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A HIERARCHY OF INTEGRABLE HAMILTONIAN SYSTEMS WITH NEUMANN TYPE CONSTRAINT**

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Abstract.

A hierarchy of integrable Hamiltonian systems with Neumann type constraint is obtained by restricting a hierarchy of evolution equations associated with $\lambda\phi_{xx}+\sum_{i=0}^{m-1}u_i\lambda^i\phi=\lambda^m\phi$ to an invariant subspace of their recursion operator. The independent integrals of motion and Hamiltonian functions for these Hamiltonian systems are constructed by using relevant recursion formula and are shown to be in involution. Thus these Hamiltonian systems are completely integrable and commute with each other.

§ 1. Introduction

Many finite-dimensional integrable Hamiltonian systems arise as restrictions of infinite-dimensional ones to finite-dimensional invariant submanifolds of their phase space (see, for example, [1—4]). In [5, 6] we proposed a straightforward way to obtain a hierarchy of finite-dimensional integrable Hamiltonian systems by restricting a hierarchy of integrable evolution equation to an invariant subspace of their recursion operator. In present paper, this approach is developed further and applied to the novel, spectral parameter dependent Schrödinger equation $\lambda\phi_{xx}+\sum_{i=1}^{m-1}\lambda^iu_i\phi=\lambda^m\phi$ for obtaining a hierarchy of finite-dimensional integrable Hamiltonian systems with Neumann type constraint on potential.

§ 2. The Constraint on Potential

Given the linear spectral problem[7]

$$\lambda\phi_{xx}+\sum_{i=0}^{m-1}u_i\lambda^i\phi=\lambda^m\phi, \tag{2.1}$$

where isospectral flows are shown to possess $(m+1)$ compatible Hamiltonian

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structures, consider time evolutions of the eigenfunction ϕ of the form:

$$\phi_{t_n}=-\frac{1}{2}B_x^{(n)}\phi+B^{(n)}\phi_x, \tag{2.2}$$

where

$$B^{(n)}=\sum_{k=0}^{n}b_k\lambda^{n-k},\ b_0=1,\ b_{k+1}=\frac{1}{2}R_{k,m-1}, \tag{2.3a}$$

$$R_k=(R_{k,0},\cdots,R_{k,m-1})^T,\ R_k=LR_{k-1}=L^ku,\ u=(u_0,\cdots,u_{m-1})^T, \tag{2.3b}$$

$$L=\begin{pmatrix} 0 & \cdots & 0 & J_0 \\ 1 & \cdots & 0 & J_1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & J_{m-1} \end{pmatrix},$$

$$J_1=\frac{1}{4}D^2+u_1-\frac{1}{2}D^{-1}u_{1x},\ J_i=u_i-\frac{1}{2}D^{-1}u_{ix},\ i=0,2,\cdots,m-1.$$

Here $D=\frac{\partial}{\partial x}$, the integral constant for the integral operator D^{-1} appearing in L is defined to be zero. Then the hierarchy of evolution equations[7] deduced from the solvability condition of (2.1) and (2.2) can be rewritten as

$$u_{t_n}=DL^nu. \tag{2.4}$$

Also, from the solvability condition it is known[7] that

$$b_k=\sum_{i=0}^{m-1}J_ib_{k-m+i}, \tag{2.5}$$

and if ϕ satisfies (2.1), then

$$D\sum_{i=0}^{m-1}\lambda^iJ_i\phi^2=\lambda^m(\phi^2)_x. \tag{2.6}$$

Indeed, (2.6) can be used to get another recursion formula for b_k. Rewrite (2.6) as

$$D\sum_{i=0}^{m-1}\lambda^iJ_iP=\lambda^mP_x. \tag{2.7}$$

Inserting the expansion

$$P=\sum_{j=0}^{\infty}P_j\lambda^{-j} \tag{2.8}$$

into (2.7), we find that P_j satisfy the same relationship (2.5) as b_j do. If take $P_0=1$, then we have $P_j=b_j$. Multiplying both sides of (2.7) by P and integrating it once yield

$$\lambda P_{xx}P-\frac{1}{2}\lambda(P_x)^2+2\sum_{i=0}^{m-1}\lambda^iu_iP^2=2\lambda^mP^2-2\lambda^m. \tag{2.9}$$

Then substituting (2.8) with $P_j=b_j$ into (2.9) gives

$$b_k=\frac{1}{2}\sum_{i=1}^{k}u_{m-i}\sum_{j=0}^{k-i}b_jb_{k-i-j}-\frac{1}{2}\sum_{j=1}^{k-1}b_jb_{k-j},\ k=1,\cdots,m-1, \tag{2.10a}$$

$$b_{k+m}=\frac{1}{4}\sum_{j=1}^{k+1}\left(b_{jxx}b_{k+1-j}-\frac{1}{2}b_{jx}b_{k+1-j,x}\right)+\frac{1}{2}\sum_{i=0}^{m-1}u_i\sum_{j=0}^{k+i}b_jb_{k+i-j}$$
$$-\frac{1}{2}\sum_{j=1}^{k+m-1}b_jb_{k+m-j},\ k=0,1,\cdots. \tag{2.10b}$$

Now, for distinct λ_j, consider following systems instead of (2.1)

$$\phi_{jxx}+\sum_{i=0}^{m-1}\lambda_j^{i-1}u_i\phi_j=\lambda_j^{m-1}\phi_j,\ j=1,\ \cdots,\ N. \tag{2.11}$$

From (2.6), we find that if $q=(q_1,\ \cdots,\ q_N)^T\equiv(\phi_1,\ \cdots,\ \phi_N)^T$ is a solution of (2.11), then

$$DL\Psi_j=\lambda_j\Psi_{jx},\ j=1,\ \cdots,\ N, \tag{2.12a}$$

where

$$\Psi_j=(\psi_{j,0},\ \cdots,\ \psi_{j,m-1})^T,$$

$$\psi_{j,0}=\lambda_j^{m-1}\phi_j^2-\sum_{i=1}^{m-1}\lambda_j^{i-1}J_i\phi_j^2,$$

$$\psi_{j,m-k}=\lambda_j^{k-1}\phi_j^2-\sum_{i=1}^{k-1}\lambda_j^{k-1-i}J_{m-i}\phi_j^2,$$

$$\psi_{j,m-1}=\phi_j^2. \tag{2.12b}$$

Throughout the paper no boundary condition on u and q is required. So (2.12a) leads to

$$L\Psi_j=\lambda_j\Psi_j+\sum_{i=0}^{m-1}\beta_ie_i, \tag{2.13a}$$

where β_i are integral constants, e_i is a vector with m components,

$$e_0=(1,\ 0,\ \cdots,\ 0)^T,\ \cdots,\ e_{m-1}=(0,\ \cdots,\ 0,\ 1)^T.$$

Note that

$$L\sum_{i=0}^{m-1}\beta_ie_i=\sum_{i=0}^{m-2}\beta_ie_{i+1}+\frac{1}{2}\beta_{m-1}u. \tag{2.13b}$$

