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PERIODIC SOLUTIONS OF FORCED LIÉNARD EQUATIONS****

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Abstarct

The existence of periodic solutions of the nonlinear system

$$\ddot{x}+f(x)\dot{x}+g(t, x)=e(t)$$

is studied by using the theory of nonhomogeneous linear periodic systems and the Schauder fixed point theorem.

§ 1. Introduction and Main Results

We consider the existence of T-periodic solution of the system

$$\ddot{x}+f(x)\dot{x}+g(t, x)=e(t), \tag{1.1}$$

where f, e: $R\to R$ and g: $R^2\to R$ are continuous functions and g, e are T-periodic in t.

Over a long period of time there are many research works about the existence of periodic solutions of Liènard equation

$$\ddot{x}+f(x)\dot{x}+g(x)=e(t) \tag{1.2}$$

(including the autonomous case of $e(t)\equiv 0$ and the periodic forcing case of $e(t)\not\equiv 0$), and the generalized Liènard equation (1.1) due to the importance both in theoretical studies and applications. Recent years, the Brouwer degree and the Leray-Schauder degree are used in the studies of equations (1.1) and (1.2) in some papers (see [3, 6—8]). Here we use the theory of nonhomogeneous linear periodic systems and the Schauder fixed point theorem to give several kinds of sufficient conditions under which (1.1) has T-periodic solution. The results are applicable, and much simple and clear as compared with some recent works (see [1, 3—8]).

Let

$$F(x)=\int_0^x f(x)\,dx, \tag{1.3}$$

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$$\tilde{g}(t, x) = g(t, x) - ax, \tag{1.4}$$

$$p = \limsup_{|x|\to\infty} \left| \frac{\tilde{g}(t, x)}{x} \right|, \tag{1.5}$$

$$q = \limsup_{|x|\to\infty} \left| \frac{F(x)}{x} \right|, \tag{1.6}$$

and make the following hypotheses:

(H_0) F, g and e are continuous, $g(t+T, x) = g(t, x)$, $e(t+T) = e(t)$;

(H_1) There exists $a>0$, such that $0<8a^{-1}p+pT^2+8qT<8-aT^2$;

(H_2) $T^2(p^2+q^2)<2(1-\cos T)$, where $a=1$.

Theorem 1. *If* (H_0), (H_1) *hold, then* (1.1) *has T-periodic solution.*

Particularly, we take $a=4p$ or $a=T^{-2}$, and then get

Corollary 1. *If* (H_0) *holds, then* $5pT^2+8qT<6$ *or* $9pT^2+8qT<7$ *implies that* (1.1) *has T-periodic solution.*

Corollary 2. *Suppose that* (H_0) *holds and* $\lim_{x\to\infty} \frac{g(t, x)}{x} = r$ *exists and satisfies the inequalities* $0 \leqslant r < 8T^{-2}$, $q<T^{-1}-rT/8$. *Then* (1.1) *has T-periodic solution.*

The proofs of Theorem 1 and Corollary 2 will be given in § 2.

Theorem 2. (H_0) *and* (H_2) *imply the existence of T-periodic solutions of* (1.1).

The proof of Theorem 2 will be given in § 3. We now abandon the traditional hypothesis that $\limsup_{|x|\to\infty} \left| \frac{g(t, x)}{x} \right|$ is sufficiently small (see [1, 3—7]), but assume that

(H_3) $T^2q^2<2(1-\cos R)$, where $R \neq 0 \pmod{2\pi}$ and $R^2 = T^2 \lim_{|x|\to\infty} \frac{g(t, x)}{x}$.

Theorem 3. (H_0) *and* (H_3) *imply the existence of T-periodic solutions of* (1.1).

Theorem 4. *Suppose that* (H_0) *holds and* $p_0 = q_0^2$, *where*

$$q_0 = \lim_{|x|\to\infty} \frac{F(x)}{x} > 0 \quad \text{and} \quad p_0 = \lim_{|x|\to\infty} \frac{g(t, x)}{x},$$

then (1.1) *has T-periodic solution.*

Theorem 3 and Theorem 4 will be proved in § 4.

