearly independent unit eigenvectors corresponding to the second eigenvalue. Hence

$$U_{3m}-U_{0m}=W_{3m}e_3+W'_{2m}e'_2+W''_{2m}e''_2+W_{1m}e_1. \tag{2.7}$$

Since U_{0m} and U_{3m} are known, $\{e_1, e'_2, e''_2, e_3\}$ is linearly independent, we can uniquely solve W_{1m}, W'_{2m}, W''_{2m} and W_{3m} from (2.7), and then U_{1m}, U_{2m} and $\varphi_{j,m+1}$ can be uniquely determined. Therefore we have

Lemma 2.1. *For* (2.1) (2.2), *and any given integer* $s\geqslant 1$, *we can define* $\{D^\alpha\varphi_j|_\Gamma (j=1, 2, 3), D^\beta U_1|_\Gamma, D^\beta U_2|_\Gamma: |\alpha|\leqslant s, |\beta|\leqslant s-1\}$ *such that the compatibility conditions on* Γ *up to the* $(s-1)$*-th order are satisfied.*

Remark. If the second class characteristic surfaces are denoted by $\Lambda(t, x, y, z)=\text{const.}$, and the surface starting from Γ is $\Lambda=0$, then $\Lambda_t=\lambda_2(U, \Lambda_x, \Lambda_y, \Lambda_z)$ must be satisfied. In this case, we still can get $\{D^\alpha\Lambda|_\Gamma\}_{|\alpha|\leqslant s}$ satisfying the compatibility conditions on Γ up to the $(s-1)$-th order.

The proof of Theorem 2.1 is the main content below. In § 3, we give a series of transformations, which make the problem simpler, and list some properties of weighted Sobolev spaces as a preparation of the further discussion. All necessary estimates of the corresponding linearized problem are established in § 4 which is the main part of this paper. Finally, in § 5 we use these estimates to prove the existence of the local solution to the nonlinear problem by an iteration process.

§ 3. Preparations

As the first step we transform the region to an angular region with two fixed boundaries by a series of coordinate transformations. Introduce

$$T_1: \begin{cases} t_1=t, \\ x_1=x, \\ y_1=y, \\ z_1=\dfrac{2z-\varphi_3(t, x, y)-\varphi_1(t, x, y)}{\varphi_3(t, x, y)-\varphi_1(t, x, y)}\cdot t, \end{cases} \tag{3.1}$$

which transforms the characteristic surface of the first class $\{z=\varphi_1\}$, the one of the third class $\{z=\varphi_3\}$, and the family of characteristic surfaces of the second class $\{\Lambda(t, x, y, z)=\text{const.}\}$ to $\{z_1=-t_1\}$, $\{z_1=t_1\}$ and $\{\Lambda_1(t_1, x_1, y_1, z_1)=\text{const.}\}$ respectively.

Introduce

$$T_2: \begin{cases} t_2=t_1, \\ x_2=x_1, \\ y_2=y_1, \\ z_2=\Lambda_1(t_1, x_1, y_1, z_1), \end{cases} \tag{3.2}$$

which transforms all characteristic surfaces of the second class to $\{z_2=\text{const.}\}$, and keeps the form of characteristic surfaces of the first and the third classes $\{z_2=\pm t_2\}$ unchanged.

By the transformation T_2T_1, (2.1) is changed to

$$B_0^{(2)}\frac{\partial U}{\partial t_2}+B_1^{(2)}\frac{\partial U}{\partial x_2}+B_2^{(2)}\frac{\partial U}{\partial y_2}+B_3^{(2)}\frac{\partial U}{\partial z_2}=0,\ t_2>0,\ -t_2<z_2<t_2, \tag{3.3}$$

where $B_0^{(2)}=B_0$, $B_1^{(2)}=B_1$, $B_2^{(2)}=B_2$, $B_3^{(2)}=\frac{\partial A_1}{\partial z_1}B_3^{(1)}+\frac{\partial A_1}{\partial t_1}B_0+\frac{\partial A_1}{\partial x_1}B_1+\frac{\partial A_1}{\partial y_1}B_2$ with

$$B_3^{(1)}=B_0\frac{\partial z_1}{\partial t}+B_1\frac{\partial z_1}{\partial x}+B_2\frac{\partial z_1}{\partial y}+B_3\frac{\partial z_1}{\partial z} \tag{3.4}$$

and $\frac{\partial z_1}{\partial t}$, $\frac{\partial z_1}{\partial x}$, $\frac{\partial z_1}{\partial y}$ and $\frac{\partial z_1}{\partial z}$ can be expressed by (3.1).

The corresponding boundary conditions are transformed into

$$U|_{z_2=t_2}=U_3|_{z_2=t_2}\ (\text{known}) \tag{3.5}$$

and

$$U|_{z_2=-t_2}=U_0|_{z_2=-t_2}\ (\text{known}), \tag{3.6}$$

Finally, we introduce

$$T_3:\begin{cases} t_3=t_2,\\ x_3=x_2,\\ y_3=y_2,\\ z_3=\begin{cases} z_2, & z_2>0,\\ -z_2, & z_2<0,\end{cases}\end{cases} \tag{3.7}$$

and set $V(t_3, x_3, y_3, z_3)=(V_1, V_2)^T=(U(t_3, x_3, y_3, z_3), U(t_3, x_3, y_3, -z_3))^T$, $A_i=\text{diag}\,(B_i^{(2)}, B_i^{(2)})$ $(i=0, 1, 2)$ and $A_3=\text{diag}(B_3^{(2)}, -B_3^{(2)})$, where the notation diag $(a_1, \cdots, a_n)$ stands for the diagonal matrix with $(a_1, \cdots, a_n)$ being its diagonal elements. Then the problem (3.3)—(3.6) is equivalent to

$$\begin{cases} LV=A_0\frac{\partial V}{\partial t_3}+A_1\frac{\partial V}{\partial x_3}+A_2\frac{\partial V}{\partial y_3}+A_3\frac{\partial V}{\partial z_3}=0,\ 0<z_3<t_3, & (3.8)\\ V_1|_{z_3=0}=V_2|_{z_3=0}, & (3.9)\\ V_1|_{z_3=t_3}=U_3,\ V_2|_{z_3=t_3}=U_0. & (3.10)\end{cases}$$

The coefficients of the system (2.8) contain an unknown function representing the characteristic surfaces of the second class. Because $A(t, x, y, z)$ satisfies $A_t+uA_x+vA_y+wA_z=0$, $A_{1t_1}+uA_{1x_1}+vA_{1y_1}+\left[\frac{\partial z_1}{\partial t}+u\frac{\partial z_1}{\partial x}+v\frac{\partial z_1}{\partial y}+w\frac{\partial z_1}{\partial z}\right]A_{1z_1}=0$ must hold for $A_1(t_1, x_1, y_1, z_1)$. If $\zeta=\phi(t, x, y, z)$ is the inverse function of $z=A_1(t, x, y, \zeta)$, then $\phi(t_2, x_2, y_2, z_2)$ must satisfy

$$\frac{\partial\phi}{\partial t_2}+a\frac{\partial\phi}{\partial x_2}+b\frac{\partial\phi}{\partial y_2}+c=0,\ t_2>0,\ -t_2<z_2<t_2, \tag{3.11}$$

where $a=u$, $b=v$, $c=-\left(\frac{\partial z_1}{\partial t}+u\frac{\partial z_1}{\partial x}+v\frac{\partial z_1}{\partial y}+w\frac{\partial z_1}{\partial z}\right)$, with the arguments (t_1, x_1, y_1, z_1) being substituted by (t_2, x_2, y_2, ϕ). Meanwhile, on $z_2=\pm t_2$, ϕ safisties

$$\phi(t_2, x_2, y_2, z_2)|_{z_2=\pm t_2}=z_2. \tag{3.12}$$

Under the transformation T_3, the problem (3.11) (3.12) is equivalent to

$$\begin{cases}\dfrac{\partial\varphi}{\partial t_3}+W_1\dfrac{\partial\varphi}{\partial x_3}+W_2\dfrac{\partial\varphi}{\partial y_3}+W_3=0,\ 0<z_3<t_3, & (3.13)\\ \varphi(t_3, x_3, y_3, z_3)|_{z_3=t_3}=z_3,\ \varphi_1|_{z_3=0}=\varphi_2|_{z_3=0}, & (3.14)\end{cases}$$

where $\varphi(t_3, x_3, y_3, z_3)=(\varphi_1, \varphi_2)^T=(\varphi(t_3, x_3, y_3, z_3), \varphi(t_3, x_3, y_3, -z_3))^T$, $W_1=\mathrm{diag}(a(V_1), a(V_2))$, $W_2=\mathrm{diag}(b(V_1), b(V_2))$ and $W_3=(c(V_1, \varphi_1), c(V_2, \varphi_2))^T$.