If take

$$u=\sum_{j=1}^{N}\alpha_j\Psi_j+\sum_{i=0}^{m-1}\tilde{\alpha}_ie_i,$$

it follows from (2.13) that the linear space M spanned by $\{\Psi_1,\ \cdots,\ \Psi_N,\ e_0,\ \cdots,\ e_{m-1}\}$ is an invariant subspace of L. In particular, we take

$$u=2\sum_{j=1}^{N}\Psi_j+2Ce_0 \tag{2.14}$$

with $\frac{dC}{dx}=0$. Moreover, we assume throughout the paper that

$$\langle\Lambda^{-1}q,\ q\rangle=1, \tag{2.15}$$

where $\langle\cdot,\ \cdot\rangle$ is the inner product in $\mathbf{R}^N$, $\Lambda=\mathrm{diag}(\lambda_1,\ \cdots,\ \lambda_N)$. Then we have following

Proposition 1. *Under the condition* (2.15), *the constraint on potential* (2.14) *is equivalent to*

$$u_{m-k}=\sum_{i=1}^{k}a_i\sum_{l_1+\cdots+l_i=k-i}\langle\Lambda^{l_1}q,\ q\rangle\cdots\langle\Lambda^{l_i}q,q\rangle,\ k=1,\ \cdots,\ m-1, \tag{2.16a}$$

$$u_0=\frac{1}{\langle\Lambda^{-2}q,\ q\rangle}\Big[\langle\Lambda^{-1}p,\ p\rangle+\sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-1-i}\langle\Lambda^{l_1}q,\ q\rangle\cdots\langle\Lambda^{l_i}q,\ q\rangle\Big], \tag{2.16b}$$

$$C=\frac{1}{2}\langle p, p\rangle+\frac{1}{2}\sum_{i=1}^{m}(-1)^{i}\sum_{l_1+\cdots+l_i=m-i}\langle \Lambda^{l_1}q, q\rangle\cdots\langle \Lambda^{l_i}q, q\rangle, \tag{2.16c}$$

where $l_1\geqslant 0, \cdots, l_i\geqslant 0$, $p=(p_1, \cdots, p_N)^T\equiv(\phi_{1x}, \cdots, \phi_{Nx})^T$, *and*

$$a_i=(-1)^{i+1}(i+1).$$

Proof It is easy to see that (2.16a) holds for $k=1$. Using the identity

$$\sum_{i=1}^{k}\beta_{k-i}\sum_{j=1}^{i}\alpha_j\gamma_{i,j}=\sum_{j=1}^{k}\alpha_j\sum_{i=0}^{k-i}\beta_i\gamma_{k-i,j} \tag{2.17}$$

and denoting $\langle \Lambda^{l_i}q, q\rangle$ by $\langle l_i\rangle$ for brevity, we find from (2.12b) and (2.14) by induction ($k\leqslant m-1$) that

$$u_{m-k}=2\langle k-1\rangle-2\sum_{i=1}^{k-1}u_{m-i}\langle k-i-1\rangle+D^{-1}\sum_{i=1}^{k-1}u_{m-i,x}\langle k-i-1\rangle,$$

where we have

$$\begin{aligned}\sum_{i=1}^{k-1}u_{m-i}\langle k-i-1\rangle&=\sum_{i=1}^{k-1}\langle k-i-1\rangle\sum_{j=1}^{i}a_j\sum_{l_1+\cdots+l_j=i-j}\langle l_1\rangle\cdots\langle l_j\rangle\\&\overset{(2.17)}{=\!=}\sum_{j=1}^{k-1}a_j\sum_{i=0}^{k-1-j}\langle i\rangle\sum_{l_1+\cdots+l_j=k-1-i-j}\langle l_1\rangle\cdots\langle l_j\rangle\\&=\sum_{j=1}^{k-1}a_j\sum_{l_1+\cdots+l_{j+1}=k-1-j}\langle l_1\rangle\cdots\langle l_{j+1}\rangle\\&=\sum_{j=2}^{k}a_{j-1}\sum_{l_1+\cdots+l_j=k-j}\langle l_1\rangle\cdots\langle l_j\rangle.\end{aligned} \tag{2.18a}$$

In the same way, one obtains

$$\begin{aligned}&D^{-1}\sum_{i=1}^{k-1}u_{m-i,x}\langle k-i-1\rangle\\&=D^{-1}\sum_{i=1}^{k-1}\langle k-i-1\rangle\sum_{j=1}^{i}ja_j\sum_{l_1+\cdots+l_j=i-j}\langle l_1\rangle_x\langle l_2\rangle\cdots\langle l_j\rangle\\&=D^{-1}\sum_{j=2}^{k}a_{j-1}(j-1)\sum_{l_1+\cdots+l_j=k-j}\langle l_1\rangle_x\langle l_2\rangle\cdots\langle l_j\rangle\\&=\sum_{j=2}^{k}\frac{(j-1)}{j}a_{j-1}\sum_{l_1+\cdots+l_j=k-j}\langle l_1\rangle\cdots\langle l_j\rangle.\end{aligned} \tag{2.18b}$$

Using (2.18) we obtain (2.16a) immediately.