§ 2. The Proof of Theorem 1

Let

$$P_T^i = \{u \mid u: R \to R^i \text{ is continuous and } u(t) = u(t+T),\ \forall t \in R\},$$

for $i=1, 2$, with the norm

$$\|n\| = \max_{0 \leqslant t \leqslant T} \{\|u(t)\|\}, \quad \forall u \in P_T^i,$$

where the vector norm is Euclidean norm as

$$\|u(t)\| = (u_1^2(t) + u_2^2(t))^{1/2}, \quad \text{if } u(t) = (u_1(t), u_2(t)) \in R^2,$$

$$\|u(t)\| = |u(t)|, \quad \text{if } u(t) \in R.$$

It is easy to know that P_T^1 is a Banach space. For $M>0$ take a subspace

$$S_M=\{u \mid u\in P_T^1,\ \|u\|\leqslant M\},$$

which is a closed convex set. Denote simply

$$u_0=\|u\|=\max_{0\leqslant t\leqslant T}\{|u(t)|\},\quad \forall u\in P_T^1.$$

and

$$\bar{e}=T^{-1}\int_0^T e(t)\,dt, \tag{2.1}$$

$$E(t)=\int_0^t e(t)\,dt-\bar{e}t. \tag{2.2}$$

(1.1) is equivelent to the 2-dimensional system

$$\begin{aligned}\dot{x}&=y-F(x)+E(t),\\ \dot{y}&=-ax-\tilde{g}(t,x)+\bar{e},\end{aligned} \tag{2.3}$$

where $\tilde{g}(t,x)=g(t,x)-ax$.

For $u(t)\in P_T^1$, we consider the nonhomogeneous linear periodic system

$$\begin{aligned}\dot{x}&=y-F(u(t))+E(t),\\ \dot{y}&=-ax-\tilde{g}(t,u(t))+\bar{e}.\end{aligned} \tag{2.4}$$

The eigenvalues of $A=\begin{pmatrix}0 & 1\\ -a & 0\end{pmatrix}$ are $\lambda=\pm\sqrt{a}\,i$. We have $\sqrt{a}T<2\pi$ by (H$_1$) and $\lambda T\not\equiv 0 \pmod{2\pi}$. Then (2.4) has a unique T-periodic solution $K(u)(t)$ for every $u\in P_T^1$ (see [2] Chapter 4, Theorem 4. 1. 1)

$$K(u)(t)=(X^{-1}(T,0)-I)^{-1}\int_0^T X(t,t+s)h(t+s,u(t+s))\,ds, \tag{2.5}$$

where $X(t,t_0)$ is the fundamental matrix of the linear system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=A\begin{pmatrix}x\\ y\end{pmatrix}$$

with $X(t_0,t_0)=I$, and

$$h(t,u(t))=\begin{pmatrix}-F(u(t))+E(t)\\ -\tilde{g}(t,u(t))+\bar{e}\end{pmatrix}. \tag{2.6}$$

Let $K(u)(t)=(x(t),y(t))$, and consider the mapping

$$T:\ u(t)\mapsto x(t). \tag{2.7}$$

We prove in the following that there exists an $M>0$, such that the mapping T has a fixed point $x(t)\in S_M$. So $(x(t),y(t))=K(x)(t)$ is a T-periodic solution of (2.3), and then $x(t)$ is a T-periodic solution of (1.1).

From (H$_1$), we may take $0\leqslant p_1-p,\ q_1-q\ll 1,\ M>M_0\gg 1$, such that

$$\begin{aligned}&(8-aT^2)^{-1}[a^{-1}(p_1+eM^{-1})(8+aT^2)+8T(q_1+E_0M^{-1})]\\ &=(1-b/2)^{-1}[a^{-1}(p_1+eM^{-1})(1+b/2)+T(q_1+E_0M^{-1})]<1\end{aligned} \tag{2.8}$$