Obviously, the proof of Theorem 2.1 is equivalent to finding the solution (V, φ) to the transformed problem (3.8)—(3.14).

Let $\Omega=\{0<z_3<t_3\}$, $\Gamma=\{t_3=z_3=0\}$, $\Sigma_1=\{z_3=0,\ t_3>0\}$, $\Sigma_2=\{z_3=t_3>0\}$, and denote the intersections of Ω, Σ_1, Σ_2 with $\{t_3<T\}$ by Ω_T, Σ_{1T}, Σ_{2T} respectively. By using the values of derivatives of φ_j and U_k on Γ given in Lemma 2.1, we may easily construct function $\varphi^{(0)}$, $V^{(0)}\in H^s$ such that

$$\begin{cases}L_0V^{(0)}=O(t_3^{s-1}) \text{ in } \Omega,\\ V_1^{(0)}=V_2^{(0)} \text{ on } \Sigma_1,\\ V_1^{(0)}=U_3,\ V_2^{(0)}=U_0 \text{ on } \Sigma_2,\\ \dfrac{\partial\varphi^{(0)}}{\partial t_3}+W_1(V^{(0)})\dfrac{\partial\varphi^{(0)}}{\partial x_3}+W_2(V^{(0)})\dfrac{\partial\varphi^{(0)}}{\partial y_3}+W_3(V^{(0)}, \varphi^{(0)})=0 \text{ in } \Omega,\\ \varphi^{(0)}=z_3 \text{ on } \Sigma_2,\\ \varphi_1^{(0)}=\varphi_2^{(0)} \text{ on } \Sigma_1,\end{cases} \tag{3.5}$$

where $L_0=L(V^{(0)}, \nabla\varphi^{(0)})$ as indicated in (3.8). The way of making $(V^{(0)}, \varphi^{(0)})$ can be found in [5].

To alleviate notations burden, we will denote (t_3, x_3, y_3, z_3) of the system (3.8)—(3.14) (the original coordinate of the system (2.1) (2.2), resp.) by (t, x, y, z) ((t', x', y', z') resp.) in the following discussion.

In order to obtain the existence of the solution to the nonlinear prodlem, we need to carry out our discussion in weighted Sobolev spaces. Denote

$$L^2_\lambda(\Omega_T)=t^\lambda L^2(\Omega_T),\ \|u\|_{L^2_\lambda(\Omega_T)}=\|t^{-\lambda}u\|_{L^2(\Omega_T)};$$

$$H^k_\lambda(\Omega_T)=\{u\,|\,\partial^\alpha u\in L^2_{\lambda-\alpha'}(\Omega_T),\ \forall\,|\alpha|\leqslant k\},$$

where $\alpha=(\alpha_t, \alpha_x, \alpha_y, \alpha_z)$, $\alpha'=\alpha_t+\alpha_z$. The norm of $H^k_\lambda(\Omega_T)$ is defined by

$$\|u\|_{k,\lambda,T}=\{\sum_{|\alpha|\leqslant k}\lambda^{2(k-\alpha')}\|\partial^\alpha u\|^2_{L^2_{\lambda-\alpha'}(\Omega_T)}\}^{1/2}.$$

By the above definitions, it is obvious that $\|t^{-\lambda+k}u\|_{H^k(\Omega_T)}\leqslant C\|u\|_{k,\lambda,T}$. Using Sobolev embedding theorem we have

$$\|t^{-\lambda}u\|_{L^\infty(\Omega_T)}\leqslant C\|u\|_{3,\lambda+3,T} \tag{3.16}$$

where C is independent of λ, T and u.

Furthermore, we introduce the tangential vector fields of $\Sigma_1\cup\Gamma$:

$$D_0=t\frac{\partial}{\partial t},\ D_1=z\frac{\partial}{\partial z},\ D_2=\frac{\partial}{\partial x},\ D_3=\frac{\partial}{\partial y} \tag{3.17}$$

and the norm

$$\|u\|'_{k,\lambda,T}=\{\sum_{|\alpha|\leq k}\lambda^{2(k-\alpha_0-\alpha_1)}\|D^\alpha u\|^2_{L^2_\lambda(\Omega_T)}\}^{1/2}$$

where $\alpha=(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$.

By direct computation it is easy to establish

Lemma 3.1. *Assume $k\geq 0$, $p\geq k+1$, and $T\leq T_0$, $\lambda\in\mathbf{R}_+$, $a\in H^p(\Omega_{T_0})$, $u\in H^k(\Omega_T)$. Then*

$$\|au\|_{k,\lambda,T}\leq C\|a\|_{H^p(\Omega_{T_0})}\|u\|_{k,\lambda,T}. \tag{3.18}$$

Lemma 3.2. *Assume $f\in C^\infty(\mathbf{R})$, $f(0)=0$, $T_0>0$, $k\geq 6$, $K>0$. Then there is a constant $C>0$ such that for any $\lambda\geq k$, $u\in H^k_\lambda(\Omega_{T_0})$ satisfying $\|u\|_{k,\lambda,T}\leq K$, we have $f(u)\in H^k_\lambda(\Omega_{T_0})$ and*

$$\|f(u)\|_{k,\lambda,T}\leq C\|u\|_{k,\lambda,T} \tag{3.19}$$

for any $T\leq T_0$.

§ 4. Energy Estimates of the Linearized Problem

For the nonlinear problem (3.8)—(3.14), we establish the following iteration process

$$\begin{cases} V^n=V^{(0)}+\xi^n, \ \xi^n=E_T(\zeta^n), \\ \varphi^n=\varphi^{(0)}+\psi^n, \ \psi^n=E_T(\theta^n), \end{cases} \tag{4.1}$$

where E_T is a bounded extending operator from $H^k_\lambda(\Omega_T)$ to $H^k_\lambda(\Omega_{T_0})$ satisfying $\|E_Tu\|_{k,\lambda,T_0}\leq K\|u\|_{k,\lambda,T}$ for any $k\leq N$, K is independent of k, λ, T, and θ^n, ζ^{n+1} are the solutions to the following linearized problem

$$\begin{cases} L(V^n, \nabla\varphi^n)\zeta^{n+1}=-L(V^n, \nabla\varphi^n)V^{(0)} \text{ in } \Omega, & (4.2)\\ \zeta_1^{n+1}=\zeta_2^{n+1}, \text{ on } \Sigma_1, & (4.3)\\ \zeta^{n+1}=0 \text{ on } \Sigma_2, & (4.4)\\ \dfrac{\partial\theta^n}{\partial t}+W_1(V^n)\dfrac{\partial\theta^n}{\partial x}+W_2(V^n)\dfrac{\partial\theta^n}{\partial y}+W_3(V^n, \varphi^{(0)}+\theta^n) & \\ \quad =-\left(\dfrac{\partial\varphi^{(0)}}{\partial t}+W_1(V^n)\dfrac{\partial\varphi^{(0)}}{\partial x}+W_2(V^n)\dfrac{\partial\varphi^{(0)}}{\partial y}\right) \text{ in } \Omega, & (4.5)\\ \theta^n=0 \text{ on } \Sigma_2. & (4.6) \end{cases}$$