In similar way, using (2.15) and (2.16a), we find from (2.11) that

$$\begin{aligned}u_0&=\frac{1}{\langle \Lambda^{-2}q, q\rangle}\Big[\langle m-2\rangle-\langle \Lambda^{-1}p_x, q\rangle\\&\qquad-\sum_{k=1}^{m-1}\langle m-k-2\rangle\sum_{i=1}^{k}a_i\sum_{l_1+\cdots+l_i=k-i}\langle l_1\rangle\cdots\langle l_i\rangle\Big]\\&=\frac{1}{\langle \Lambda^{-2}q, q\rangle}\Big[\langle m-2\rangle+\langle \Lambda^{-1}p, p\rangle\\&\qquad-\sum_{i=1}^{m-1}a_i\sum_{l_1+\cdots+l_{i+1}=m-1-i}\langle l_1\rangle\cdots\langle l_i\rangle\langle l_{i+1}-1\rangle\Big]\\&=\frac{1}{\langle \Lambda^{-2}q, q\rangle}\Big[\langle m-2\rangle+\langle \Lambda^{-1}p, p\rangle\\&\qquad-\sum_{i=1}^{m-1}a_i\sum_{l_1+\cdots+l_i=m-1-i,\, l_{i+1}=0}\langle l_1\rangle\cdots\langle l_i\rangle\end{aligned}$$

$$-\sum_{i=1}^{m-2} a_i \sum_{l_1+\cdots l_{i+1}=m-2-i,\, l_{i+1}=1+l'_{i+1}} \langle l_1\rangle\cdots\langle l_i\rangle\langle l'_{i+1}\rangle \Big]$$

$$=\frac{1}{\langle \Lambda^{-2}q, q\rangle}\Big[\langle \Lambda^{-1}p, p\rangle - \langle m-2\rangle$$

$$-\sum_{i=2}^{m-1}(a_i+a_{i-1})\sum_{l_1+\cdots+l_i=m-i-1}\langle l_1\rangle\cdots\langle l_i\rangle\Big],$$

which leads to (2.16b).

Finally, in order to determine C, we compute the first component of (2.14):

$$u_0=2\langle m-1\rangle-2\sum_{i=1}^{m-1}u_{m-i}\langle m-i-1\rangle$$

$$+D^{-1}\sum_{i=1}^{m-1}u_{m-i,x}\langle m-i-1\rangle-\langle p, p\rangle-\langle p_x, q\rangle+2C$$

which, by substituting (2.11), becomes

$$u_0=\langle m-1\rangle-\sum_{i=1}^{m-1}u_{m-i}\langle m-i-1\rangle$$

$$+D^{-1}\sum_{i=1}^{m-1}u_{m-i,x}\langle m-i-1\rangle-\langle p, p\rangle+u_0+2C.$$

By means of (2.18) we obtain

$$C=\frac{1}{2}\langle p, p\rangle+\frac{1}{2}\sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_i\rangle.$$

Indeed we can prove that C is a constant of motion for (2.11) under the constraint on potential (2.16a, b) and (2.15), i.e.,

$$\frac{dC}{dx}=0.$$

§ 3. A Completely Integrable Hamiltonian System

Under the constraint on potential (2.15) and (2.16), in the same way as we did in (2.18a), (2.11) is reduced to

$$p_{jx}=-\frac{1}{\langle \Lambda^{-2}q, q\rangle}\Big[\langle \Lambda^{-1}p, p\rangle+\sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-i-1}\langle \Lambda^{l_1}q,q\rangle\cdots\langle \Lambda^{l_i}q, q\rangle\Big]\lambda_j^{-1}q_j$$

$$-\sum_{i=1}^{m}(-1)^i i\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_{i-1}\rangle\lambda_j^{l_i}q_j. \qquad (3.1a)$$

It can be written in canonical constrained Hamiltonian system

$$p_x=-\frac{\partial H_0}{\partial q},\quad q_x=\frac{\partial H_0}{\partial p} \qquad (3.1b)$$

which is defined on the tangent bundle of sphere:

$$TS^{N-1}=\{(p, q)\in R^{2N}\,|\,F=\frac{1}{2}(\langle \Lambda^{-1}q, q\rangle-1)=0,\ G=\langle \Lambda^{-1}p, q\rangle=0\}, \qquad (3.1c)$$

with

$$H_0=\frac{1}{2}\frac{1}{\langle\Lambda^{-2}q,q\rangle}\Big[\langle\Lambda^{-1}p,p\rangle$$
$$+\sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-i-1}\langle\Lambda^{l_1}q,q\rangle\cdots\langle\Lambda^{l_i}q,q\rangle\Big](\langle\Lambda^{-1}q,q\rangle-1)$$
$$+\frac{1}{2}\sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle\Lambda^{l_1}q,q\rangle\cdots\langle\Lambda^{l_i}q,q\rangle+\frac{1}{2}\langle p,p\rangle. \tag{3.1d}$$

The formulae (2.14) and (2.10) enable us to construct the simple integrals of motion for system (3.1). Assuming (p, q) satisfies (3.1), we have from (2.12), (2.13) and (2.14)

$$DLu|_A=2\sum_{j=1}^{N}\lambda_j\Psi_{jx},$$

which leads to

$$Lu|_A=2\sum_{j=1}\lambda_i\Psi_j+2C_2e_{m-1}+\sum_{i=0}^{m-2}\beta_i^{(1)}e_i,$$

where subscript A means to substitute (2.14) or (2.16) into the expression, hereafter C_i, $\beta_i^{(k)}$ are integral constants. Furthermore, we get by induction

$$DL^ku|_A=2\sum_{i=0}^{k}C_i\sum_{j=1}^{N}\lambda_j^{k-i}\Psi_{jx},\ (C_0=1,\ C_1=0), \tag{3.2a}$$

$$L^ku|_A=2\sum_{i=1}^{k}C_i\sum_{j=1}^{N}\lambda_j^{k-i}\Psi_j+2C_{k+1}e_{m-1}+\sum_{i=0}^{m-2}\beta_i^{(k)}e_i, \tag{3.2b}$$

which, along with (2.3), implies that

$$b_k|_A=\sum_{i=0}^{k-1}C_i\langle\Lambda^{k-i-1}q,q\rangle+C_k,\ k=1,2,\cdots. \tag{3.3}$$

It is obvious that C_i are the integrals of motion for system (3.1a). By inserting (3.3) into (2.10a), a straightforward calculation by induction gives

$$C_1=\cdots=C_{m-1}=0,\ C_m=C. \tag{3.4}$$

Substituting (3.3) into both sides of (2.10b) yields the following

Lemma 1.

$$C_{k+m}=F_{k+m}+\sum_{i=1}^{k+m-2}C_i\sum_{j=1}^{k+m-1-i}C_jF_{k+m-i-j}+2\sum_{j=1}^{k+m-1}C_jF_{k+m-j}-\frac{1}{2}\sum_{j=1}^{k+m-1}C_jC_{k+m-j}, \tag{3.5a}$$

with $F_1=\cdots=F_{m-1}=0$, $F_m=C_m=C$,

$$F_{m+l}=\frac{1}{2}\langle\Lambda^lp,p\rangle+\frac{1}{2}\sum_{i=0}^{l-1}\Big[\langle\Lambda^ip,p\rangle\langle\Lambda^{l-i-1}q,q\rangle-\langle\Lambda^ip,q\rangle\langle\Lambda^{l-i-1}p,q\rangle\Big]$$
$$+\frac{1}{2}\sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle\Lambda^{l_1}q,q\rangle\cdots\langle\Lambda^{l_{i-1}}q,q\rangle\langle\Lambda^{l_i+l}q,q\rangle,\ l=0,1,\cdots, \tag{3.5b}$$

where $l_1\geqslant0,\cdots,l_i\geqslant0$, F_{m+l} *are the integrals of motion for* (3.1a).