(where $b=aT^2/4,\ e=|\bar{e}|$), and

$$|\tilde{g}(t, x)| < p_1|x|, \quad \text{as } 0 \leqslant t < T, \ |x| > M_0, \tag{2.9}$$
$$|\tilde{g}(t, x)| < p_1 M, \quad \text{as } 0 \leqslant t < T, \ |x| \leqslant M_0, \tag{2.10}$$
$$|F(x)| < q_1|x|, \quad \text{as } |x| > M_0, \tag{2.11}$$
$$|F(x)| < q_1 M, \quad \text{as } |x| \leqslant M_0. \tag{2.12}$$

Suppose now $u \in S_M$. Since $(x(t), y(t)) = K(u)(t)$ is T-periodic, there exist times t_0, t_1, t_2, such that

$$0 \leqslant t_0 \leqslant t_1 < t_2 \leqslant t_0 + T,$$
$$|x(t_0)| = x_0 = \max_{0<t<T} \{|x(t)|\}, \tag{2.13}$$
$$\dot{y}(t_i) = 0, \ i = 1, 2.$$

From $(t_1 - t_0) + (t_0 + T - t_2) < T$, we have either

$$t_1 - t_0 < T/2, \tag{2.14}$$

or

$$t_0 + T - t_2 < T/2. \tag{2.15}$$

Without loss of generality, we may assume (2.14) holds. By the definition of t_1, $-ax(t_1) - \tilde{g}(t_1, u(t_1)) + \bar{e} = \dot{y}(t_1) = 0$, so $x(t_1) = a^{-1}(\bar{e} - \tilde{g}(t_1, u(t_1)))$, and then

$$|x(t_1)| \leqslant a^{-1}(p_1 M + e) \triangleq c_1. \tag{2.16}$$

It is easily seen that $\dot{x}(t_0) = 0$, so $y(t_0) = F(u(t_0)) - E(t_0)$,

$$|y(t_0)| \leqslant q_1 M + E_0 \triangleq c_2. \tag{2.17}$$

By using (2.4) again and again we get

$$x_0 = |x(t_0)| = |x(t_1) + \int_{t_1}^{t_0} \dot{x}(t)dt| \leqslant c_1 + \int_{t_0}^{t_1} |\dot{x}(t)|dt$$
$$\leqslant c_1 + \int_{t_0}^{t_1} |y - F(u) + E|dt \leqslant c_1 + \int_{t_0}^{t_1}\Big(\Big|\int_{t_0}^{s_1} \dot{y}(t)dt\Big| + |y(t_0)|\Big)ds_1 + T(q_1 M + E_0)/2$$
$$\leqslant \int_{t_0}^{t_1}\int_{t_0}^{s_1} |\dot{y}(t)|dtds_1 + c_1 + Tc_2 = \int_{t_0}^{t_1}\int_{t_0}^{s_1} |-ax(t) - \tilde{g}(t, u(t)) + \bar{e}|dtds_1 + c_1 + Tc_2$$
$$\leqslant a\int_{t_0}^{t_1}\int_{t_0}^{s_1} |x(t)|dtds_1 + c_1 + Tc_2 + bc_1/2 \leqslant bx_0/2 + c_1(1 + b/2) + Tc_2,$$

and then

$$x_0 \leqslant (1 - b/2)^{-1}[c_1(1 + b/2) + Tc_2]$$
$$= (1 - b/2)^{-1}[a^{-1}(p_1 + eM^{-1})(1 + b/2) + T(q_1 + E_0 M^{-1})]M < M. \tag{2.18}$$

It is proved that

$$TS_M \subset S_M. \tag{2.19}$$

Besides, from (2.4) and (2.17) we have

$$|\dot{x}(t)| \leqslant |y(t)| + q_1 M + E_0 = |y(t_0) + \int_{t_0}^{t} \dot{y}(t)dt| + c_2$$
$$\leqslant \int_{t_0}^{t} |ax(t) - \tilde{g}(t, u(t)) - \bar{e}|dt + 2c_2$$
$$< T(aM + p_1 M + e) + 2c_2. \tag{2.20}$$

We know that TS_M is a equicontinuous family of functions, so the image set TS_M is relatively compact by the Ascoli–Arzela theorem. From the continuity of F,

g and the continuous dependence of the solutions for the right hand side of the system, we know that T is continuous with respect to u. By the Schauder fixed point theorem, there exists an $x \in S_M$, such that $x(t) = T(x(t))$ is a T-periodic solution of (1.1). The proof of Theorem 1 is completed.