The purpose of this section is to establish prior estimates of this linearized problem, the existence of its solutions is naturally obtained. First, by the choice of $(V^{(0)}, \varphi^{(0)})$ in § 3, we confirm that for any integer k and constant $\Delta>0$, there is $T_0>0$ such that $|z|_{k_0,T_0}+|U_0|_{k_0}, T_0+|U_3|_{k_0,T_0}<\Delta$, where the notation $|\ |_{k_0,T_0}$ is the norm of the Sobolev space $H^{k_0}(\Sigma_{T_0})$. From (3.15) we know

$$\|V^{(0)}\|_{k_0,T_0}, \ \|\varphi^{(0)}\|_{k_0,T_0}<2\Delta, \tag{4.7}$$

$$\|L_0V^{(0)}\|_{k_0,\lambda+2,T_0}<\Delta \tag{4.8}$$

by possibly decreasing T_0 and setting $s=\lambda+3$, where $\|\ \|_{k_0,T_0}$ represents the norm of the Sobolev space $H^{k_0}(\Omega_{T_0})$.

For $k\geq 7$, let $s_T=\{(\xi, \psi)\in H^k_\lambda(\Omega_T)\times H^k_\lambda(\Omega_T)\,|\,\lambda(\|\xi\|_{k,\lambda,T}+\|\psi\|_{k,\lambda,T})\leq K^{-1}\Delta\}$, and its norm is defined by $\|(\xi, \psi)\|_{k,\lambda,T}=\lambda(\|\xi\|_{k,\lambda,T}+\lambda\|\psi\|_{k,\lambda,T})$.

A prior estimate of the problem (4.5) (4.6) is

Theorem 4.1. *If $k_0\geqslant k+4$, $(\xi^n, 0)\in\varepsilon_T$, λ is sufficiently large, then the solution θ^n of (4.5) (4.6) satisfies*

$$\lambda\|\theta^n\|_{k,\lambda,T}\leqslant C\|\xi^n\|_{k,\lambda-1,T}. \tag{4.9}$$

Proof Because $\dfrac{\partial\varphi^{(0)}}{\partial t}+W_1(V^{(0)})\dfrac{\partial\varphi^{(0)}}{\partial x}+W_2(V^{(0)})\dfrac{\partial\varphi^{(0)}}{\partial y}+W_3(V^{(0)}, \varphi^{(0)})=0$, (4.5) is equivalent to

$$\frac{\partial\theta^n}{\partial t}+W_1(V^n)\frac{\partial\theta^n}{\partial x}+W_2(V^n)\frac{\partial\theta^n}{\partial y}=e=e_1+e_2, \tag{4.10}$$

where $e_1=W_3(V^{(0)}, \varphi^{(0)})-W_3(V^n, \varphi^{(0)}-\theta^n)$ and

$$e_2=(W_1(V^{(0)})-W_1(V^n))\frac{\partial\varphi^{(0)}}{\partial x}+(W_2(V^{(0)})-W_2(V^n))\frac{\partial\varphi^{(0)}}{\partial y}.$$

From Lemmas 3.1 and 3.2, we have

$$\|e_2\|_{k,\lambda,T}\leqslant C\|\xi^n\|_{k,\lambda,T}. \tag{4.11}$$

Furthermore,

$$\|e_1\|_{k,\lambda,T}\leqslant C(\Delta)\|\theta^n\|_{k,\lambda,T}. \tag{4.12}$$

Multiplying $t^{-2\lambda+1}\theta^n$ on both sides of (4.10) integrating it on Ω_T, and then by using the condition (4.5), we have

$$\lambda\|\theta^n\|_{L^2_\lambda(\Omega_T)}\leqslant\|e\|_{L^2_{\lambda-1}(\Omega_T)}. \tag{4.13}$$

Suppose D is the tangential vector indicated in (3.17). Acting D^α on (4.5) (4.6) and using the fact that Σ_2 is a noncharacteristic surface of (4.5), we have

$$\begin{cases}\left(\dfrac{\partial}{\partial t}+W_1(V^n)\dfrac{\partial}{\partial x}+W_2(V^n)\dfrac{\partial}{\partial y}\right)D^\alpha\theta^n\\ \quad=\left[\dfrac{\partial}{\partial t}+W_1(V^n)\dfrac{\partial}{\partial x}+W_2(V^n)\dfrac{\partial}{\partial y},\ D^\alpha\right]\theta^n-D^\alpha e, \text{ in } \Omega\\ D^\alpha\theta^n|_{\Sigma_2}=0.\end{cases}$$

By the similar way we can establish

$$\begin{aligned}\lambda\|D^\alpha\theta^n\|_{L^2_\lambda(\Omega_T)}&\leqslant\left\|\left[\frac{\partial}{\partial t}+W_1(V^n)\frac{\partial}{\partial x}+W_2(V^n)\frac{\partial}{\partial y},\ D^\alpha\right]\theta^n-D^\alpha e\right\|_{L^2_{\lambda-1}(\Omega_T)}\\ &\leqslant\left\|\left[\frac{\partial}{\partial t}+W_1(V^n)\frac{\partial}{\partial v}+W_2(V^n)\frac{\partial}{\partial y},\ D^\alpha\right]\theta^n\right\|_{L^2_{\lambda-1}(\Omega_T)}+\|D^\alpha e\|_{L^2_{\lambda-1}(\Omega_T)},\end{aligned}$$

which yields

$$\lambda\|\theta^n\|_{k,\lambda,T}\leqslant C(\|\xi^n\|_{k,\lambda-1,T}+C(\Delta)\|\theta^n\|_{k,\lambda,T}) \tag{4.14}$$

as λ is sufficiently large.

Combining (4.14) with (4.5) immediately leads to (4.9).

Next, we are going to consider the problem (4.2)—(4.4). The system (4.2) is symmetric hyperbolic one, then it has strong solution if its boundary conditions are maximal dissipative. The conditions (4.3) and (4.4) seem to be too many, but they are in fact equivalent to maximal dissipative ones. For instance, let us show the fact for (4.3). Because the coefficient matrix of $\frac{\partial}{\partial z}$ in the operator L is diag $(B_3^{(2)}, -B_3^{(2)})$, by using a suitable transformation of ζ_1^{n+1} we can reduce $B_3^{(2)}$ to diag $(0, \beta^*)$ where β^* is a 3×3 symmtric matrix. Similarly, $-B_3^{(2)}$ can be changed to diag$(0, -\beta^*)$ by the same transformation of ζ_2^{n+1}. It is easy to verify that the condition

$$(\zeta_1^{n+1})^* = (\zeta_2^{n+1})^* \tag{4.15}$$

is maximal dissipative on Σ_1, if ξ_i^{n+1} is rewritten as $((\zeta_i^{n+1})', (\zeta_i^{n+1})^*)^T$.

Now, let us point out that (4.15) implies (4.3). Because $\frac{\partial}{\partial z}\zeta^{n+1}$ does not appear in the first and the fifth equations of this system after the above transformation, these two equations can be regarded as differential equations of functions (ζ^{n+1}) on Σ_1. Therefore, $(\zeta_1^{n+1})' = (\zeta_2^{n+1})'$ holds on Σ_1, for it is true at $t=0$. Combining this with (4.15) leads to (4.3), and thus we have shown that (4.3) is equivalent to a maximal dissipative boundary condition on Σ_1. The same method can be applied to showing that (4.4) is equivalent to a maximal dissipative one $(\zeta^{n+1})^* = 0$ on Σ_2.