Proof It is clear that in order to get F_{k+m}, we just need to replace b_j in both sides of (2.10b) by $\langle\Lambda^{j-1}q,q\rangle$ which is the term without containing $C_i(i>0)$ in (3.3):

$$F_{k+m}=\underbrace{\frac{1}{2}\sum_{j=1}^{k+1}[\langle\Lambda^{j-1}p_x,q\rangle+\langle\Lambda^{j-1}p,p\rangle]\langle k-j\rangle}_{\langle 1\rangle\quad\langle 2\rangle}-\underbrace{\frac{1}{2}\sum_{j=1}^{k}\langle\Lambda^{j-1}p,q\rangle\langle\Lambda^{k-j}p,q\rangle}_{\langle 2\rangle}$$

$$+\underbrace{\frac{1}{2}\sum_{i=0}^{m-1}u_i\sum_{l=0}^{k+i}\langle l-1\rangle\langle k+i-l-1\rangle}_{\langle 3\rangle}$$

$$-\underbrace{\frac{1}{2}\sum_{j=1}^{k+m-1}\langle j-1\rangle\langle k+m-j-1\rangle}_{\langle 4\rangle}-\underbrace{\langle k+m-1\rangle}_{\langle 5\rangle},$$

$$\langle 2\rangle=\frac{1}{2}\langle\Lambda^k p,p\rangle+\frac{1}{2}\sum_{j=0}^{k-1}[\langle\Lambda^j p,p\rangle\langle\Lambda^{k-1-j}q,q\rangle-\langle\Lambda^j p,q\rangle\langle\Lambda^{k-1-j}p,q\rangle],$$

$$\langle 1\rangle\overset{(2.11)}{=\!=}\underbrace{-\frac{1}{2}\sum_{j=1}^{k+1}\sum_{i=0}^{m-1}u_i\langle i+j-2\rangle\langle k-j\rangle}_{\langle 6\rangle}+\underbrace{\frac{1}{2}\sum_{j=1}^{k+1}\langle m+j-2\rangle\langle k-j\rangle}_{\langle 7\rangle},$$

$$\langle 3\rangle=\frac{1}{2}\sum_{i=0}^{m-1}u_i\sum_{l=0}^{k}\langle l-1\rangle\langle k+i-l-1\rangle+\frac{1}{2}\sum_{i=1}^{m-1}u_i\sum_{l=k+1}^{k+i}\langle l-1\rangle\langle k+i-l-1\rangle$$

$$=\frac{1}{2}\sum_{i=0}^{m-1}u_i\sum_{j=1}^{k+1}\langle k-j\rangle\langle i+j-2\rangle$$

$$+\frac{1}{2}\sum_{i=1}^{m-1}u_{m-i}\sum_{l=0}^{m-1-i}\langle k+l\rangle\langle m-i-l-2\rangle$$

$$\langle 3\rangle+\langle 6\rangle\overset{(2.16)}{=\!=}\frac{1}{2}\sum_{i=1}^{m-1}\sum_{j=1}^{i}a_j\sum_{l_1+\cdots+l_j=i-j}\langle l_1\rangle\cdots\langle l_j\rangle\sum_{l=0}^{m-1-i}\langle k+l\rangle\langle m-i-l-2\rangle$$

$$\overset{(2.17)}{=\!=}\frac{1}{2}\sum_{j=1}^{m-1}a_j\sum_{i=0}^{m-1-j}\sum_{l_1+\cdots+l_j=m-1-j-i}\langle l_1\rangle\cdots\langle l_j\rangle\sum_{l=0}^{i}\langle k+l\rangle\langle i-l-1\rangle$$

$$=\frac{1}{2}\sum_{j=1}^{m-1}a_j\sum_{l_1+\cdots+l_{j+2}=m-1-j}\langle l_1\rangle\cdots\langle l_j\rangle\langle l_{j+1}+k\rangle\langle l_{j+2}-1\rangle$$

$$=\frac{1}{2}\sum_{j=1}^{m-1}a_j\sum_{l_1+\cdots+l_{j+2}=m-1-j,\,l_{j+2}=0}\langle l_1\rangle\cdots\langle l_j\rangle\langle l_{j+1}+k\rangle$$

$$+\frac{1}{2}\sum_{j=1}^{m-2}a_j\sum_{l_1+\cdots+l_{j+2}=m-2-j}\langle l_1\rangle\cdots\langle l_j\rangle\langle l_{j+1}+k\rangle\langle l_{j+2}\rangle\,(l_{j+2}\to l_{j+2}+1)$$

$$=\underbrace{\frac{1}{2}\sum_{j=2}^{m}(-1)^j\sum_{l_1+\cdots+l_j=m-j}\langle l_1\rangle\cdots\langle l_{j-1}\rangle\langle l_j+k\rangle+\frac{1}{2}\sum_{l_1+l_2=m-2}\langle l_1\rangle\langle l_2+k\rangle}_{\langle 8\rangle},$$

$$\langle 4\rangle+\langle 5\rangle+\langle 7\rangle+\langle 8\rangle=-\frac{1}{2}\langle k+m-1\rangle.$$

Then it is easy to see that $\langle 1\rangle+\cdots+\langle 5\rangle$ amount to (3.5b). Furthermore, a straightforward computation shows that F_{k+m} are the integrals of the motion for (3.1a), i.e., if (p, q) satisfies (3.1a), then

$$\frac{dF_{k+m}}{dx}=0,\quad k=0,1,\cdots. \tag{3.6}$$

Finally, substituting (3.3) into (2.7b), and computing the terms containing either $\langle\Lambda^j p, p\rangle$ or $\langle\Lambda^j p, q\rangle$, we find that

$$\frac{1}{4}\sum_{j=1}^{k+1}\left(b_{jxx}b_{k+1-j}-\frac{1}{2}b_{jx}b_{k+1-j,x}\right)$$
$$=\sum_{i=0}^{k}C_i\sum_{j=0}^{k-i}\Big\{C_j\langle A^{k-i-j}p,p\rangle+\sum_{l=0}^{k-i-j-1}[\langle A^lp,p\rangle\langle A^{k-i-j-1-l}q,q\rangle$$
$$-\langle A^lp,q\rangle\langle A^{k-i-j-1-l}p,q\rangle]\Big\}+E_1,$$
$$-\frac{1}{2}\sum_{j=1}^{k+m-1}b_jb_{k+m-j}=-\frac{1}{2}\sum_{j=1}^{k+m-1}C_jC_{k+m-j}+E_2,$$

where E_1 is a sum of terms like $C_iC_j\sum_{l_1+\cdots+l_n=k+m-i-j-n}\langle l_1\rangle\cdots\langle l_n\rangle$.

