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COMBINED FINITE ELEMENT AND PSEUDOSPECTRAL METHOD FOR THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS**

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Abstract

This paper analyzes a combined method with artificial compression for solving the three-dimensional evolutionary Navier-Stokes equations with periodic and no-slip boundary condition. A Fourier pseudospectral method with a control operator is used in the periodic direction and a standard finite element method in the two others. The generalized stability of the scheme and optimal rate of convergence of the velocity in L^2-norm are proved on the assumption that the BB condition of the finite element approximation for the two-dimensional Stokes equations is satisfied.

§ 1. Introduction

Let $x=(x_1, x_2)\in Q$, which is a convex polygon of $\mathbf{R}^2$, $y\in I=(0, 2\pi)$ and $\Omega=Q\times I$. We consider the evolutionary Navier-Stokes equations as follows

$$\begin{cases}\partial_t U+(U\cdot\nabla)U-\nu\nabla^2 U+\nabla P=f, & \text{in } \Omega\times(0, T],\\ \nabla\cdot U=0, & \text{in } \Omega\times[0, T],\\ U(x, y, 0)=U_0(x, y), & \text{in } \Omega,\end{cases}\tag{1.1}$$

where $U=(U^{(1)}, U^{(2)}, U^{(3)})$ is the velocity, P the ratio of pressure over density, $\nu>0$ the viscosity constant, Vector function $U_0(x, y)$ and the body force $f(x, y, t)$ are given with period 2π for the variable y. We assume that the problem (1.1) is submitted to semi-periodic boundary condition: periodic condition in y direction and no-slip boundary condition in the two others, i. e., $U(x, y, t)=0$ for all $x\in Q$, $y\in I$ and $t\in[0, T]$. For fixing the pressure P, we also require that

$$\int_\Omega P(x, y, t)\,dxdy=0, \quad t\in(0, T].$$

Many efforts have been done on the numerical approximations of the Navier-Stokes equations. The early work is mainly concerning finite difference methods (FDM)[1-2] and finite element methods (FEM)[3-4]. Recent advances of spectral

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methods show that for the problems with periodic boundary conditions, the Fourier spectral method, using the trigonometric polynomials as basis, usually gives much better results than the standard FDM and FEM [5-8]. To study the boundary-layer stability, the unsteady separation, the flow past a suddenly heated vertical plate and some other problems, we have to deal with semi-peoriodic problems such as (1.1) (see [9—12]). If the domain Q is rectangular, we can use the combined Chebyshev (or Legendre)-Fourier spectral methods [13-14]. However, for a general domain Q the spectral methods can not be used directly. Although FEM is suitable for complex geometrics, it may not benefit from periodicity of the problem. So many papers are devoted to combined schemes: Fourier spectral methods in the periodic directions and finite difference or finite element methods in the others[15-19]. The authors also proposed combined finite element and Fourier pseudospectral methods for solving two-dimensional vorticity equations and Navier-Stokes equations, in which a control operator is used to prevent nonlinear instability due to aliasing appeared in pseudospectral methods. The numerical results show the advanteges of such methods [20-21].

This paper aims to generalize the work of [18, 21] to solve the three-dimensional Navier-Stokes equations (1.1) by using the artificial compressibility method. We also take into account the effect of numerical quadrature. The generalized stability of the scheme and optimal rate of convergence of the velocity in L^2-norm are proved.

To obtain the convergence, we first disscuss the numerical approxi mation of the Stokes problem. We prove that if the Babuška-Brezzi (BB) condition holds for the FEM of the two-dimensional problem, then it also holds for the combined finite element and pseudospectral method (FPM) of the three-dimensional problem, provided that the additional finite dimensionalt space, to which the third component of the approximate velocity belongs, contains the continuous piece-wise linear polynomials space. This result improves that given in [8].

In Section 2, we formulate the combined scheme. Section 3 gives some lemmas used in this paper. Section 4 is devoted to error estimates for the combined FPM of the Stokes problem. In Section 5, we prove the generalized stability and convergence.

§ 2. Finite Element-Pseudospectral Scheme

Let $D\subset\mathbf{R}^n$ ($n=1$, 2 or 3) be an open convex bounded set. For $r\geqslant 0$, we denote by $H^r(D)$, $H^r_0(D)$ and $L^q(D)$ $(1\leqslant q\leqslant\infty)$ the classical Sobolev spaces. The norm and semi-norm of $[H^r(D)]^n$ are denoted by $\|\cdot\|_{r,D}$ and $|\cdot|_{r,D}$. The norm of $[L^\infty(D)]^n$

is $\|\cdot\|_{\infty,D}$ and the inner product of $[L^2(D)]^n$ is $(\cdot,\ \cdot)_D$. In the case of $r=0$ or $D=\Omega$, we drop the subscript r or D respectively. Let

$$\mu_D(u)=\int_D u(z)dz$$

and set

$$H^1_{0,p}(\Omega)=\{u\in H^1(\Omega)\,|\,u(\cdot,\ y)\,|_{\partial\Omega}=0,\ \forall y\in I;\ u(x,\ 0)=u(x,\ 2\pi),\ \forall x\in Q\},$$
$$\tilde{L}^2(D)=\{u\in L^2(D)\,|\,\mu_D(u)=0\},$$
$$V=[H^1_{0,p}(\Omega)]^3,\quad W=\tilde{L}^2(\Omega).$$

We consider the weak foamulation of (1.1): find $(U,\ P)\in V\times W$ such that

$$\begin{cases}(\partial_t U,\ v)+((U\cdot\nabla)U,\ v)+a(U,\ v)-b(v,\ P)=(f,\ v), & \forall v\in V,\ t\in(0,\ T],\\ b(U,\ w)=0, & \forall w\in W,\ t\in[0,\ T],\end{cases}\tag{2.1}$$

where

$$a(u,\ v)=\nu(\nabla u,\ \nabla v),\ b(u,\ w)=(\nabla\cdot u,\ w).$$

Let $(\mathscr{T}_h)$ be a family of triangulation of $\bar{Q}$ by triangles K of diamater$\leqslant h$ such that $\bar{Q}=\bigcup_{K\in\mathscr{T}_h}K$. Assume that there are constants $\mu_1,\ \mu_2>0$ such that for all h and $K\in\mathscr{T}_h$,

$$h_K\leqslant\mu_1\rho_K,\quad h\leqslant\mu_2 h_K,\tag{2.2}$$

where h_K is the diameter of K, and ρ_K is the diameter of the largest circle contained in K. For m$\geqslant$0, a fixed integer, set finite dimensional spaces