We now prove Corollary 2.

If $r > 0$, we take $a = r$, $p = 0$, then the conclusion is given by Theorem 1. For the case $r = 0$, we consider the system

$$\dot{x} = y - F(x) + E(t),$$
$$\dot{y} = -g_n(t, x) + \bar{e}, \qquad (2.21)_n$$

where $g_n(t, x) = g(t, x) + r_n x$, $r_n > 0$.

Obviously $\lim\limits_{|x| \to \infty} \frac{g_n(t, x)}{x} = r_n$. Take a sequence $\{r_n\}$ which tends monotonously to zero, and r_n sufficiently small such that, for any $n \in Z^+$, $q < T^{-1} - r_n T/8$. Since $\tilde{g}_n(t, x) = g_n(t, x) - r_n x = g(t, x)$, from the proof of Theorem 1 and the Corollary 2 of the case $r > 0$, there exists an $M > 0$, independent of n, such that for any $u \in S_M$, system $(2.21)_n$ has T-periodic solution $(x_n(t), y_n(t))$ with $x_n(t) \in S_M$. Let $x_n(t) = T_n(x_n(t))$. Similar to (2.20), we may get

$$|\dot{x}_n(t)| \leqslant T(r_n M + p_1 M + e) + 2c_2,$$

where $0 < p_1 \ll 1$, $0 < q_1 - q \ll 1$.

Hence the set $\{T_n(x_n)\} \subset S_M$ is relatively compact. We denote a cluster point of $\{T_n(x_n)\}$ by $\bar{x}(t) \in S_M$. Similarly, we may prove that there exists an $M_1 > 0$, such that $\{y_n(t)\}$ has a cluster point $\bar{y}(t)$ in S_{M_1}. Then $(\bar{x}, \bar{y})$ is a T-periodic solution of the limit system of $(2.21)_n$

$$\dot{x} = y - F(x) + E(t),$$
$$\dot{y} = -g(t, x) + \bar{e}.$$

So $\bar{x}(t)$ is a T-periodic solution of (1.1). Corollary 2 is proved.

§ 3. The Proof of Theorem 2

As in § 2, $\|\cdot\|$ is still the Euclidean norm. The norm $\|A\|$ of a matrix A is taken by the square root of the largest eigenvalues of the matrix AA^*, where A^* is the transposition of A.

By (H_2), $T \neq 2k\pi$, we may take $p_1 > p$, $q_1 > q$, $M_0 \gg 1$, such that as $M > M_0$, (2.9)—(2.12) and the formula

$$T(2 - 2\cos T)^{-1/2}[(q_1 + E_0 M^{-1})^2 + (p_1 + eM^{-1})^2]^{1/2} < 1 \qquad (3.1)$$

hold, where $e = |\bar{e}|$,

$$\tilde{g}(t, x) = g(t, x) - x. \qquad (3.2)$$

For any $u \in P_T^1$, the system

$$\dot{x}=y-F(u(t))+E(t),$$
$$\dot{y}=-x-\tilde{g}(t,u(t))+\bar{e} \tag{3.3}$$

has a unique T-periodic solution

$$K(u)(t)=(X^{-1}(T,0)-I)^{-1}\int_0^T X(t,t+s)h(t+s)ds, \tag{3.4}$$

where

$$h(t)=\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}=\begin{pmatrix} -F(u(t))+\bar{E}(t) \\ -\tilde{g}(t,u(t))+\bar{e} \end{pmatrix}; \tag{3.5}$$

and $X(t,s)$ is the fundamental matrix with $X(s,s)=I$ of the linear system

$$\dot{x}=y, \quad \dot{y}=-x. \tag{3.6}$$

We prove that for the mapping T: $u(t)\mapsto x(t)$, $TS_M\subset S_M$. Then T has a fixed point $x(t)\in S_M$, such that $K(x(t))=(x(t),y(t))$ is a T-periodic solution of the system

$$\dot{x}=y-F(x)+E(t),$$
$$\dot{y}=-g(t,x)+\bar{e}. \tag{3.7}$$

Hence $x(t)$ is a T-periodic solution of (1.1).