Thus we assert that after inserting (3.3), (2.10b) can be rewritten as

$$C_{k+m}=\sum_{i=0}^{k}C_i\sum_{j=0}^{k-i}C_jF_{k+m-i-j}-\frac{1}{2}\sum_{i=1}^{k+m-1}C_iC_{k+m-i}+E_3. \tag{3.7}$$

Since C_i and F_i are the integrals of the motion, E_3 must be an integral of the motion of (3.1a), too. However E_3 does not contain $\langle A^ip,p\rangle$ or $\langle A^ip,q\rangle$ explicitly. So E_3 must be zero. Observe that $C_1=\cdots=C_{m-1}=F_1=\cdots=F_{m-1}=0$, (3.7) can be reformulated as (3.5a).

In Appendix A, we show that F_{k+m} are in involution with respect to the ordinary Poisson bracket defined as

$$\{f,g\}=\sum_{j=1}^{N}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right),$$

i.e., we have

$$\{F_k,F_l\}=0,\quad k,l=1,2,\cdots. \tag{3.8}$$

We now restrict the integrals of motion F_l to $TS^{N-1}=\{(p,q)\in R^{2N}\,|\,F=\frac{1}{2}(\langle A^{-1}q,q\rangle-1)=0,\ G=\langle A^{-1}p,q\rangle=0\}$ to construct the integrals of motion F_l^* for (3.1b, c, d) by requiring that the vectorfield $X^{F_l^*}$ be tangential to TS^{N-1}. For this purpose, we have to set [10,11]

$$F_l^*=F_l-\mu_lF-\tilde{\mu}_lG,$$

and require that

$$\{F_l^*,F\}|_{TS^{N-1}}=\{F_l^*,G\}|_{TS^{N-1}}=0. \tag{3.9}$$

It follows from (3.9) that the Lagrangian multipliers μ_l and μ_l^* are determined by

$$\mu_l=\frac{\{F_l,G\}}{\{F,G\}}\Big|_{TS^{N-1}},\quad \tilde{\mu}_l=\frac{\{F_l,F\}}{\{G,F\}}\Big|_{TS^{N-1}}.$$

Then it is easy to calculate that

$$\tilde{\mu}_l=0,\ \mu_1=\cdots=\mu_{m-1}=0,$$
$$\mu_{m+l}=-\frac{1}{\langle A^{-2}q,q\rangle}\Big[[\langle A^{-1}p,p\rangle\langle A^{l-1}q,q\rangle$$
$$+\langle A^{l-1}q,q\rangle\sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-i-1}\langle A^{l_1}q,q\rangle\cdots\langle A^{l_i}q,q\rangle],\ l=0,1,2,\cdots,$$

and we have

$$
\begin{aligned}
F_{m+l}^{*}=&\frac{1}{2}\langle \Lambda^{l}p,\ p\rangle+\frac{1}{2}\sum_{i=0}^{l-1}[\langle \Lambda^{i}p,\ p\rangle\langle \Lambda^{l-i-1}q,\ q\rangle-\langle \Lambda^{i}p,\ q\rangle\langle \Lambda^{l-i-1}p,\ q\rangle]\\
&+\frac{1}{2}\sum_{i=1}^{m}(-1)^{i}\sum_{l_1+\cdots+l_i=m-i}\langle \Lambda^{l_1}q,\ q\rangle\cdots\langle \Lambda^{l_{i-1}}q,\ q\rangle\langle \Lambda^{l_i+l}q,\ q\rangle\\
&+\frac{1}{2}\frac{1}{\langle \Lambda^{-2}q,\ q\rangle}\Big[\langle \Lambda^{-1}p,\ p\rangle\langle \Lambda^{l-1}q,\ q\rangle\\
&+\langle \Lambda^{l-1}q,\ q\rangle\sum_{i=1}^{m-1}(-1)^{i}\sum_{l_1+\cdots+l_i=m-i-1}\langle \Lambda^{l_1}q,\ q\rangle\cdots\langle \Lambda^{l_i}q,\ q\rangle\Big](\langle \Lambda^{-1}q,\ q\rangle-1),\\
&\qquad l=0,\ 1,\ 2,\ \cdots,
\end{aligned}
\tag{3.10a}
$$

and we find that

$$H_0=F_m^{*}. \tag{3.10b}$$

It is obvious that F_l^* are the integrals of motion for system (3.1b, c, d). (3.10b) and (3.9) guarantees that the flow defined by (3.1b, d) is on the tangent bundle TS^{N-1} defined by (3.1c). Also, one verifies readily

$$\{F_l,\ \mu_k F\}\,|_{TS^{N-1}}=\mu_k\{F_l,\ F\}\,|_{TS^{N-1}}=0,$$

which together with (3.8) means that

$$\{F_l^*,\ F_k^*\}\,|_{TS^{N-1}}=0,\ l,\ k=1,\ 2,\ \cdots. \tag{3.11}$$

Thus F_l^*, $l=1,\ 2,\ \cdots$ are integrals of motion in involution for the constrained Hamiltonian system (3.1b, c, d). Note that λ_k are distinct and the Vandermonde determinant of $\lambda_1,\ \cdots,\ \lambda_N$ is not zero. This guarantees that grad F_m, $\cdots$, grad F_{m+N-1} are independent. When constructing F_l^* from F_l under the constraint condition $\langle \Lambda^{-1}q,\ q\rangle=1$, we find that $F_m^*,\ \cdots,\ F_{m+N-2}^*$ are functionally independent. Therefore we have the following

Proposition 2. *The Hamiltonian system* (3.1) *is completely integrable in the sense of Liouville*[8].

In the case $m=2$, similar system (3.1) was given in [11].

Proposition 3. *If* $(p,\ q)$ *solves* (3.1), *then* u *given by* (2.16) *satisfies a certain higher order stationary equaiion*

$$DL^N u+\sum_{k=0}^{N-1}d_k DL^k u=0, \tag{3.12}$$

where coefficients d_k *are determined by* $\lambda_1,\ \cdots,\ \lambda_N,\ C_1,\ \cdots,\ C_N$.