$$L_{m,h}=\{u\,|\,u|_K\in\mathbf{P}_m,\ \forall K\in\mathscr{T}_h\},\quad S_{m,h}=L_{m,h}\cap H^1(Q),\quad S^0_{m,h}=L_{m,h}\cap H^1_0(Q),$$
$$L^{(l)}_h=\{u\,|\,u|_K\in\mathbf{P}^{(l)}_K,\quad \mathbf{P}_{m+1}\subseteq\mathbf{P}^{(l)}_K\subseteq\mathbf{P}_{m+2},\ \forall K\in\mathscr{T}_h\},\quad l=1,\ 2,\ 3,$$
$$\mathbf{P}_K=\bigcup_{l=1}^3\mathbf{P}^{(l)}_K,\quad L_h=\bigcup_{l=1}^3 L^{(l)}_h,\quad \mathring{S}^{(l)}_h=L^{(l)}_h\cap H^1_0(Q),$$

where $\mathbf{P}_m$ is the space of polynomials of degree m or less.

Using the notations as in [23], let

$$F_K:\ \hat{x}\in\hat{K}\to F_K(\hat{x})=B_K\hat{x}+b_K$$

be the invertible affine mapping which maps $\hat{K}$ onto K such that the Jacobian of the mapping F_K is positive. Consider a quadrature scheme

$$\int_{\hat{K}}\hat{u}(\hat{x})d\hat{x}\sim\sum_{l=1}^L\hat{\omega}_l\hat{u}(\hat{b}_l),\quad \hat{\omega}_l>0,\ 1\leqslant l\leqslant L,$$

which induces a quadrature scheme over the set K

$$\int_K u(x)dx\sim\sum_{l=1}^L\omega_{l,K}u(b_{l,K}),\quad \omega_{l,K}>0,\ 1\leqslant l\leqslant L,$$

where

$$\omega_{l,K}=\det(B_K)\hat{\omega}_l,\ b_{l,K}=F_K(\hat{b}_l).$$

We define

$$(u,\ v)_{h,Q}=\sum_{K\in\mathscr{T}_h}(u,\ v)_{h,K},\quad (u,\ v)_{h,K}=\sum_{l=1}^L\omega_{l,K}u(b_{l,K})v(b_{l,K})$$

and assume that

$$E_K(uv)=(u,v)_K-(u,v)_{h,K}=0,\ \forall u,v\in \mathbf{P}_K,\ K\in\mathscr{T}_h. \tag{2.3}$$

We also define

$$(u,v)_h=\int_I dy(u,v)_{h,Q}=\sum_{K\in\mathscr{T}_h}\int_I dy(u,v)_{h,K}, \tag{2.4}$$

$$E_h(uv)=(u,v)-(u,v)_h=\sum_{K\in\mathscr{T}_h}\int_I dyE_K(uv). \tag{2.5}$$

We introduce the points $y_j=2\pi j/(2N+1)$ $(0\leqslant j\leqslant 2N)$, and set

$$S_N=\text{span}\ \{e^{ily}\mid -N\leqslant l\leqslant N\}.$$

$\mathring{S}_N$ is the subset of S_N, containing all real-valued functions. Let P_C: $C(\bar{I})\to S_N$ be an interpolation operator, i.e.,

$$P_Cu(y_j)=u(y_j),\ 0\leqslant j\leqslant 2N.$$

If u is a vector, then $P_Cu=(P_Cu^{(1)}, P_Cu^{(2)}, P_Cu^{(3)})$ and so on. For $\gamma\geqslant 1$ and $u\in S_N$, we define a control operator $R=R(\gamma)$ by

$$Ru(y)=\sum_{|l|<N}\left(1-\left|\frac{l}{N}\right|^{\gamma}\right)\hat{u}_l e^{ily},\ \hat{u}_l=\frac{1}{2\pi}(u, e^{ily})_I.$$

If $u, v, w\in V$ and $\nabla\cdot v=0$, then

$$((v\cdot\nabla)u,w)=\frac{1}{2}((v\cdot\nabla)u,w)-\frac{1}{2}((v\cdot\nabla)w,u)\equiv J(u,v,w). \tag{2.6}$$

In order to approximate (2.6), we define

$$J_\delta(u,v,w)=\frac{1}{2}(P_C[(v\cdot\nabla)u],\ w)_h-\frac{1}{2}(P_C[(v\cdot\nabla)w],\ u)_h.$$

It is clear that

$$J_\delta(u,v,w)+J_\delta(w,v,u)=0. \tag{2.7}$$

Let τ be the mesh size in time t and $\lambda=\tau(h^{-2}+N^2)\leqslant C$. $u^k(x)=u(x,k\tau)$, which is denoted usually by u^k for simplicity. We define $u_t^k=\frac{1}{\tau}(u^{k+1}-u^k)$.

We now define the approximation spaces

$$V_\delta^{(l)}=\mathring{S}_h^{(l)}\otimes\mathring{S}_N,\quad V_\delta=\prod_{l=1}^{3}V_\delta^{(l)},\quad W_\delta=(L_{m,h}\otimes\mathring{S}_N)\cap\tilde{L}^2(\Omega).$$

Let β be a small positive parameter of artificial compression (see [3]). The finite element-pseudospectral scheme for (1.1) is to find $(u_\delta^k, p_\delta^k)\in V_\delta\times W_\delta$ such that

$$\begin{cases} L_1(u_\delta^k, p_\delta^k, v)\equiv(u_{\delta t}^k, v)+J_\delta(Ru_\delta^{k+\theta}, Ru_\delta^k, Rv)+a(u_\delta^{k+\sigma}, v)-b(v, p_\delta^{k+\alpha}) \\ \qquad =(RP_C\Pi^{m+1}f^k, v),\quad \forall v\in V_\delta,\ k\geqslant 0, \\ L_2(u_\delta^k, p_\delta^k, w)\equiv\beta(p_{\delta t}^k, w)+b(u_\delta^{k+\alpha}, w)=0,\quad \forall w\in W_\delta,\ k\geqslant 0, \\ u_\delta^0=P_C\Pi^{m+1}U_0,\ p_\delta^0\in W_\delta, \end{cases} \tag{2.8}$$

where $u_\delta^{k+\theta}=u_\delta^k+\theta\tau u_{\delta t}^k$, etc., θ, $\sigma\geqslant 0$, $\alpha>1/2$ and Π^{m+1} is the usual Lagrange interpolation operator of degree $m+1$ (see [23]). p_δ^0 is arbitrarily chosen, but $\|p_\delta^0\|\leqslant C$.