It is easy to calculate that

$$X(t,s)=\begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}. \tag{3.8}$$

$$X^{-1}(T,0)=\begin{pmatrix} \cos T & -\sin T \\ \sin T & \cos T \end{pmatrix},$$

$$C\triangleq(X^{-1}(T,0)-I)^{-1}=\frac{1}{2-2\cos T}\begin{pmatrix} \cos T-1 & \sin T \\ -\sin T & \cos T-1 \end{pmatrix},$$

$$CC^*=\frac{1}{2-2\cos T}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\|C\|=(2-2\cos T)^{-1/2}. \tag{3.9}$$

From (3.8) we have

$$X(t,t+s)X^*(t,t+s)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So

$$\|X(t,t+s)\|=1. \tag{3.10}$$

From (2.9)—(2.12), (3.4), (3.5), (3.9), (3.10) and (3.1), we know that for $u\in S_M$,

$$\|x(t)\|\leqslant\|K(u)(t)\|\leqslant T(2-2\cos T)^{-1/2}[(q_1+E_0M^{-1})^2+(p_1+eM^{-1})^2]^{1/2}M<M.$$

Hence $TS_M\subset S_M$. Similar to Theorem 1, we may prove that T: $S_M\to S_M$ has a fixed point $x(t)$ which is a T-periodic solution of (1.1). The Theorem 2 is proved.

§ 4. The Proofs of Theorem 3 and Theorem 4

By (H_3), we may take $0<q_1-q\ll 1$, $0<s\ll 1$, $M_0\gg 1$, such that as $M>M_0$,

$$T^2(q_1^2+E_0M^{-1})^2+(sR+eT^2M^{-1}R^{-1})^2<2-2\cos R. \tag{4.1}$$

Consider the system

$$\dot{x}=y-F(x)+E(t),\quad \dot{y}=-g(t,x)+\bar{e}. \tag{4.2}$$

Let $g^*(t,x)=g(TR^{-1}t,x)$, $E^*(t)=E(TR^{-1}t)$, and change the variables $t=TR^{-1}\tau$, $z=TR^{-1}y$. Denote still τ, z by t, y. Then (4.2) becomes

$$\dot{x}=y-TR^{-1}(F(x)-E^*(t)),\quad \dot{y}=-T^2R^{-2}(g^*(t,x)-\bar{e}). \tag{4.3}$$

Consider the linear R-periodic system

$$\dot{x}=y-TR^{-1}(F(u(t))-E^*(t)),\quad \dot{y}=-x-T^2R^{-2}(g^*(t,x)-\bar{e})+u(t), \tag{4.4}$$

where $u\in P_R^1$. (4.4) has a unique R-periodic solution

$$K(u)(t)=(X^{-1}(R,0)-I)^{-1}\int_0^R X(t,t+s)h(t+s)ds, \tag{4.5}$$

where $X(t,s)$ is given by (3.8), and

$$h(t)=\begin{pmatrix} h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} -TR^{-1}(F(u(t))-E^*(t)) \\ -T^2R^{-2}(g^*(t,u(t))-\bar{e})+u(t)\end{pmatrix}. \tag{4.6}$$

From (H_3) and the continuity and periodicity of $g(t,x)$, we may take $M>M_1\geqslant M_0$, such that as $|x|\geqslant M_1$,

$$\left|\frac{T^2R^{-2}g(t,x)}{x}-1\right|<s, \tag{4.7}$$

$$\left|\frac{F(x)}{x}\right|<q_1 \tag{4.8}$$

and as $|x|<M_1$,

$$|T^2R^{-2}g(t,x)-x|<sM. \tag{4.9}$$

For any $u\in S_M$, from (3.9), (3.10) and (4.7)—(4.9) we get

$$\|K(u)(t)\|<(2-2\cos R)^{-1/2}[T^2(q_1+E_0M^{-1})^2+(sR+eT^2M^{-1}R^{-1})^2]^{1/2}M<M.$$

So the first component of $K(u)(t)$, $x(t)\in S_M$. Similar to § 2, § 3, we may prove that the mapping T: $u(t)\mapsto x(t)$ has a fixed point $x\in S_M$. $K(x)(t)=(x(t),y(t))$ is an R-periodic solution of (4.3), and $(\bar{x}(t),\bar{y}(t))=(x(T^{-1}Rt),y(T^{-1}Rt))$ is a T-periodic solution of (4.2). Theorem 3 is proved.