Proof Set

$$(\lambda-\lambda_1)\cdots(\lambda-\lambda_N)=\lambda^N+\sum_{k=1}^{N}g_k\lambda^{N-k}.$$

It follows from (3.2a) that

$$
\begin{aligned}
\sum_{k=0}^{N}d_k DL^k u\,|_A&=2\sum_{j=1}^{N}\Psi_{jx}\sum_{k=0}^{N}d_k\sum_{i=0}^{k}C_i\lambda_j^{k-i}\\
&=2\sum_{j=1}^{N}\Psi_{jx}\sum_{k=0}^{N}\lambda_j^{N-k}\sum_{i=0}^{k}C_i d_{N-k+i}.
\end{aligned}
$$

Taking $d_N=1$ and

$$d_{N-k}=g_k-\sum_{i=1}^{k}C_i d_{N-k+i},\ k=1,\ \cdots,\ N,$$

we obtain (3.12) immediately.

§ 4. The Hamiltonian Systems Deduced From the Time Part

The formula (3.5a) allows us to obtain

$$C_i=\sum_{j=1}^{i}\alpha_j\sum_{m_1+\cdots+m_j=i}F_{m_1}\cdots F_{m_j},\ i=1,\ 2,\ \cdots, \tag{4.1a}$$

where $m_1\geqslant 1,\ \cdots,\ m_j\geqslant 1,\ \alpha_1=1,\ \alpha_2=\frac{3}{2},\ \ \alpha_3=\frac{5}{2}$,

$$\alpha_j=\sum_{l=1}^{j-2}\alpha_l\alpha_{j-l-1}+2\alpha_{j-1}-\frac{1}{2}\sum_{l=1}^{j-1}\alpha_l\alpha_{j-l}. \tag{4.1b}$$

The details are given in Appendix B. By using F_l^* instead of F_l, C_i can be rewritten as

$$C_i=\sum_{j=1}^{i}\alpha_j\sum_{m_1+\cdots+m_j=i}F^*_{m_1}\cdots F^*_{m_j}+(\langle\Lambda^{-1}q,\ q\rangle-1)\widetilde{C}_i.$$

When restricting C_i on TS^{N-1}, we can drop the last term $(\langle\Lambda^{-1}q,\ q\rangle-1)\widetilde{C}_i$ since it vanishes on TS^{N-1}, and we have

$$C_i|_{TS^{N-1}}=\sum_{j=1}^{i}\alpha_j\sum_{m_1+\cdots+m_j=i}F^*_{m_1}\cdots F^*_{m_j},\ i=1,\ 2,\ \cdots. \tag{4.1'}$$

Now consider systems obtained from (2.2)

$$\phi_{jt_n}=-\frac{1}{2}B^{(n)}_{jx}\phi_j+B^{(n)}_j\phi_{jx},\ B^{(n)}_j=B^{(n)}|_{\lambda=\lambda_j},\ j=1,\ \cdots,\ N, \tag{4.2}$$

which, under the constraint condition (2.15), (2.16) and (3.1), can be reformulated as

$$g_{jt_n}|_{TS^{N-1}}\overset{(3.3)}{=\!=\!=}\sum_{k=1}^{n}\lambda_j^{n-k}\sum_{i=0}^{k-1}C_i|_{TS^{N-1}}(\langle\Lambda^{k-1-i}q,\ q\rangle p_j-\langle\Lambda^{k-1-i}p,\ q\rangle q_j)+\sum_{k=0}^{n}\lambda_j^{n-k}C_k|_{TS^{N-1}}p_j$$

$$\overset{(2.17)}{=\!=\!=}\sum_{i=0}^{n-1}C_i|_{TS^{N-1}}\left[\sum_{k=0}^{n-i-1}\lambda_j^k(\langle\Lambda^{n-k-1-i}q,\ q\rangle p_j-\langle\Lambda^{n-k-1-i}p,\ q\rangle q_j)+\lambda_j^{n-i}p_j\right]+C_n|_{TS^{N-1}}p_j$$

$$\overset{(4.1')}{=\!=\!=}\sum_{i=0}^{m+n}\sum_{l=0}^{i}\alpha_l\sum_{m_1+\cdots+m_l=i}F^*_{m_1}\cdots F^*_{m_l}\frac{\partial F^*_{m+n-i}}{\partial p_j}\Big|_{TS^{N-1}}$$

$$(\alpha_0=1,\ F_0^*=1,\ F_1^*=\cdots=F^*_{m-1}=0)$$

$$=\sum_{i=0}^{m+n}\alpha_i\sum_{l=1}^{m+n-i}\sum_{m_1+\cdots+m_i=m+n-l}F^*_{m_1}\cdots F^*_{m_i}\frac{\partial F^*_l}{\partial p_j}\Big|_{TS^{N-1}}\quad(m_1\geqslant 1,\ \cdots,\ m_i\geqslant 1)$$

$$=\left(\frac{\partial}{\partial p_j}\sum_{i=0}^{m+n}\frac{\alpha_i}{i+1}\sum_{m_1+\cdots+m_{i+1}=m+n}F^*_{m_1}\cdots F^*_{m_{i+1}}\right)\Big|_{TS^{N-1}}$$

$$=\left(\frac{\partial}{\partial p_j}\sum_{i=0}^{n}\frac{\alpha_i}{i+1}\sum_{m_1+\cdots+m_{i+1}=m+n}F^*_{m_1}\cdots F^*_{m_{i+1}}\right)_{TS^{N-1}},\quad(F_1^*=\cdots=F^*_{m-1}=0). \tag{4.3}$$

Also, it is found by a direct computation that

$$\frac{d}{dx}\frac{\partial F_k^*}{\partial p}\Big|_{TS^{N-1}}=-\frac{\partial F_k^*}{\partial q}\Big|_{TS^{N-1}},\ k=1,\ 2,\ \cdots. \tag{4.4}$$

Then (4.3) and (4.4) imply that under the constraint condition (2.15), (2.16) and (3.1), (4.2) can be written in canonical Hamiltonian form

$$q_{t_n}=\frac{\partial H_n}{\partial p},\quad p_{t_n}=-\frac{\partial H_n}{\partial q}, \tag{4.5a}$$

with

$$(p,q)\in TS^{N-1}, \tag{4.5b}$$

$$H_n=\sum_{i=0}^{n}\frac{\alpha_i}{i+1}\sum_{m_1+\cdots+m_{i+1}=n+m}F^*_{m_1}\cdots F^*_{m_{i+1}}, \tag{4.5c}$$

where $m_l\geqslant 1$, $F^*_1=\cdots=F^*_{m-1}=0$, $\alpha_0=1$, α_i are given by (4.1b). The condition (3.9) guarantees that the vectorfield defind by (4.5a) is tangential to TS^{N-1}.