§ 3. Lemmas

Throughout the paper, C will denote various positive constants independent of h, N, τ and any functions. The notation $\hookrightarrow$ will denote the continuous imbedding of spaces. We assume that $h\leqslant CN^{-1}$. This is not an unrealistic restriction, because the scheme is more accurate in the periodic direction than in the others.

1) Notations and lemmas about functions defined on Q.

In this subsection, all functions are defined on Q. Thus we drop the subscript Q in the notations of norms and inner product.

Let $\mathscr{L}$: $L^2(Q)\to L_{m,h}$, $\mathscr{P}$: $L^2(Q)\to S_{m+1,h}$ and $\mathscr{P}_0$: $L^2(Q)\to S^0_{m+1,h}$ be the L^2-orthogonal projection operators. Let $\mathscr{P}_1$: $H^1_0(Q)\to S^0_{m+1,h}$ be the H^1-projection operator, i.e.,

$$(\nabla_x(p_1u-u),\ \nabla_x v)=0,\quad \forall v\in S^1_{m+1,h},$$

where $\nabla_x=(\partial_{x_1},\ \partial_{x_2})$.

Lemma 1. *If* $u\in H^r(Q)$, *then*

$$\|u-\mathscr{L}u\|\leqslant Ch^r|u|_r,\quad 0\leqslant r\leqslant m+1,\tag{3.1}$$

Lemma 2[20] *If* $u\in H^r(Q)$, *then*

$$\|u-\mathscr{P}u\|_\mu\leqslant Ch^{r-\mu}|u|_r,\quad 0\leqslant\mu\leqslant\min(1,\ r),\ r\leqslant m+2.\tag{3.2}$$

If $u\in H^r(Q)\cap H^d_0(Q)$, $d=\min(1,\ r)$, *then*

$$\|u-\mathscr{P}_0u\|_\mu\leqslant Ch^{r-\mu}|u|_r,\quad 0\leqslant\mu\leqslant\min(1,\ r),\ r\leqslant m+2,\tag{3.3}$$

$$\|u-\mathscr{P}_1u\|_\mu\leqslant Ch^{r-\mu}|u|_r,\quad 0\leqslant\mu\leqslant 1\leqslant r\leqslant m+2,\tag{3.4}$$

Lemma 3. *If* $u\in \mathbf{P}_{2m+4}(K)$ *and* $v\in\mathbf{P}_K$, *then*

$$|E_K(uv)|\leqslant Ch^r|u|_{r,K}\|v\|_{0,K},\quad 0\leqslant r\leqslant m+2.\tag{3.5}$$

If $E_K(u)=0$ *in addition, then*

$$|E_K(uv)|\leqslant Ch^{r+\mu}|u|_{r,K}|v|_{\mu,K},\quad 0\leqslant r\leqslant m+2,\ 0\leqslant\mu\leqslant 1.\tag{3.6}$$

Proof The results can be proved by using the methods as in [23].

Lemma 4. *Let* $H^{-1}(Q)=(H^1(Q))'$ *and* h *be suitably small. Then there exists a linear operator* $\widetilde{\mathscr{P}}_0$: $L^2(Q)\to S^0_{1,h}$ *such that for* $u\in H^r_0(Q)$ $(0\leqslant r\leqslant 1)$, *we have* $\mu_Q(u-\widetilde{\mathscr{P}}_0u)=0$ *and*

$$\|u-\widetilde{\mathscr{P}}_0u\|_\mu\leqslant Ch^{r-\mu}\|u\|_r,\quad -1\leqslant\mu\leqslant r,\ 0\leqslant r\leqslant 1.\tag{3.7}$$

Proof Let the vertices of all $K\in\mathscr{T}_h$ be denoted by $\{b_l\}$ and $\{w_l\}$ be the basis for $S_{1,h}$ defined by $w_j(b_l)=\delta_{jl}$. We define $\Lambda=\{l\,|\,b_l\in\partial Q\}$. For each vertex $b_l\in\partial Q$, we take one of its adjacent verties $b_{l'}\in Q$ and set $\widetilde{w}_l=w_l-c_l w_{l'}$ with $c_l=\mu_Q(w_l)/\mu_Q(w_{l'})$. It is easy to see that by (2.2),

$$|c_l|\leqslant(\mu_1\mu_2)^2,\quad \|\widetilde{w}_l\|\leqslant Ch.\tag{3.8}$$

Let $\widetilde{\mathscr{P}}$: $L^2(Q)\to S_{1,h}$ be the L^2-orthogonal projection operator. Then define

$$\widetilde{\mathscr{P}}_0u=\widetilde{\mathscr{P}}u-\eta,\quad \eta=\sum_{l\in\Lambda}(\widetilde{\mathscr{P}}u)(b_l)\widetilde{w}_l.$$

Obviously, $\mathscr{P}_0 u \in S^0_{1,h}$. Because $\mu_Q(\tilde{w}_l)=0$, $\mu_Q(\mathscr{P}_0 u)=\mu_Q(\tilde{\mathscr{P}} u)=\mu_Q(u)$.

For any $v\in H^1(Q)$, we have

$$|(u-\tilde{\mathscr{P}}u,\ v)|=|(u-\tilde{\mathscr{P}}u,\ v-\tilde{\mathscr{P}}v)|\leqslant\|u-\tilde{\mathscr{P}}u\|\|v-\tilde{\mathscr{P}}v\|\leqslant Ch^{r+1}|u|_r|v|_1.$$

Therefore, it is well known that

$$\|u-\tilde{\mathscr{P}}u\|_\mu\leqslant Ch^{r-\mu}|u|_r,\quad -1\leqslant\mu\leqslant r. \tag{3.9}$$

Next, we shall prove that

$$\|\eta\|_\mu\leqslant Ch^{r-\mu}|u|_r,\quad -1\leqslant\mu\leqslant r. \tag{3.10}$$

By the inverse property, we have

$$\|\eta\|_\mu\|\eta\|\leqslant Ch^{-\mu}(\eta,\ \eta)\leqslant Ch^{-\mu}\|\eta\|_{-1}\|\eta\|_1\leqslant Ch^{-1-\mu}\|\eta\|_{-1}\|\eta\|,\quad \mu\geqslant 0.$$