In the sequel we still take the form of the boundary problem as (4.2)—(4.4), and try to establish energy estimates of its solutiou, i. e., to estimate ζ^{n+1} by the norms of ξ^n and θ^n. First, denoting $-L(V^n, \nabla\varphi^n)V^{(0)}$ by G, we have

Theorem 4.2. *If $(\xi^n, \psi^n)\in s_T$, λ is sufficiently large, then the solution ζ^{n+1} of (4.2)—(4.4) satisfies*

$$\lambda\|\zeta^{n+1}\|_{0,\lambda,T} \leqslant C\|G\|_{0,\lambda-1,T}. \tag{4.16}$$

Proof We will use the method of dyadic decomposition to derive the energy estimates in the angular region Ω_T (This method can be refered to [3], [7]). Introduce a decomposition

$$\sum_{j=-\infty}^{\infty} \chi(2^j t) = 1, \ \forall t>0$$

where $\chi\in C_0^\infty(\mathbf{R}_+)$, supp $\chi\subset(1/2, 2)$. Let $\zeta^{n+1,j}(t,x,y,z) = \chi(2^j t)t^{-\lambda}\zeta^{n+1}(t,x,y,z)$ and $\tilde\zeta^{n+1,j}(t,x,y,z) = 2^{-j}\zeta^{n+1,j}(2^{-j}t, x, y, 2^{-j}z)$, where the index j of $\zeta^{n+1,j}$ satisfies $2^{-j-1}<T_0$. Then $2^{-1}<t<\min(2, 2^jT)$ in supp $\tilde\zeta^{n+1,j}$. Obviously, $\|\zeta^{n+1,j}\|_{L^2} = \|\tilde\zeta^{n+1,j}\|_{L^2}$, and $\|\zeta^{n+1}\|_{0,\lambda,T}$ is equivalent to $\sum_{2^{-j-1}<T_0}\|\zeta^{n+1,j}\|_{L^2(\Omega_T)}$.

Since ζ^{n+1} satisfies the problem (4.2)—(4.4), $\tilde\zeta^{n+1,j}$ satisfies

$$\begin{cases} \tilde L_\lambda^j\zeta^{n+1,j} = \left(t\tilde A_0\frac{\partial}{\partial t} + 2^{-j}t\tilde A_1\frac{\partial}{\partial x} + 2^{-j}t\tilde A_2\frac{\partial}{\partial y} + t\tilde A_3\frac{\partial}{\partial z} + \lambda\tilde A_0\right)\zeta^{n+1,j} = \tilde F^j, \\ \tilde\zeta^{n+1,j} = \tilde\zeta^{n+1,j} \text{ on } \Sigma_1,\ 2^{-1}<t<\min(2, 2^jT_0), \\ \tilde\zeta^{n+1,j} = 0 \text{ on } \Sigma_2, \end{cases}$$

where $\tilde A_i(t,x,y,z) = \tilde A_i(2^{-j}t, x, y, 2^{-j}z)$ $(i=0,1,2,3)$, $\tilde F^j = -\tilde F_1^j + \tilde F_2^j$ with $F_1^j = \chi(2^jt)t^{-\lambda+1}G$, $F_2^j = 2^jA_0\chi'(2^jt)t^{-\lambda+1}\zeta^{n+1}$.

For the above problem, the theory of symmetric hyperbolic systems shows

$$\lambda\|\zeta^{n+1,j}\|_{L^2}\leqslant C\|\tilde{F}^j\|_{L^2},$$

which is equivalent to

$$\lambda\|\zeta^{n+1}\|_{0,\lambda,T}\leqslant C(\|G\|_{0,\lambda-1,T}+\|\zeta^{n+1}\|_{0,\lambda,T}).$$

Thus we are led to (4.16) as λ is large enough.

Before giving a higher order energy estimate of the problem (4.2)—(4.4), we introduce

Lemma 4.1. *For any integer $s_1\geqslant 0$, there is an integer $s_2\geqslant 0$ such that the values of $\{D^\alpha\zeta^{n+1}|_{\Sigma^2}: |\alpha|\leqslant s_1\}$ can be uniquely determined by the $\{D^\beta U_1|_\Gamma, D^\beta U_2|_\Gamma: |\beta|\leqslant s_2\}$ given in Lemma 2.1.*

Let us sketch the proof of this lemma. First, the tangential derivatives of ζ_1^{n+1} on Σ_2 can be directly obtained because the value of ζ_1^{n+1} on Σ_2 is known. Since Σ_2 is the simple characteristic surface of L in (4.2), the restriction of the coefficient of $\partial_n\zeta^{n+1}$ to Σ_2 may be changed into $\mathrm{diag}(0,\beta^*)$ in (4.2) by a suitable regular transformation of ζ_1^{n+1}, where β^* is a 3×3 invertible matrix, n is the normal direction of Σ_2. If ζ_1^{n+1} is written as $(((\zeta_1^{n+1})', (\zeta_1^{n+1})^*)^T$ correspondingly, then $\partial_n(\zeta_1^{n+1})^*|_{\Sigma^2}$ can be solved directly from (4.2), and $\partial_n(\zeta_1^{n+1})'$ does not appear in the first equation of (4.2). By acting ∂_n on both sides of the first equation of (4.2), we obtain a differential equation of $\partial_n(\zeta_1^{n+1})'$ on Σ_2. Taking $\partial_n(\zeta_1^{n+1})'|_\Gamma$ as its initial data, we can uniquely determine the value of $\partial_n(\zeta_1^{n+1})'|_{\Sigma^2}$ from this Cauchy problem. Again, by using the fact $\det(\beta^*)\neq 0$ we obtain the value of $\partial_n^2(\zeta_1^{n+1})^*|_{\Sigma^2}$; by solving a differential equation on Σ_2 we obtain the value of $\partial_n^2(\zeta^{n+1})'|_{\Sigma^2}$. Alternatively, we can uniquely determine the values of all derivatives of ζ_1^{n+1} on Σ_2. The results for ζ_2^{n+1} can be similarly obtained.

Theorem 4.3. *Suppose $k\geqslant 7$, $k'<k$, λ is large enough. Then the solution ζ^{n+1} to the problem (4.2)—(4.4) saisfies*

$$\lambda\|\zeta^{n+1}\|'_{k',\lambda,T}\leqslant C(\|\zeta^{n+1}\|_{k',\lambda,T}+\|G\|_{k',\lambda-1,T}). \tag{4.17}$$

Proof Act $D^\alpha(|\alpha|\leqslant k')$ on both sides of the problem (4.2)—(4.4):

$$\begin{cases} L(V^n,\nabla\varphi^n)D^\alpha\zeta^{n+1}=[L(V^n,\nabla\varphi^n),D^\alpha]\zeta^{n+1}+D^\alpha G,\\ D^\alpha\zeta_1^{n+1}=D^\alpha\zeta_2^{n+1}\ \text{on}\ \Sigma_1,\\ D^\alpha\zeta^{n+1}\ \text{is known on}\ \Sigma_2.\end{cases}$$

For this problem, by using Theorem 4.2 we have

$$\lambda\|D^\alpha\zeta^{n+1}\|_{0,\lambda,T}\leqslant C(\|[L(V^n,\nabla\varphi^n),D^\alpha]\zeta^{n+1}\|_{0,\lambda-1,T}+\|D^\alpha G\|_{0,\lambda-1,T}). \tag{4.18}$$

Set $$[L(V^n,\nabla\varphi^n),D^\alpha]\zeta^{n+1}=\sum_{\substack{|\beta_1|+|\beta_2|+|\beta_3|=|\alpha|\\|\beta_3|<|\alpha|}}hD^{\beta_1}V^nD^{\beta_2}\partial\varphi^nD^{\beta_3}\partial\zeta^{n+1}.$$

According to different cases of the index β_i, carefully applying Sobolev embedding theorem we can establish

$$\|[L(V^n,\nabla\varphi^n),D^\alpha]\zeta^{n+1}\|_{0,\lambda-1,T}\leqslant C(\Delta)\|\zeta^{n+1}\|_{|\alpha|,\lambda,T}.$$

Combining this estimate with (4.18) immediately leads to (4.17).