Proposition 4. *The Hamiltonian systems* (4.5) ($n=1, 2, \cdots$, *call* $t_0=x$) *and* (3.1) *are completely integrable and commute with each other. If* (p, q) *is a solution of* (3.1) *and* (4.5) (*for a fixed* n), *then* u *given by* (2.16) *satisfies equation* (2.4).

Proof Since F^*_k are in involution, it follows from (4.5c) and (3.11) that

$$\{H_l, H_k\}|_{TS^{N-1}}=0,\quad \left.\frac{dF^*_k}{dt_n}\right|_{TS^{N-1}}=\{F^*_k, H_n\}|_{TS^{N-1}}=0,\ l, k, n=0, 1, \cdots.$$

This indicates that systems (4.5) and (3.1) are completely integrable and commute with each other. Observe that (2.4) is deduced from the solvability condition of (2.11) and (4.2), (3.1) and (4.5) are obtained by inserting (2.15) and (2.16) into (2.11) and (4.2), respectively. Hence we assert that if (p, q) satisfies (3.1) and (4.5) (for a fixed n), then u given by (2.16) solves equation (2.4).

Appendix A. Involution integrals of motion

Set

$$G_k=\frac{1}{2}\sum_{i=0}^{k-1}[\langle\Lambda^i p, p\rangle\langle\Lambda^{k-i-1}q, q\rangle-\langle\Lambda^i p, q\rangle\langle\Lambda^{k-i-1}p, q\rangle],$$

$$Q_k=\frac{1}{2}\langle\Lambda^k p, p\rangle+\frac{1}{2}\sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle\Lambda^{l_1}q, q\rangle\cdots\langle\Lambda^{l_{i-1}}q, q\rangle\langle\Lambda^{l_i+k}q, q\rangle.$$

Using the following identity

$$\begin{aligned}&\sum_{i=0}^{l}\langle\Lambda^{l+k+j-i}p, p\rangle\langle\Lambda^i q, q\rangle+\sum_{i=0}^{k}\langle\Lambda^i p, p\rangle\langle\Lambda^{l+k+j-i}q, q\rangle\\&=\sum_{i=0}^{l+k+j}\langle\Lambda^{l+k+j-i}p, p\rangle\langle\Lambda^i q, q\rangle\\&-\sum_{i=l+1}^{l+j-1}\langle\Lambda^{l+k+j-i}p, p\rangle\langle\Lambda^i q, q\rangle,\ j=0, 1, \cdots,\end{aligned} \tag{A1}$$

we have shown in [9] that

$$\{G_k, G_l\}=0.$$

It is found that

$$\{G_k, Q_l\}+\{G_l, Q_k\}$$

$$=\sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_{i-2}\rangle\left\{\sum_{j=0}^{k-1}[(i-1)\langle\Lambda^{k-1-j}p, q\rangle\langle j+l_{i-1}\rangle\langle l+l_i\rangle\right.$$

$$-\underset{\langle 3\rangle}{(i-1)\langle k-1-j\rangle\langle \Lambda^{j+l_{i-1}}p,q\rangle\langle l+l_i\rangle}+\underset{\langle 2\rangle}{\langle \Lambda^{k-1-j}p,\ q\rangle\langle l_{i-1}\rangle\langle l+j+l_i\rangle}$$

$$-\underset{\langle 1\rangle}{\langle k-1-j\rangle\langle l_{i-1}\rangle\langle \Lambda^{l+j+l_i}p,\ q\rangle}]+\sum_{j=0}^{l-1}[-\underset{\langle 4\rangle}{(i-1)\langle \Lambda^{l-1-j}p,\ q\rangle\langle j+l_{i-1}\rangle\langle k+l_i\rangle}$$

$$+\underset{\langle 4\rangle}{(i-1)\langle l-1-j\rangle\langle \Lambda^{j+l_{i-1}}p,\ q\rangle\langle k+l_i\rangle}-\underset{\langle 1\rangle}{\langle \Lambda^{l-1-j}p,\ q\rangle\langle l_{i-1}\rangle\langle k+j+l_i\rangle}$$

$$+\underset{\langle 2\rangle}{\langle l-1-j\rangle\langle l_{i-1}\rangle\langle \Lambda^{k+j+l_i}p,\ q\rangle}]\Big\},$$

$$\begin{aligned}
\langle 1\rangle+\langle 2\rangle &\overset{(A.1)}{=\!=\!=} \sum_{i=1}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_{i-1}\rangle\\
&\quad\times\Big[-\sum_{n=0}^{k+l+l_i-1}\langle \Lambda^{k+l+l_i-n-1}p,\ q\rangle\langle n\rangle\\
&\quad+\sum_{n=k}^{k+l_i-1}\langle \Lambda^{k+l+l_i-n-1}p,\ q\rangle\langle n\rangle\Big]-\text{cycle}(k,\ l)\\
&\overset{l_i\to l_i+1}{=\!=\!=} \sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-1-i}\langle l_1\rangle\cdots\langle l_{i-1}\rangle\\
&\quad\times\sum_{n=0}^{l_i}[\langle \Lambda^{l+l_i-n}p,\ q\rangle\langle k+n\rangle-\langle \Lambda^{k+l_i-n}p,\ q\rangle\langle l+n\rangle]\\
&=\sum_{i=1}^{m-1}(-1)^i\sum_{l_1+\cdots+l_{i+1}=m-1-i}\langle l_1\rangle\cdots\langle l_{i-1}\rangle\\
&\quad\times[\langle \Lambda^{l+l_{i+1}}p,\ q\rangle\langle k+l_i\rangle-\langle \Lambda^{k+l_{i+1}}p,\ q\rangle\langle l+l_i\rangle]\\
&=\sum_{i=2}^{m}(-1)^{i-1}\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_{i-2}\rangle\\
&\quad\times[\langle \Lambda^{l+l_i}p,\ q\rangle\langle k+l_{i-1}\rangle-\langle \Lambda^{k+l_i}p,\ q\rangle\langle l+l_{i-1}\rangle],
\end{aligned}$$