So it is sufficient to prove (3.10) for $\mu=-1$. For any $v\in H^1(Q)$, letting $v_h=\tilde{\mathscr{P}}v$, we have

$$|(\eta,\ v)|\leqslant\sum_{l\in A}|\tilde{\mathscr{P}}u(b_l)(\tilde{w}_l,\ v_h)|\leqslant\Big(\sum_{l\in A}|\tilde{\mathscr{P}}u(b_l)|^2\Big)^{1/2}\Big(\sum_{l\in A}|(\tilde{w}_l,\ v_h)|^2\Big)^{1/2}. \tag{3.11}$$

Suppose that $b_l\in K_l\in\mathscr{T}_h$ and supp $\tilde{w}_l=Q_l$. By the inverse property, we have

$$|\tilde{\mathscr{P}}u(b_l)|=|\tilde{\mathscr{P}}u(b_l)-\mathscr{P}_0u(b_l)|\leqslant Ch^{-1}\|\tilde{\mathscr{P}}u-\mathscr{P}_0u\|_{0,K_l}. \tag{3.12}$$

On the other hand, since $\mu_{Q_l}(\tilde{w}_l)=0$, we have from (3.8) that

$$|(\tilde{w}_l,\ v_h)|=|(\tilde{w}_l,\ v_h-v_h(b_l))_{Q_l}|\leqslant Ch\|v_h-v_h(b_l)\|_{0,Q_l}. \tag{3.13}$$

By the equivalence of norms on $\mathbf{P}_1$, we get for any $x,\ x'\in K$

$$|v_h(x)-v_h(x')|C\inf_{w\in\mathbf{P}_0}\|\hat{v}_h+w\|_{1,\hat{K}}\leqslant C|\hat{v}_h|_{1,\hat{K}}\leqslant C|v_h|_{1,K}.$$

Therefore, for any $x\in Q_l$, we have

$$|v_h(x)-v_h(b_l)|\leqslant C|v_h|_{1,Q_l}, \tag{3.14}$$

which implies

$$\|v_h-v_h(b_l)\|_{0,Q_l}\leqslant C(\text{meas}(Q_l))|^{1/2}v_h|_{1,Q_l}\leqslant Ch|v_h|_{1,Q_l}. \tag{3.15}$$

Hence we obtain from (3.11) ~ (3.15) and (3.3)

$$|(\eta,\ v)|\leqslant Ch\Big(\sum_{l\in A}\|\tilde{\mathscr{P}}r-\mathscr{P}_0u\|^2_{0,K_l}\Big)^{1/2}\Big(\sum_{l\in A}|v_h|^2_{1,Q_l}\Big)^{1/2}$$

$$\leqslant Ch\|\tilde{\mathscr{P}}u-\mathscr{P}_0u\|\|v_h\|_1\leqslant Ch^{r+1}|u|_r\|v\|_1. \tag{3.16}$$

Thus we get (3.10). The combination of (3.9) with (3.10) leads to (3.7).

Lemma 5. *If* $u\in L^\infty(K)\cap H^r(K)$ *for all* $K\in\mathscr{T}_h$, $r<1/2$ *and is fixed, then* $u\in H^r(Q)$. *In particular*,

$$L_{0,h}\subset H^r(Q),\quad S_{1,h}\subset H^{1+r}(Q). \tag{3.17}$$

Proof The result can be got by proving that[22,17]

$$|u|^2_r=\int_Q\int_Q|u(x)-u(y)|^2|x-y|^{-2-2r}dx\,dy<\infty. \tag{3.18}$$

2) Notations and lemmas about functions defined on $I=(0,\ 2\pi)$.

For $s\geqslant 0$, let $H^s_p(I)\subset H^s(I)$ denote the Sobolev space of periodic distributions, i.e.,

$$H^s_p(I)=u\Big\{\in L^2(I)\Big|\ \sum_{l=-\infty}^{\infty}(1+|l|^2)^s|\hat{u}_l|^2<\infty,\ \hat{u}_l=\frac{1}{2\pi}(u,\ e^{ilx})_I\Big\}. \tag{3.19}$$

For $s<0$, let $H^s_p(I)=(H^{-s}_p(I))'$.

In this subsection, we drop the subscript I. It is easy to show that for any u, $w\in S_N$ and $v\in C(\bar{I})$, we have

$$(P_C(uv),\ w)=(u,\ P_C(vw)),\tag{3.20}$$

Let P_N: $L^2(I)\to S_N$ be the L^2-orthogonal projection operator.

Lemma 6[6, 20]. *If $u\in H_p^s(I)$ and $s\geq 0$, then*

$$\|P_Nu-u\|_\mu\leq CN^{\mu-s}|u|_s,\quad |P_Nu|_s\leq|u|_s,\quad \mu\leq s,\tag{3.21}$$

and if $s>1/2$ in addition, then

$$\|P_Cu-u\|_\mu\leq CN^{\mu-s}|u|_s,\quad |P_Cu|_s\leq C|u|_s,\quad 0\leq\mu\leq s.\tag{3.22}$$

Lemma 7[20]. *If $u,\ v\in S_N$, then*

$$|u|_s\leq N^{s-\mu}|u|_\mu,\quad 0\leq\mu\leq s,\tag{3.23}$$

$$|P_C(uv)|_s\leq C|uv|_s,\quad s\geq 0.\tag{3.24}$$

Lemma 8[20]. *If $u\in S_N$, $0\leq s-\mu\leq\gamma$ and $s\geq 0$, then*

$$\|Ru-u\|_\mu\leq CN^{\mu-s}|u|_s,\quad |Ru|_s\leq|u|_s.\tag{3.25}$$

Lemma 9[20]. *If $-2\leq\mu\leq s$, $s>1/2$, $\gamma\geq\max(s-\mu,\ 1)$, $r>0$ and $u,\ v\in S_N$, then*

$$\|RP_C(RuRv)-RuRv\|_\mu\leq CN^{\mu-s}(|u|_s\|v\|_{1/2+r}+\|u\|_{1/2+r}|v|_s).\tag{3.26}$$

3) Notations and lemmas about functions defined on $\Omega=Q\times I$.