Theorem 4.3 gives an estimate of tangential derivatives of ζ^{n+1} with order lower than k. Before giving an estimate of the k-th order derivatives of ζ^{n+1}, we point out how to get the estimate of normal derivatives from the estimate of tangential derivatives by using the special form of the gas dynamic system.

Theorem 4.4. *If* $(\xi^n, \psi^n)\in s_T$, $0<k'<k$, λ *is large enough, then the solution* ζ^{n+1} *of* (4.2)—(4.4) *satisfies*

$$\lambda\|\zeta^{n+1}\|_{k',\lambda,T}\leqslant C(\lambda\|D(\zeta^{n+1})\|_{k'-1,\lambda,T}+\lambda\|G\|_{k'-1,\lambda-1,T}+T). \tag{4.19}$$

Proof For the system (4.2), the coefficient of $\frac{\partial}{\partial z}\zeta^{n+1}$ is $A=\mathrm{diag}\,(B_3^{(2)}, -B_3^{(2)})$, where

$$B_3^{(2)}=\begin{bmatrix}\rho_n E & & & \frac{\partial z}{\partial x'}\\ & \rho_n E & & \frac{\partial z}{\partial y'}\\ & & \rho_n E & \frac{\partial z}{\partial z'}\\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} & \rho_n E\end{bmatrix},$$

with $E=\frac{\partial z}{\partial t'}+u_n\frac{\partial z}{\partial x'}+v_n\frac{\partial z}{\partial y'}+w_n\frac{\partial z}{\partial z'}$.

Notice that φ^n satisfies (4.5), then all diagonal elements of $B_3^{(2)}$ vanish.

Since $\frac{\partial z}{\partial z'}=\frac{\partial A^{(2)}}{\partial z_2}\cdot\frac{\partial z}{\partial z_1}=\frac{\partial z}{\partial z_1}\Big/\frac{\partial\phi}{\partial z}$, $\left.\frac{\partial\phi}{\partial z}\right|_{\Sigma_2}=1$, then there is $T_0>0$, such that $\frac{\partial z}{\partial z'}\geqslant C>0$ on Ω_{T_0}. Hence

$$\|\zeta_{14}^{n+1}\|_{k',\lambda,T}+\left\|\frac{\partial z}{\partial x'}\zeta_{11}^{n+1}+\frac{\partial z}{\partial y'}\zeta_{12}^{n+1}+\frac{\partial z}{\partial z'}\zeta_{13}^{n+1}\right\|_{k',\lambda,T}$$
$$\leqslant C(\|D(\zeta^{n+1})\|_{k'-1,\lambda,T}+\|\zeta^{n+1}\|_{k'-1,\lambda,T}+\|G\|_{k'-1,\lambda-1,T}). \tag{4.20}$$

In order to estimate $\|\zeta_{11}^{n+1}\|_{k',\lambda,T}, \|\zeta_{12}^{n+1}\|_{k',\lambda,T}$ and $\|\zeta_{13}^{n+1}\|_{k',\lambda,T}$, we estimate the rotation of velocity vector, and come back to the original physical coordinates. In this coordinate system the iteration process (4.1)—(4.4) has the form

$$\begin{bmatrix}\rho_n & & & \\ & \rho_n & & \\ & & \rho_n & \\ & & & c_n^{2}\rho_n^{-1}\end{bmatrix}\frac{\partial}{\partial t'}\begin{bmatrix}u_{n+1}\\ v_{n+1}\\ w_{n+1}\\ p_{n+1}\end{bmatrix}+\begin{bmatrix}\rho_n u_n & & & 1\\ & \rho_n u_n & & \\ & & \rho_n u_n & \\ 1 & & & c_n^{-2}\rho_n^{-1}u_n\end{bmatrix}\frac{\partial}{\partial x'}\begin{bmatrix}u_{n+1}\\ v_{n+1}\\ w_{n+1}\\ p_{n+1}\end{bmatrix}$$
$$+\begin{bmatrix}\rho_n v_n & & & \\ & \rho_n v_n & & 1\\ & & \rho_n v_n & \\ & 1 & & c_n^{-2}\rho_n^{-2}v_n\end{bmatrix}\frac{\partial}{\partial y'}\begin{bmatrix}u_{n+1}\\ v_{n+1}\\ w_{n+1}\\ p_{n+1}\end{bmatrix}$$

$$+\begin{bmatrix} \rho_n w_n & & & \\ & \rho_n w_n & & \\ & & \rho_n w_n & 1 \\ & & 1 & c_n^{-2}\rho_n^{-1}w_n \end{bmatrix} \frac{\partial}{\partial z'} \begin{bmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \\ p_{n+1} \end{bmatrix} = 0. \tag{4.21}$$

Differentiating the first and the third equations with respect to z' and x' respectively, and then substracting one from other leads to

$$\left(\frac{\partial}{\partial t'} + u_n \frac{\partial}{\partial x'} + v_n \frac{\partial}{\partial y'} + w_n \frac{\partial}{\partial z'}\right)\left(\frac{\partial u_{n+1}}{\partial z'} - \frac{\partial w_{n+1}}{\partial x'}\right) = -h_n, \tag{4.22}$$

where

$$h_n = \rho_n^{-1}\left[\frac{\partial \rho_n}{\partial z'}\frac{\partial u_{n+1}}{\partial t'} - \frac{\partial \rho_n}{\partial x'}\frac{\partial w_{n+1}}{\partial t'} + \frac{\partial(\rho_n u_n)}{\partial z'}\frac{\partial u_{n+1}}{\partial x'} - \frac{\partial(\rho_n u_n)}{\partial z'}\frac{\partial w_{n+1}}{\partial x'} \right.$$
$$\left. + \frac{\partial(\rho_n v_n)}{\partial z'}\frac{\partial u_{n+1}}{\partial y'} - \frac{\partial(\rho_n v_n)}{\partial x'}\frac{\partial w_{n+1}}{\partial y'} + \frac{\partial(\rho_n w_n)}{\partial z'}\frac{\partial u_{n+1}}{\partial z'} - \frac{\partial(\rho_n w_n)}{\partial x'}\frac{\partial w_{n+1}}{\partial z'}\right].$$

Denote $\left(\frac{\partial v_{n+1}}{\partial z'} - \frac{\partial w_{n+1}}{\partial y'}, \ \frac{\partial w_{n+1}}{\partial x'} - \frac{\partial u_{n+1}}{\partial z'}, \ \frac{\partial u_{n+1}}{\partial y'} - \frac{\partial v_{n+1}}{\partial x'}\right)$ by $\mathbf{rot}_{n+1}$. Then each argument of $\mathbf{rot}_{n+1}$ satisfies an equality which is analogue of (4.22). Summing up those, we can obtain

$$\left(\frac{\partial}{\partial t'} + u_n \frac{\partial}{\partial x'} + v_n \frac{\partial}{\partial y'} + w_n \frac{\partial}{\partial z'}\right)\mathbf{rot}_{n+1} = \boldsymbol{H}_n. \tag{4.23}$$

Under the transformation $T = T_3 T_2 T_1$, (4.23) is changed into

$$\left(\frac{\partial}{\partial t} + u_n \frac{\partial}{\partial x} + v_n \frac{\partial}{\partial y}\right)\mathbf{rot}_{n+1} = \boldsymbol{H}_n. \tag{4.24}$$

Similarly, starting from (3.15) satisfied by $U^{(0)}$ we can obtain

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}\right)\mathbf{rot}_0 = \boldsymbol{H}_0 + \delta \tag{4.25}$$

where $s = O(t^{\lambda+1})$.