$$\begin{aligned}
\langle 3\rangle+\langle 4\rangle &= \sum_{i=2}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-i,\,l_i\geq 1}(i-1)\langle l_1\rangle\cdots\langle l_{i-2}\rangle\langle l+l_{i-1}\rangle\\
&\quad\times\Big[\sum_{n=l_i}^{k+l_i-1}\langle n\rangle\langle \Lambda^{k+l_i-n-1}p,\ q\rangle-\sum_{n=0}^{k-1}\langle n\rangle\langle \Lambda^{k+l_i-n-1}p,\ q\rangle\Big]-\text{cycle}(k,\ l)\\
&\overset{l_i\to l_i+1}{=\!=\!=} \sum_{i=2}^{m-1}(-1)^i\sum_{l_1+\cdots+l_i=m-1-i}(i-1)\langle l_1\rangle\cdots\langle l_{i-2}\rangle\langle l+l_{i-1}\rangle\\
&\quad\times\Big[\sum_{n=k}^{k+l_i}\langle n\rangle\langle \Lambda^{k+l_i-n}p,\ q\rangle-\sum_{n=0}^{l_i}\langle n\rangle\langle \Lambda^{k+l_i-n}p,\ q\rangle\Big]-\text{cycle}(k,\ l)\\
&=\sum_{i=2}^{m-1}(-1)^i\sum_{l_1+\cdots+l_{i+1}=m-i-1}(i-1)\langle l_1\rangle\cdots\langle l_{i-2}\rangle\langle l+l_{i-1}\rangle\\
&\quad\times[\langle k+l_i\rangle\langle \Lambda^{l_{i+1}}p,\ q\rangle-\langle l_i\rangle\langle \Lambda^{k+l_{i+1}}p,\ q\rangle]-\text{cycle}\ (k,\ l)\\
&=\sum_{i=3}^{m}(-1)^{i-1}\sum_{l_1+\cdots+l_i=m-i}(i-2)\langle l_1\rangle\cdots\langle l_{i-2}\rangle\\
&\quad\times[\langle k+l_{i-1}\rangle\langle \Lambda^{l+l_i}p,\ q\rangle-\langle l+l_{i-1}\rangle\langle \Lambda^{k+l_i}p,\ q\rangle],
\end{aligned}$$

$$\begin{aligned}
\{Q_k,\ Q_l\} &= \sum_{i=2}^{m}(-1)^i\sum_{l_1+\cdots+l_i=m-i}\langle l_1\rangle\cdots\langle l_{i-2}\rangle[(i-1)\langle \Lambda^{l+l_{i-1}}p,\ q\rangle\langle l_i+k\rangle\\
&\quad+\langle l_{i-1}\rangle\langle \Lambda^{k+l+l_i}p,\ q\rangle]-\text{cycle}(k,\ l)
\end{aligned}$$

$$=\sum_{i=1}^{m}(-1)^{i}\sum_{l_1+\cdots+l_i=m-i}(i-1)\langle l_1\rangle\cdots\langle l_{i-2}\rangle[\langle k+l_{i-1}\rangle\langle \Lambda^{l+l_i}p,\ q\rangle$$
$$-\langle l+l_{i-1}\rangle\langle \Lambda^{k+l_i}p,\ q\rangle].$$

Hence we have $\{G_k,\ Q_l\}+\{G_l,\ Q_k\}+\{Q_k,\ Q_l\}=0$. Then it follows that $\{F_{k+m},\ F_{l+m}\}=0$.

Appendix B. The proof of formula (4.1)

We show (4.1) by induction. First we have from (3.5a)

$$2\sum_{j=1}^{k+m-1}C_jF_{k+m-j}=2\sum_{j=1}^{k+m-1}\sum_{l=1}^{j}\alpha_l\sum_{m_1+\cdots+m_l=j}F_{m_1}\cdots F_{m_l}F_{k+m-j}$$
$$\overset{(2.17)}{=\!=\!=}2\sum_{j=1}^{k+m-1}\alpha_j\sum_{l=1}^{k+m-j}\sum_{m_1+\cdots+m_j=k+m-l}F_{m_1}\cdots F_{m_j}F_l$$
$$=2\sum_{j=1}^{k+m-1}\alpha_j\sum_{m_1+\cdots+m_{j+1}=k+m}F_{m_1}\cdots F_{m_{j+1}}\quad(m_1\geqslant 1,\ \cdots,\ m_{j+1}\geqslant 1)$$
$$=2\sum_{j=2}^{k+m}\alpha_{j-1}\sum_{m_1+\cdots+m_j=k+m}F_{m_1}\cdots F_{m_j}.$$

A direct computation shows that

$$\sum_{j=1}^{k-1}\left(\sum_{i=1}^{j}\alpha_i\beta_{i,j}\right)\left(\sum_{l=1}^{k-j}\alpha_l\gamma_{l,k-j}\right)=\sum_{j=2}^{k}\sum_{l=1}^{j-1}\alpha_l\alpha_{j-1}\sum_{i=l}^{k+l-j}\beta_{l,i}\gamma_{j-l,k-i}.$$

Then we obtain

$$\frac{1}{2}\sum_{j=1}^{k+m-1}C_jC_{k+m-j}=\frac{1}{2}\sum_{j=1}^{k+m-1}\left(\sum_{i=1}^{j}\alpha_i\sum_{m_1+\cdots+m_i=j}F_{m_1}\cdots F_{m_i}\right)$$
$$\times\left(\sum_{l=1}^{k+m-j}\alpha_l\sum_{m_1+\cdots+m_l=k+m-j}F_{m_1}\cdots F_{m_l}\right)$$
$$=\frac{1}{2}\sum_{j=2}^{k+m}\left(\sum_{l=1}^{j-1}\alpha_l\alpha_{j-l}\right)$$
$$\times\sum_{i=l}^{k+m+l-j}\sum_{m_1+\cdots+m_l=i}F_{m_1}\cdots F_{m_l}\sum_{m_1+\cdots+m_{j-l}=k+m-i}F_{m_1}\cdots F_{m_{j-l}}$$
$$=\frac{1}{2}\sum_{j=2}^{k+m}\left(\sum_{l=1}^{j-1}\alpha_l\alpha_{j-l}\right)\sum_{m_1+\cdots+m_j=k+m}F_{m_1}\cdots F_{m_j}.$$

Similarly we find that

$$\sum_{i=1}^{k+m-2}C_i\sum_{j=1}^{k+m-1-i}C_jF_{k+m-i-j}=\sum_{j=3}^{k+m}\left(\sum_{l=1}^{j-2}\alpha_l\alpha_{j-1-l}\right)\sum_{m_1+\cdots+m_j=k+m}F_{m_1}\cdots F_{m_j}.$$

Then we obtain (4.1) from (3.5) immediately.

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