Let A be a Banach space. We denote by $C(a,\ b;\ A)$ $(a<b)$ the space of strongly continuous functions from $[a,\ b]$ to A, and by L^2 $(a,\ b;\ A)$ the space of measurable functions $u(z)$ from $(a,\ b)$ to A, satisfying

$$\|u\|_{L^2(a,b;A)}=\left(\int_a^b\|u(z)\|_A^2dz\right)^{1/2}<\infty.$$

For any nonnegative integer j, let

$$H^j(a,\ b;\ A)=\{u(z)\in A\,|\,\partial_z^k u\in L^2(a,\ b;\ A),\quad 0\leq k\leq j\},$$

equipped respectively with the norm and semi-norm

$$\|u\|_{H^j(a,b;A)}=\left(\sum_{k=0}^{j}\int_a^b\|\partial_z^k u(z)\|_A^2dz\right)^{1/2},\quad |u|_{H^j(a,b;A)}=\left(\int_a^b|\partial_z^j u(z)|_A^2dz\right)^{1/2}.$$

For any positive real s, $H^s(a,\ b;\ A)$ is defined by interpolation. We define the non-isotropic Sobolev space for real $r,\ s\geq 0$ (see [22])

$$H^{r,s}(\Omega)=L^2(I;\ \mathrm{H}^r(Q))\cap H^s(I;\ L^2(Q)),$$

equipped respectively with the norm and semi-norm

$$\|u\|_{H^{r,s}(\Omega)}=(\|u\|^2_{L^2(I;H^r(Q))}+\|u\|^2_{H^s(I;L^2(Q))})^{1/2},$$

$$|u|_{H^{r,s}(\Omega)}=(|u|^2_{L^2(I;H^r(Q))}+|u|^2_{H^s(I;L^2(Q))})^{1/2}.$$

Let $C_p^\infty(I;\ C^\infty(Q))$ be the set of restrictions to Ω of infinitely differentiable functions with period 2π for y. If r and s are non-negative integers, we define by $H_p^{r,s}(\Omega)$ the closure of $C_p^\infty(I;\ C^\infty(Q))$ in $H^{r,s}(\Omega)$ and by $H_p^s(I;\ H^r(Q))$ the closure of C_p^∞ $(I;\ C^\infty(Q))$ in H^s $(I;\ H^r(Q))$. For real r, $s\geq 0$, they are defined by interpolation. Let $\mathrm{H}^{-1}(\Omega)=(H^1_{0,p}(\Omega))'$. Moreover, we define

$$H^{r,s}_{0,p}(\Omega)=H_p^{r,s}(\Omega)\cap L^2(I;\ H_0^1(Q)),$$

$$A^{r,s}=H_p^{r,s}(\Omega)\cap H_p^{s-1}(I;\ H_0^1(Q))\cap H_p^1(I;\ H^{r-1}(Q)),\quad r,\ s\geq 1.$$

For simplicity, we use the notations

$$\|\cdot\|_{r,s}=\|\cdot\|_{[H^{r,s}(\Omega)]^n},\quad n=1,2,3,$$

$$\|\cdot\|_{s(r)}=\|\cdot\|_{[H^s(I;H^r(\Omega))]^n},\quad \|\cdot\|_{A^{r,s}}=\|\cdot\|_{r,s}+\|\cdot\|_{s-1(1)}+\|\cdot\|_{1(r-1)}.$$

Let $|\cdot|_A$ be the semi-norm corresponding to $\|\cdot\|_A$ and $\|\mu\|_A=\max\limits_{0<k\tau<T}\|u^k\|_A$.

Hereafter, let $\bar{r}=\min(r, m+2)$.

Lemma 10[17]. *The following continuous imbedding holds:*

$$H^{r,s}(\Omega)\subsetneq H^{s'}(I;H^{r'}(Q)),\quad \text{if } r'/r+s'/s=1, \tag{3.27}$$

$$H^{r,s}(\Omega)\subsetneq C(\bar{\Omega}),\quad \text{if } 2/r+1/s<2. \tag{3.28}$$

Lemma 11[17]. *If $u\in H_p^{r,s}(\Omega)$ and $2/r+1/s<2$, then*

$$\|u-\Pi^{m+1}P_Cu\|\leqslant C(h_{\bar r}+N^{-s})|u|_{\bar r,s}. \tag{3.29}$$

Lemma 12[20]. *If $u\in H_{0,p}^{r,s}(\Omega)$ and $r, s\geqslant 0$, then*

$$\|u-\mathscr{P}_0P_Nu\|\leqslant C(h_{\bar r}+N^{-s})|u|_{\bar r,s}. \tag{3.30}$$

If $u\in A^{r,s}$ and $r, s\geqslant 1$, then

$$\|u-\mathscr{P}_0P_Nu\|_1\leqslant C(h_{\bar r}^{-1}+N^{1-s})|u|_{\bar r,s}. \tag{3.31}$$

Lemma 13. *If $u\in H_p^{r,s}(\Omega)$ and $s\geqslant 0$, $0\leqslant r\leqslant m+1$, then*

$$\|u-\mathscr{L}P_Nu\|\leqslant C(h^r+N^{-s})|u|_{r,s}. \tag{3.32}$$

Remark 1. If $u\in\tilde{L}^2(\Omega)$, then $\mathscr{L}P_Nu\in\tilde{L}^2(\Omega)$.

For $u\in(H^1(Q))^2$ and $w\in\tilde{L}^2(Q)$, let

$$\underline{a}(u,v)=\nu(\nabla_x u,\nabla_x v)_Q,\quad \underline{b}(u,w)=(\nabla_x\cdot u,w)_Q.$$

We set $S_h=S_h^{(1)}\times S_h^{(2)}$ and $\tilde{L}_h=L_{m,h}\cap\tilde{L}^2(Q)$. Assume that the following BB condition holds:

$$\sup_{v\in S_h}\frac{\underline{b}(v,w)}{\|v\|_{1,Q}}\geqslant C\|w\|_{0,Q},\quad \forall w\in\tilde{L}_h, \tag{3.33}$$

which is equivalent to the following condition[4]

$$\sup_{w\in\tilde{L}_h}\frac{\underline{b}(v,w)}{\|w\|_{0,Q}}\geqslant C\|v\|_{1,Q},\quad \forall v\in V_h^{\perp}, \tag{3.34}$$

where

$$V_h=\{u\in S_h\,|\,\underline{b}(u,w)=0,\ \forall w\in\tilde{L}_h\}$$

and $V_h^{\perp}$ is the orthogonal complement of V_h in S_h. Another equivalent condition is as follows[24]

$$\sup_{(\xi,\varphi)\in S_h\times\tilde{L}_h}\frac{|\underline{a}(u,\xi)+\underline{b}(\xi,q)+\underline{b}(u,\varphi)|}{\|\xi\|_{1,Q}+\|\varphi\|_{0,Q}}\geqslant C(\|u\|_{1,Q}+\|q\|_{0,Q}),\ \forall(u,q)\in S_h\times\tilde{L}_h, \tag{3.35}$$

which can be derived from (3.33)—(3.34).