Combining (4.24) with (0.25), we have

$$\left(\frac{\partial}{\partial t} + u_n \frac{\partial}{\partial x} + v_n \frac{\partial}{\partial y}\right)(\mathbf{rot}_{n+1} - \mathbf{rot}_0) = \mathbf{F} \tag{4.26}$$

where $\mathbf{F} = \boldsymbol{H}_n - \boldsymbol{H}_0 - s - \left(\xi_{11}^n \frac{\partial}{\partial x} + \xi_{12}^n \frac{\partial}{\partial y}\right)\mathbf{rot}_0$.

From Lemma 4.1, the boundary condition of (4.26) is

$$(\mathbf{rot}_{n+1} - \mathbf{rot}_0)|_{\Sigma_2} \text{ is known.} \tag{4.27}$$

For the problem (4.26) (4.27), by a similar method as used in Theorem 4.1 we deduce

$$\lambda\left(\left\|\frac{\partial \zeta_{11}^{n+1}}{\partial z'} - \frac{\partial \zeta_{13}^{n+1}}{\partial x'}\right\|_{k'-1,\lambda,T} + \left\|\frac{\partial \zeta_{12}^{n+1}}{\partial z'} - \frac{\partial \zeta_{13}^{n+1}}{\partial y'}\right\|_{k'-1,\lambda,T}\right)$$
$$\leqslant C(\|\xi^n\|_{k',\lambda,T} + \|\zeta^{n+1}\|_{k',\lambda+1,T} + T). \tag{4.28}$$

From $\frac{\partial \zeta_{11}^{n+1}}{\partial z'} - \frac{\partial \zeta_{13}^{n+1}}{\partial x'}$, $\frac{\partial \zeta_{12}^{n+1}}{\partial z'} - \frac{\partial \zeta_{13}^{n+1}}{\partial y'}$, $\frac{\partial z}{\partial x'}\frac{\partial \zeta_{11}^{n+1}}{\partial z} + \frac{\partial z}{\partial y'}\frac{\partial \zeta_{12}^{n+1}}{\partial z} + \frac{\partial z}{\partial z'}\frac{\partial \zeta_{13}^{n+1}}{\partial z}$, we can uniquely determine the values of $\frac{\partial \zeta_{11}^{n+1}}{\partial z}$, $\frac{\partial \zeta_{12}^{n+1}}{\partial z}$ and $\frac{\partial \zeta_{13}^{n+1}}{\partial z}$. Hence

$$\lambda(\|\zeta_{11}^{n+1}\|_{k',\lambda,T}+\|\zeta_{12}^{n+1}\|_{k',\lambda,T}+\|\zeta_{13}^{n+1}\|_{k',\lambda,T})$$
$$\leqslant C(\lambda\|D(\zeta^{n+1})\|_{k'-1,\lambda,T}+\|\zeta^{n+1}\|_{k',\lambda,T}+\lambda\|G\|_{k'-1,\lambda-1,T}+T). \quad (4.29)$$

The same discussion can be applied to ζ_2^{n+1}. Combining (4.29) with (4.20) immediately deduces the conclusion.

For $G=-L(V^n, \nabla\varphi^n)V^{(0)}$ we have

Lemma 4.2. *If $(\xi^n, \psi^n)\in s_T$, $k_0\geqslant k+3$, λ is sufficiently large, $0\leqslant k'\leqslant k$, then*

$$\lambda\|G\|_{k',\lambda-1,T}\leqslant CT. \quad (4.30)$$

Proof Let $L_n=L(V^n, \nabla\varphi^n)$. Then

$$G=-L_nV^{(0)}=-(L_n-L_0)V^{(0)}-L_0V^{(0)}.$$

By using Theorem 4.1, we get

$$\|G\|_{k',\lambda-1,T}\leqslant C(\Delta)(\|\xi^n\|_{k',\lambda-1,T}+\|\nabla\psi^n\|_{k',\lambda-1,T})+\|L_0V^{(0)}\|_{k',\lambda-1,T}$$
$$\leqslant C(\Delta)\|\xi^n\|_{k'+1,\lambda-1,T}+\|L_0V^{(0)}\|_{k',\lambda-1,T}.$$

Therefore

$$\lambda\|G\|_{k',\lambda-1,T}\leqslant CT.$$

Combining Theorem 4.3 with Lemma 4.2 we obtain the following inequality

$$\lambda\|\zeta^{n+1}\|'_{k',\lambda,T}\leqslant C(\|\zeta^{n+1}\|_{k',\lambda,T}+T) \quad (4.31)$$

under the assumptions of Theorem 4.3. By virture of Theorem 4.4, we have

Theorem 4.5. *Suppose $(\xi^n, \psi^n)\in s_T$, $k_0\geqslant k+3$, $0\leqslant k'<k$, λ is large enough. Then for the problem (4.2)—(4.4), we have*

$$\lambda\|\zeta^{n+1}\|_{k',\lambda,T}<CT. \quad (4.32)$$

Now, let us derive the estimate of order k. First, we have

Theorem 4.6. *For the problem (4.2)—(4.4), if λ is sufficiently large, $|\alpha|\leqslant k$, then*

$$\lambda\left\|D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right\|_{0,\lambda,T}\leqslant C(T+\|\zeta^{n+1}\|_{k,\lambda-1,T}) \quad (4.33)$$

holds.

Proof (4.2) can be rewritten as

$$L^1(V^n, \nabla\phi^n)\zeta_1^{n+1}=\left(B_0^{(2)}\frac{\partial}{\partial t}+B_1^{(2)}\frac{\partial}{\partial x}+B_2^{(2)}\frac{\partial}{\partial y}+B_3^{(2)}\frac{\partial}{\partial z}\right)\zeta_1^{n+1}=G_1 \quad (4.34)$$

and

$$L^2(V^n, \nabla\phi^n)\zeta_2^{n+1}=\left(B_0^{(2)}\frac{\partial}{\partial t}+B_1^{(2)}\frac{\partial}{\partial x}+B_2^{(2)}\frac{\partial}{\partial y}-B_3^{(2)}\frac{\partial}{\partial z}\right)\zeta_2^{n+1}=G_2. \quad (4.35)$$

Without loss of generality, we only consider (4.34), and omit the index "1" in the following discussion for simplicity. The coefficients of L have the special form: $B_0^{(2)}=B_0$, $B_1^{(2)}=B_1$, $B_2^{(2)}=B_2$, $B_3^{(2)}=(B_3^{(1)}-B_0\phi_t^n-B_1\phi_x^n-B_2\phi_y^n)/\phi_z^n$, where $B_3^{(1)}$ has been shown in (3.4). Differentiating both sides of (4.34) with respect to x deduces

$$\frac{\partial}{\partial x}(L\zeta^{n+1})=L\zeta_x^{n+1}+L'_x\zeta^{n+1}+\mathscr{L}V_x^n-(L\phi_x^n)\frac{\zeta_z^{n+1}}{\phi_z^n},$$

where L'_x is the operator derived from L by differentiating its coefficients with respect to x, and

$$\mathscr{L}V^n_x=\frac{\partial B_0^{(2)}}{\partial V}V^n_x\frac{\partial\zeta^{n+1}}{\partial t}+\frac{\partial B_1^{(2)}}{\partial V}V^n_x\frac{\partial\zeta^{n+1}}{\partial x}+\frac{\partial B_2^{(2)}}{\partial V}V^n_x\frac{\partial\zeta^{n+1}}{\partial y}+\frac{\partial B_3^{(2)}}{\partial V}V^n_x\frac{\partial\zeta^{n+1}}{\partial z}.$$