Turn to Theorem 4. Let $q_0=RT^{-1}$. Then under the transformation $\tau=q_0t$, $y=q_0z$, (4.2) becomes (4.3). Consider the R-periodic system

$$\dot{x}=-x+y-\tilde{F}(u(t))+q_0^{-1}E^*(t),\quad \dot{y}=-x-\tilde{g}(t,u(t))+\bar{e}q_0^{-2}, \tag{4.10}$$

where $\widetilde{F}(x)=q_0^{-1}F(x)-x$, $\widetilde{g}(t,x)=q_0^{-2}g^*(t,x)-x$.

From (H_4), $\lim\limits_{|x|\to\infty}\frac{\widetilde{F}(x)}{x}=0$, $\lim\limits_{|x|\to\infty}\frac{\widetilde{g}(t,x)}{x}=0$.

The matrix

$$X(t,s)=\exp\{-(t-s)/2\}$$
$$\times\begin{pmatrix}\cos\frac{\sqrt{3}}{2}(t-s)-\frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}(t-s) & \frac{2\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}(t-s)\\ -\frac{2\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}(t-s) & \cos\frac{\sqrt{3}}{2}(t-s)\end{pmatrix}$$

is the fundamental matrix of system

$$\dot{x}=-x+y,\quad \dot{y}=-x.$$

Let $v=\frac{\sqrt{3}}{2}R$, $D=e^{-R}\left(\cos^2 v-\frac{\sqrt{3}}{3}\sin v\cos v+\frac{4}{3}\sin v\right)$ and d be the determinant of $B=X^{-1}(R,0)-I$. Then $D>0$,

$$X^{-1}(R,0)=D^{-1}e^{-R/2}\begin{pmatrix}\cos v & (-2\sqrt{3}\sin v)/3\\ \frac{2\sqrt{3}}{3}\sin v & \cos v-(\sqrt{3}\sin v)/3\end{pmatrix},$$

$$d=D^{-2}e^{-R}\cos^2 v-\frac{\sqrt{3}}{3}D^{-2}e^{-R}\sin v\cos v+\frac{4}{3}D^{-2}e^{-R}\sin^2 v-2D^{-1}e^{-R/2}\cos v$$
$$+\frac{\sqrt{3}}{3}D^{-1}e^{-R/2}\sin v+1$$
$$=D^{-1}\left\{\left[1-\frac{1}{2}e^{-R/2}\left(2\cos v-\frac{\sqrt{3}}{3}\sin v\right)\right]^2+\frac{5}{4}e^{-R}\sin^2 v\right\}>0,$$

$$B^{-1}=d^{-1}\begin{pmatrix}D^{-1}e^{-R/2}\left(\cos v-\frac{\sqrt{3}}{3}\sin v\right)-1 & \frac{2\sqrt{3}}{3}D^{-1}e^{-R/2}\sin v\\ -\frac{2\sqrt{3}}{3}D^{-1}e^{-R/2}\sin v & D^{-1}e^{-R/2}\cos v-1\end{pmatrix}.$$

It is easily seen that $\|B^{-1}\|$ and $\|X(t,t+s)\|$ are finite as $0\leqslant s<R$. Similar to above we may prove that for

$$(x(t),y(t))\triangleq K(u)(t)=B^{-1}\int_0^R X(t,t+s)\begin{pmatrix}-\widetilde{F}(u(t+s))+q_0^{-1}E^*(t+s)\\ -\widetilde{g}(t+s,u(t+s))+\bar{e}q_0^{-2}\end{pmatrix}ds,$$

there exists a sufficiently large $M>0$, such that the mapping $T: u(t)\mapsto x(t)$ has a fixed point $x(t)\in S_M$, then $x(RT^{-1}t)$ is a T-periodic solution of (1.1).

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