For a vector $v=(v^{(1)},v^{(2)},v^{(3)})$, we denote $\underline{v}=(v^{(1)},v^{(2)})$ and $\underset{=}{v}=v^{(3)}$.

Lemma 14. *Assume that $V_1=V_\delta^{(1)}\times V_\delta^{(2)}\times(S_{1,h}^0\otimes\dot{S}_N)$, $hN\leqslant C$ and (3.33) holds, Then*

$$\sup_{v_\delta\in V_1}\frac{b(v_\delta,w)}{\|v_\delta\|_1}\geqslant C\|w\|,\quad \forall w\in W_\delta. \tag{3.36}$$

Proof Let

$$V_0=\{u\in [H_0^1(\Omega)]^3 | \operatorname{div} u=0\}.$$

According to Lemma 3.2 of Chapter I in [4], the divergence operator is an isomorphism from the orthogonal complement of the set V_0 in $[H_0^1(\Omega)]^3$ onto W. So for $w\in W_\delta$, there exists $v\in V$ such that

$$\operatorname{div} v=w, \quad \|v\|_1\leqslant C\|w\|. \tag{3.37}$$

Let $P_N v\equiv v^N\equiv(\underline{v}^N, \underline{v}^N)$, $\underline{v}_\delta=\mathscr{P}_0\underline{v}^N$ and $\tilde{v}=\partial_y(\underline{v}^N-\underline{v}_\delta)$. By (3.7), we have

$$\|\underline{v}_\delta\|_1\leqslant C\|\underline{v}^N\|_1\leqslant C\|v\|_1, \tag{3.38}$$

$$\|\tilde{v}\|_{\mu,Q}\leqslant Ch^{r-\mu}\|\partial_y\underline{v}^N\|_{r,Q}, \quad -1\leqslant\mu\leqslant r,\ 0\leqslant r\leqslant 1. \tag{3.39}$$

Next, for each $y\in I$, let $(v_\delta, q_\delta)\in S_h\times\tilde{L}_h$ be determined by

$$\begin{cases}\underline{a}(\underline{v}_\delta, \xi)-\underline{b}(\xi, q_\delta)=\underline{a}(\underline{v}^N, \xi), & \forall\xi\in S_h,\\ \underline{b}(\underline{v}_\delta, \varphi)=\underline{b}(\underline{v}^N, \varphi)+(\tilde{v}, \varphi)_Q, & \forall\varphi\in\tilde{L}_h,\end{cases} \tag{3.40}$$

which has a unique solution $(\underline{v}_\delta, q_\delta)$ thanks to (3.3?)[4]. It is easy to see that $(P_N\underline{v}_\delta, p_N q_\delta)$ is also the solution of (3.40). Thus we know that $(v_\delta, q_\delta)\in V_1\times W_\delta$. Let $u=\mathscr{P}\underline{v}_1{}^N-\underline{v}_\delta\in V_\delta^{(1)}\times V_\delta^{(2)}$. From (3.40) we have

$$|\underline{a}(u, \xi)+\underline{b}(\xi, q_\delta)+\underline{b}(u, \varphi)|\leqslant|(\tilde{v}, \varphi)_Q|+|i(\underline{v}^N-\mathscr{P}_1\underline{v}^N, \varphi)|$$
$$\leqslant(\|\tilde{v}\|_{0,Q}+\|\underline{v}^N-\mathscr{P}_1\underline{v}^N\|_{1,Q})\|\varphi\|_{0,Q}.$$

Hence by (3.35), (3.39) and (3.4), we get

$$\|\underline{v}^N-\underline{v}_\delta\|_{1,Q}+\|q_\delta\|_{0,Q}\leqslant\|\underline{v}^N-\mathscr{P}_1\underline{v}^N\|_{1,Q}+\|u\|_{1,Q}+\|q_\delta\|_{0,Q}$$
$$\leqslant C(\|\tilde{v}\|_{0,Q}+\|\underline{v}^N-\mathscr{P}_1\underline{v}^N\|_{1,Q})\leqslant C(\|\partial_y\underline{v}^N\|_{0,Q}+\|\underline{v}^N\|_{1,Q}). \tag{3.41}$$

Now consider an auxiliary problem: for $g\in L^2(Q)$, find $(u_g, p_g)\in[H_0^1(Q)]^2\times\tilde{L}^2(Q)$ such that

$$\begin{cases}\underline{a}(\xi, u_g)-\underline{b}(\xi, p_g)=(g, \xi)_Q, & \forall\xi\in[H_0^1(Q)]^2,\\ \underline{b}(u_g, \varphi)=0, & \forall\varphi\in\tilde{L}^2(Q).\end{cases} \tag{3.42}$$

We know from the regularity theorem (see [4; Theorem 5.2 of Chapter I]) that

$$\|u_g\|_{2,Q}+\|p_g\|_{1,Q}\leqslant C\|g\|_{0,Q}. \tag{3.43}$$

Letting $\xi=\underline{v}^N-\underline{v}_\delta$ and $\varphi=q_\delta$ in (3.42), we get

$$(g, \underline{v}^N-\underline{v}_\delta)_Q={}_L(\underline{v}^N-\underline{v}_\delta, u_g)-\underline{b}(\underline{v}^N-\underline{v}_\delta, p_g)+\underline{b}(u_g, q_\delta).$$

By (3.40), for any $(\xi, \varphi)\in S_h\times\tilde{L}_h$, we have

$$-\bar{v}(\underline{v}^N-\underline{v}_\delta, \xi)-\underline{b}(\xi, q_\delta)+\underline{b}(a^N-\underline{v}_\delta, \varphi)+(\tilde{v}, \varphi)_Q=0.$$

Therefore,

$$(g, \underline{v}^N-\underline{v}_\delta)_Q=\underline{a}(\underline{v}^N-\underline{v}_\delta, u_g-\xi)-\underline{b}(\underline{v}^N-\underline{v}_\delta, p_g-\varphi)+\underline{b}(u_g-\xi, q_\delta)$$
$$+(\tilde{v}, \varphi-p_g)_Q-(\tilde{v}, q_g)_Q.$$