Therefore

$$L\zeta^{n+1}_x+L'_x\zeta^{n+1}+\mathscr{L}V^n_x-(L\psi^n_x)\frac{\zeta^{n+1}_z}{\phi^n_z}-(L\phi^{(0)}_x)\frac{\zeta^{n+1}_z}{\phi^n_z}=\frac{\partial G}{\partial x}.$$

Through subtle computation (see [4]), we may obtain

$$L\left(\zeta^{n+1}_x-\frac{V^{n+1}_z}{\phi^n_z}\psi^n_x\right)=Q_2-\frac{\partial}{\partial x}(L_0V^{(0)}),\tag{4.36}$$

where $Q_2=(L\phi^{(0)}_x)\frac{V^{n+1}_z}{\phi^n_z}-(L_0\phi^{(0)}_x)\frac{V^{(0)}_z}{\phi^{(0)}_z}+\frac{\psi^n_x}{\phi^n_z}L'_zV^{n+1}+\frac{\psi^n_x}{\phi^n_z}(\mathscr{L}+\mathscr{L}_0)V^n_z-L'_xV^{n+1}+L'_{0x}V^{(0)}-\mathscr{L}V^n_x-\mathscr{L}_0V^n_x+l_0V^{(0)}_x+(L_0-L)V^{(0)}_x$, $\mathscr{L}_0$ is obtained from $\mathscr{L}$ by replacing $V^{(0)}$ for ζ^{n+1}, and l_0 is obtained by replacing $\phi^{(0)}$, $V^{(0)}$, $V^{(0)}$ for $\phi^{(n)}$, $V^{(n)}$, ζ^{n+1} respectively.

Similarly, if D_i is the tangential operator introduced in (3.17), then we have

$$L\left(D_i\zeta^{n+1}-\frac{D_i\psi^n}{\phi^n_z}V^{n+1}\right)=Q_i-D_i(L_0V^{(0)}),\ i=0,\ 1,\ 3,\tag{4.37}$$

where $Q_i(i=0,\ 1,\ 2,\ 3)$ is a function of $\nabla\zeta^{n+1}$, $\nabla\psi^n$, $\nabla\psi^{n+1}$, $\nabla^2V^{(0)}$ and $\nabla^2\phi^{(0)}$, it vanishes as $\zeta^{n+1}=\nabla\zeta^{n+1}=\psi^n=\nabla\psi^n=0$. Successively, we can get

$$L\left(D^\alpha\zeta^{n+1}-\frac{V^{n+1}_z}{\phi^n_z}D^\alpha\psi^n\right)=Q^{(\alpha)}-D^\alpha(L_0V^{(0)}),\tag{4.38}$$

where $|\alpha|=k'\leqslant k$, $Q^{(\alpha)}$ depends on $\nabla^\beta\zeta^{n+1}$, $\nabla\zeta\psi^n(|\beta|\leqslant k')$, and $Q^{(\alpha)}=0$ as $\nabla^\beta\zeta^{n+1}=\nabla^\beta\psi^n=0$.

Obviously, the form of (4.35) is similar to that of (4.34), so if L is taken as diag $(L^1,\ L^2)$, then

$$D^\alpha\zeta^{n+1}-\frac{V^{n+1}_z}{\varphi^n_z}D^\alpha\psi^n=\left(D^\alpha\zeta^{n+1}_1-\frac{V^{n+1}_{1z}}{\varphi^n_{1z}}D^\alpha\psi^n_1,\ D^\alpha\zeta^{n+1}_2-\frac{V^{n+1}_{2z}}{\varphi^n_{2z}}D^\alpha\psi^n_2\right)^T$$

also satisfies a symmetric hyperbolic system like (4.38).

Next, let us deduce the boundary conditions of $D^\alpha\zeta^{n+1}-\frac{V^{n+1}_z}{\varphi^n_z}D^\alpha\psi^n$ on Σ_1 and Σ_2. By (4.2) we know

$$\left(B_0\frac{\partial}{\partial t}+B_1\frac{\partial}{\partial x}+B_2\frac{\partial}{\partial y}+\frac{1}{\phi^n_z}B_3^{(2)\prime}\frac{\partial}{\partial z}\right)V^{n+1}=0,$$

where $B_3^{(2)\prime}=B_3^{(1)}-B_0\phi^n_t-B_1\phi^n_x-B_2\phi^n_z$, $B_3^{(1)}$ is indicated in (3.4), so

$$B_3^{(2)\prime}\frac{V^{n+1}_z}{\phi^n_z}=-B_0V^{n+1}_t-B_1V^{n+1}_x-B_2V^{n+1}_y.\tag{4.39}$$

The right side of (4.39) is continuous on Σ_1, because only tangential derivatives of V^{n+1} appear in it. Therefore, on Σ_1 we have

$$B_3^{(2)\prime}\left[\left(D^\alpha\zeta^{n+1}-\frac{V^{n+1}_z}{\varphi^n_z}D^\alpha\psi^n\right)_1-\left(D^\alpha\zeta^{n+1}-\frac{V^{n+1}_z}{\varphi^n_z}D^\alpha\psi^n\right)_2\right]=0.\tag{4.40}$$

The condition (4.40) is also maximal dissipative. In fact, by using the similar argument ahead of Theorem 4.2, (4.40) can be rewritten as

$$\beta^*(\sigma_1^*-\sigma_2^*)=0$$

if $B_3^{(2)\prime}$ has been transformed to diag $(0, \beta^*)$. The symmetricity of β^* deduces

$$(\sigma_2', \sigma_1^*, \sigma_2', \sigma_2^*)\begin{bmatrix} 0 & & & \\ & \beta^* & & \\ & & 0 & \\ & & & -\beta^* \end{bmatrix}\begin{bmatrix} \sigma_1' \\ \sigma_1^* \\ \sigma_2' \\ \sigma_2^* \end{bmatrix}=\sigma_1^*\beta^*\sigma_1^*-\sigma_2^*\beta^*\sigma_2^*=0.$$

Hence (4.40) is maximal dissipative.

Besides, since $\psi^n=0$ on Σ_2, by using the argument in the proof of Theorem 4.1 we have $D^\alpha\psi^n=0$ in Σ_2. Therefore $D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n$ is known on Σ_2.

Hence, for the system (4.2) we have ($|\alpha|\leqslant k'\leqslant k$)

$$\begin{cases} L\left(D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right)=Q^{(\alpha)}-D^\alpha(L_0V^{(0)}) \text{ in } \Omega, & (4.41)\\ B_3^{(2)\prime}\left[\left(D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right)_1-\left(D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right)_2\right]=0 \text{ on } \Sigma_1, & (4.42)\\ D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n \text{ is known on } \Sigma_2. & (4.43)\end{cases}$$

For the above problem, using the method of Theorem 4.2 we get

$$\lambda\left\|D^\alpha\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right\|_{0,\lambda,T}\leqslant C(\|Q^{(\alpha)}\|_{0,\lambda-1,T}+\|D^\alpha(L_0V^{(0)})\|_{0,\lambda-1,T}). \quad (4.44)$$

Because $\|Q^{(\alpha)}\|_{0,\lambda-1,T}\leqslant C(T+\|\zeta^{n+1}\|_{k',\lambda,T})$, (4.44) implies(4.33) immidiately.