Hence by putting $\xi=\mathscr{P}_0 u_g$ and $\varphi=\mathscr{L}p_g$, we obtain fom (3.1), (3.3), (3.41), (3.39) and (3.43)

$$|(g, \underline{v}^N-\underline{v}_\delta)_Q|\leqslant C\{h(\|\underline{v}^N-\underline{v}_\delta\|_{1,Q}+\|q_\delta\|_{0,Q}+\|\tilde{v}\|_{0,Q})(\|u_g\|_{2,Q}$$
$$+\|p_g\|_{1,Q})+\|\tilde{v}\|_{-1,Q}\|p_g\|_{1,Q}\}$$
$$\leqslant Ch(\|\partial_y\underline{v}^N\|_{0,Q}+\|\underline{v}^N\|_{1,Q})\|g\|_{0,Q}$$

which implies

$$\|\underline{v}^N-\underline{v}_\delta\|_{0,Q}\leqslant Ch(\|\partial_y\underline{v}^N\|_{0,Q}+\|\underline{v}^N\|_{1,Q}).$$

The above inequality and (3.41) lead to

$$\|\underline{v}_\delta\|_1\leqslant\|\underline{v}_\delta\|_{0(1)}+\|\underline{v}_\delta\|_{1(0)}\leqslant C(\|v^N\|_1+hN\|v^N\|_1)\leqslant C\|v\|_1. \tag{3.44}$$

Finally, let $\tilde{w}=w-\mu_Q(w)$ and take $\varphi=\tilde{w}$ in the second equation of (3.40). Because $\mu_Q(\tilde{v})=0$, we have

$$\underline{b}(\underline{v}_\delta, w)=\underline{b}(\underline{v}^N, w)+(\tilde{v}, w)_Q=\underline{b}(\underline{v}^N, w)+(\partial_y(\underline{v}^N-\underline{v}_\delta), w)_Q.$$

So by putting $v_\delta\equiv(\underline{v}_\delta, \underline{v}_\delta)$, we get from (3.37), (3.38) and (3.44)

$$b(v_\delta, w)=b(v^N, w)=b(v, w)=\|w\|^2\geqslant C\|v\|_1\|w\|\geqslant C\|v_\delta\|_1\|w\|,$$

which completes the proof.

Lemma 15. *There exists a positive constant C_I such that for any $u\in V_\delta$*

$$\|u\|_\infty^2\leqslant C_I h^{-2}N\|u\|^2, \tag{3.45}$$

$$|u|_1^2\leqslant C_I(h^{-2}+N^2)\|u\|^2. \tag{3.46}$$

Lemma 16. *If $u\in H^{1/2+\mu}(I; H^\mu(Q))$ $(0<\mu<1)$ and $v\in H^1(\Omega)$, then*

$$\|uv\|\leqslant C\|u\|_{1/2+\mu(\mu)}\|v\|_1. \tag{3.47}$$

Lemma 17. *If $\bar{s}>3/2$, then there exist $0<\bar{\mu}<3/2$ and $\mu>0$ such that*

$$H^{\bar{\mu},\bar{s}}(\Omega)\subsetneq H^{3/2+\mu}(I; H^\mu(Q))\cap H^{1/2+\mu}(I:H^{1+\mu}(Q))\cap C(\bar{\Omega}). \tag{3.48}$$

Proof The results can be got from Lemma 10.

Now let $\partial_1=\partial_{x_1}$, $\partial_2=\partial_{x_2}$, $\partial_3=\partial_y$.

Lemma 18. *If $u\in L_{2m+4,h}\otimes\mathring{S}_N$ and $v\in L_h\otimes\mathring{S}_N$, then*

$$|D_h(uv)|\leqslant C\|u\|\|v\|, \tag{3.49}$$

$$|(u,v)_h|\leqslant C\|u\|\|v\|. \tag{3.50}$$

If $u\in(L_{2m+4,h}\otimes\mathring{S}_N)\cap H^1(\Omega)$ and $v\in\mathring{S}_h^{(1)}\otimes\mathring{S}_N$, then

$$|E_h(u\partial_j v)|\leqslant C|u|_{0(1)}\|v\|, \quad j=1, 2, \tag{3.51}$$

$$|(u, \partial_j v)_h|\leqslant C|u|_{0(1)}\|v\|, \quad j=1, 2. \tag{3.52}$$

Proof The desired results (3.49)—(3.52) can be got from (2.5), (3.5) and the inverse property.

Lemma 19. *If $u, v, w\in L_h\otimes\mathring{S}_N$, then*

$$(P_C(vu), w)_h=(u, P_C(vw))_h, \tag{3.53}$$

$$E_h(P_C(vu)w)=E_h(uP_C(vw)). \tag{3.54}$$

$$(P_C(v\partial_y u), w)_h=(u, \partial_y P_C(vw))_h, \tag{3.55}$$

$$E_h(P_C(v\partial_y u)w)=E_h(u\partial_y P_C(vw)). \tag{3.56}$$

Proof It can be shown that (3.53)—(3.56) hold by using (2.4), (2.5) and (3.20).

Lemma 20. *Assume that $u, v, w\in\bigcup_{l=1}^{3}V_\delta^{(l)}$, $\bar{s}>3/2$ and*

$$M=|(P_C(v\partial_j u), w)_h|+|(P_C(v\partial_l w), u)_h|, \quad j, l=1, 2, 3.$$

Then there exists $0<\bar{\mu}<3/2$ such that

$$M\leqslant C\|u\|_{\bar{\mu},\bar{s}}\|v\||w|_1, \tag{3.57}$$

$$M \leqslant C\|u\|\,\|v\|_{\bar{\mu},s}|w|_1, \tag{3.58}$$

$$M \leqslant C(\|u\|_\infty |v|_1 + |u|_1 \|v\|_\infty)\|w\|. \tag{3.59}$$

And if $x, v \in H^1_{0,p}(\Omega) \cap H_{\bar{\mu},s}(\Omega)$ *and* $w \in H^1_{0,p}(\Omega)$, *then*

$$|(v\partial_j u, w)| + |(v\partial_l w, u)| \leqslant C \min\{\|u\|_{\bar{\mu},s}\|v\|, \|u\|\,\|v\|_{\bar{\mu},s}\}\|w\|_1. \tag{3.60}$$

Proof Firstly, we get from (3.53), (3.50), (3.23) and (3.47)

$$M = |(v, P_C(\partial_j uw))_h| + |(v, P_C(u\partial_l w))_h| \leqslant C\|v\|(\|\partial_j uw\| + \|u\partial_l w\|)$$
$$\leqslant C\|v\|(\|\partial_j u\|_{1/2+\mu(\mu)} + \|u\|_\infty)\|w\|_1.$$

Then (3.57) follows from (3.48). Next, if $j \neq 3$, by (3.53) and (3.52) we have

$$|(P_C(v\partial_j u), w)_h| = |(\partial_j u, P_C(vw))_h| \leqslant C\|u\|\,|P_C(vw)|_{0(1)}.$$

If $j=3$, then by (3.55) and (3.50) we have

$$|(P_C(v\partial_y u), w)_h| = |(u, \partial_y P_C(vw))_h| \leqslant C\|u\|\,|P_C(vw)|_{0(1)}.$$

Thus we get from (3.24)

$$M \leqslant C\|u\|(|vw|_1 + \|v\partial_l w\|)$$

aud (3.58) follows. Finally, by the samc argument as above we can prove (3.59).