Corollary 4.1. *As λ is sufficiently large,*

$$\lambda\|\zeta^{n+1}\|_{k,\lambda,T}\leqslant C(T+\|\zeta^{n+1}\|_{k,\lambda,T}) \quad (4.45)$$

is valid.

Proof Since $|\varphi_z^{(0)}|\geqslant C>0$ and $(\xi^n, \psi^n)\in s_T$, there is $T>0$ such that $|\varphi_z^n|\geqslant C/2>0$ is satisfied in Ω_T. Meanwhile, Theorem 4.5 indicates $\|V_z^{n+1}\|_{L^\infty}\leqslant C$, so

$$\lambda\left\|\frac{V_z^{n+1}}{\varphi_z^n}D^\alpha\psi^n\right\|_{0,\lambda,T}\leqslant C\lambda\|D^\alpha\psi^n\|_{0,\lambda,T}\leqslant C'\|\xi^n\|_{k,\lambda,T}$$

holds. Substituting these equalities into (4.33) we obtain (4.45).

Making use of the technique which is applied to estimating the normal derivatives of the solution by its tangential derivatives in Theorem 4.4 we can come to the following conclusion:

Theorem 4.7. *Under the assumtipions of Theorem 4.6, the solution ζ^{n+1} to (4.2)—(4.4) satisfies*

$$\lambda\|\zeta^{n+1}\|_{k,\lambda,T}\leqslant CT, \quad (4.46)$$

where $\lambda>0$ is large enough.

§ 5. Proof of Theorem 2.1

Now let us prove our main result Theorem 2.1. For the nonlinear problem (3.8)—(3.14), we can construct a sequence $\{(\xi^n, \psi^n)\}_{n\geq 1}$ in s_T by the iteration process (4.1)—(4.6) starting from an approximate solution $(V^{(0)}, \phi^{(0)})$ given in § 3. In fact, if $(\xi^n, \psi^n)\in \varepsilon_T$, then ξ^{n+1} can be obtaind from (4.2)—(4.4) and (4.1), so does ψ^{n+1} from (4.5) (4.6) and (4.1). By using Theorems 4.1 and 4.7, we have $(\xi^{n+1}, \psi^{n+1})\in \varepsilon_T$ as T is sufficiently small. Therefore, $\{(\xi^n, \psi^n)\}$ is a bounded sequence in ε_T.

Since θ^n satisfies

$$\frac{\partial\theta^n}{\partial t}+W_1(V^n)\frac{\partial\theta^n}{\partial x}+W_2(V^n)\frac{\partial\theta^n}{\partial y}+W_3(V^n, \varphi^{(0)}+\theta^n)$$
$$=-\left(\frac{\partial\varphi^{(0)}}{\partial t}+W_1(V^n)\frac{\partial\varphi^{(0)}}{\partial x}+W_2(V^n)\frac{\partial\varphi^{(0)}}{\partial y}\right)$$

for any $n\geqslant 1$, $\theta^{n+1}-\theta^n$ satisfies

$$\begin{cases}\left[\frac{\partial}{\partial t}+W_1(V^{n+1})\frac{\partial}{\partial x}+W_2(V^{n+1})\frac{\partial}{\partial y}\right](\theta^{n+1}-\theta^n)+W_3(V^{n+1}, \varphi^{(0)}+\theta^{n+1}) \\ \quad -W_3(V^n, \varphi^{(0)}+\theta^n) \\ \quad =[W_1(V^n)-W_1(V^{n+1})]\frac{\partial}{\partial x}(\varphi^{(0)}+\theta^n)+[W_2(V^n)-W_2(V^{n+1})]\frac{\partial}{\partial y}(\varphi^{(0)}+\theta^n) \text{ in } \Omega, \\ \theta^{n+1}-\theta^n=0 \text{ on } \Sigma_2. \end{cases} \tag{5.1}$$

Applying Theorem 4.1 to this problem, we obtain

$$\lambda\|\theta^{n+1}-\theta^n\|_{0,\lambda,T}\leqslant C\|\zeta^{n+1}-\zeta^n\|_{0,\lambda-1,T}. \tag{5.2}$$

Furthermore,

$$L_{n+1}(\zeta^{n+2}-\zeta^{n+1})=-[L_{n+1}-L_n]V^{n+1}$$
$$=-[L_{n+1}-L(V^n, \nabla\varphi^{n+1})]V^{n+1}-[L(V^n, \nabla\varphi^{n+1})-L_n]V^{n+1}.$$

Omitting the subscript 1, the first four components of $-[L(V^n,\nabla\varphi^{n+1})-L_n]V^{n+1}$ can be written as

$$-[L(V^n, \nabla\varphi^{n+1})-L_n]V^{n+1}=L_{n+1}\left[\frac{V_z^{n+1}}{\varphi_z^n}(\varphi^n-\varphi^{n+1})\right]+R_1,$$

where $R_1=\left[-L_{n+1}\left(\frac{V_z^{n+1}}{\varphi_z^n}\right)\right](\varphi^{n+1}-\varphi^n)+\frac{V_z^{n+1}}{\varphi_z^n}[L_{n+1}-L(V^n, \nabla\varphi^{n+1})](\varphi^n-\varphi^{n+1})$.

Hence,

$$\begin{cases} L(V^{n+1}, \nabla\varphi^{n+1})\left[(\zeta^{n+2}-\zeta^{n+1})-\frac{V_z^{n+1}}{\varphi_z^n}(\theta^{n+1}-\theta^n)\right]=R \text{ in } \Omega, \\ B_3^{(2)}\left\{\left[\zeta^{n+2}-\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}(\theta^{n+1}-\theta^n)\right]_1-\left[\zeta^{n+2}-\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}(\theta^{n+1}-\theta^n)\right]_2=0 \text{ on } \Sigma_1, \\ \zeta^{n+2}-\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}(\theta^{n+1}-\theta^n)=0 \text{ on } \Sigma_2, \end{cases} \tag{5.3}$$

where $R=-[L_{n+1}-L(V^n, \nabla\varphi^{n+1})]V^{n+1}+R_1$ satisfies $\|R\|_{0,\lambda,T}\leqslant C\|\zeta^{n+1}-\zeta^n\|_{0,\lambda,T}$.

Noticing that the boundary conditions in (5.3) are still equivalent to maximal dissipative ones, we have the following estimate

$$\lambda\left\|\zeta^{n+2}-\zeta^{n+1}-\frac{V_z^{n+1}}{\varphi_z^n}(\theta^{n+1}-\theta^n)_{0,\lambda,T}\right\|\leqslant C\|R\|_{0,\lambda-1,T}\leqslant C\|\zeta^{n+1}-\zeta^n\|_{0,\lambda-1,T}$$
$$\leqslant CT\|\zeta^{n+1}-\zeta^n\|_{0,\lambda,T}. \tag{5.4}$$

By the boundedness of V_z^{n+1} and the positive definiteness of φ_z^n, (5.2) and (5.4) imply

$$\lambda\|\zeta^{n+2}-\zeta^{n+1}\|_{0,\lambda,T}\leqslant CT\|\zeta^{n+1}-\zeta^n\|_{0,\lambda,T}. \tag{5.5}$$

Thus, we have known the convergence of the sequence $\{(\xi^n, \psi^n)\}_{n\geqslant 1}$ in $L^2_\lambda(\Omega_T)\times L^2_\lambda(\Omega_T)$ under the assumptions of Theorem 2.1. In view of Banach-Saks Theorem and the boundedness of $\{(\xi^n, \psi^n)\}_{n\geqslant 1}$ in s_T, its limit is also in s_T. Taking the limit for the linearized problem (4.2)—(4.6), we confirm that the limit (V, ϕ) of the sequence $\{(V^n, \phi^n)\}$ is the solution of (3.8)—(3.14).

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