It is now clear that (3.60) can b3 got also.

Lemma 21. *Asumme that* $u, v, w \in \bigcup_{l=1}^{3} V_\delta^{(l)}$, $s > 3/2$ *and*

$$M_1 = |(RP_C - I)(Rv\partial_j Ru), w)|,$$
$$M_2 = |(P_C(Rv\partial_j Rw) - Rv\partial_j w, Ru)|,$$

where $j=1, 2, 3$. *Then there exists* $1 < \bar{\mu} < 3/2$ *such that*

$$M_1 + M_2 \leqslant CN^{-s}\|u\|_{A^{\bar{\mu},s}}\|v\|_{A^{\bar{\mu},s}}\|w\|_1. \tag{3.61}$$

Proof By (3.26), Holder inequality and the imbedding thoerem, we get

$$M_1 \leqslant \|(RP_C - I)(Rv\partial_j Ru)\|_{-1}\|w\|_1$$
$$\leqslant CN^{-s}\left\{\int_Q dx(|\partial_j u|^2_{s-1,I}\|v\|^2_{1/2+r,I} + \|\partial_j u\|^2_{1/2+r,I}|v|^2_{s-1,I})\right\}^{1/2}\|w\|_1$$
$$\leqslant CN^{-s}(\|\partial_j u\|_{s-1(0)}\|v\|_{1/2+r(1+r)} + \|\partial_j u\|_{1/2+r(r)}\|v\|_{s-1(1)})\|w\|_1,$$
$$M_2 = |((RP_C - I)(RvRu), \partial_j w)|$$
$$\leqslant CN^{-s}\left\{\int_Q dx(|u|^2_{s,I}\|v\|^2_{1/2+r,I} + \|u\|^2_{1/2+r,I}|v|^2_{s,I})\right\}^{1/2}\|w\|_1$$
$$\leqslant CN^{-s}(\|u\|_{s(0)}\|v\|_{1/2+r(1+r)} + \|u\|_{1/2+r(1+r)}\|v\|_{s(0)})\|w\|_1.$$

§ 4. Error Estimation for the Stokes Problem

For any $(u, p) \in V \times W$, let $(\eta, \xi) \in V' \times \tilde{L}^2(\Omega)$ such that

$$\begin{cases} a(u, v) - b(v, p) = \langle \eta, v \rangle, & \forall v \in V, \\ b(u, w) = (\xi, w), & \forall w \in W. \end{cases} \tag{4.1}$$

We approximate (u, p) by $(u^*, p^*) \in V_\delta \times W_\delta$ satisfying

$$\begin{cases} a(u^*, v) - b(v, p^*) = \langle \eta, v \rangle, & \forall v \in V_\delta, \\ b(u^*, w) = (\xi, w), & \forall w \in W_\delta. \end{cases} \tag{4.2}$$

Problem (4.2) is a combined finite element and spectral apprpximation to the Stokes problem (4.1). On use of an abstract approximation result for this problem (see [4; Theorem 1.1 of Chapter II]) and the BB condition (3.36), we know that the problem (4.2) has a unique solution. We then define a linear operator $P_\delta: V\times W\to V_\delta\times W_\delta$ by $P_\delta(u, p)\equiv(P_\delta u, P_\delta p)=(u^*, p^*)$. Obviously, we have then (4.1) that

$$\begin{cases} a(P_N u, v)-b(v, P_N p)=\langle\eta, v\rangle, & \forall v\in V_\delta,\\ b(P_N u, w)=(\xi, w), & \forall w\in W_\delta,\end{cases}$$

which implies that

$$P_\delta P_N u=P_\delta u,\quad P_\delta P_N p=P_\delta p. \tag{4.3}$$

Lemma 24. *If* $(u, p)\in[A^{r,s}]^3\times(H_p^{r-1,s-1}(\Omega)\cap W)$ *and* $r, s\geqslant 1$, *then*

$$\|u-P_\delta u\|_1+\|p-P_\delta p\|\leqslant C(h^{\bar r-1}+N^{1-s})(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1}), \tag{4.4}$$

$$\|P_N u-P_\delta u\|_1+\|P_N p-P_\delta p\|\leqslant C_h^{\bar r-1}(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1}). \tag{4.5}$$

Proof The result is a consequence of [4; Theorem 1.1 of Chapter II] due to (3.36), (3.31)—(3.32) and (3.3).

Lemma 25. *If the conditions of Lemma 24 are fulfilled, andthen*

$$\|u-P_\delta u\|\leqslant C(h^{\bar r}+N^{-s})(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1}). \tag{4.6}$$

Proof For $g\in L^2(\Omega)$, let $(u_g, p_g)\in V\times W$ be the solution of

$$\begin{cases} a(v, u_g)-b(v, p_g)=(g, v), & \forall v\in V,\\ b(u_g, w)=0, & \forall w\in W.\end{cases} \tag{4.7}$$

By the regularity result[19], we know that $(u_g, p_g)\in[\mathrm{H}_p^{2,2}(\Omega)]^3\times H_p^{1,1}(\Omega)$ and

$$\|u_g\|_2+\|p_g\|_1\leqslant C\|g\|. \tag{4.8}$$

Denote $\eta^N=P_N\eta$ and $\eta^*=P_\delta\eta$ for simplicity. Let $v=u^N-u^*$ and $w=p^N-p^*$ in (4.7). Then $(g, u^N-u^*)=a(u^N-u^*, u_g^N)-b(u^N-u^*, p_g^N)-b(u_g^N, p^N-p^*)$.

By (4.1) and (4.2), we get for any $(v, w)\in V_\delta\times W_\delta$

$$a(u^N-u^*, v)-b(v, p^N-p^*)-b(u^N-u^*, w)=0.$$

Therefore,

$$(g, u^N-u^*)=a(u^N-u^*, u_g^N-v)-b(u^N-u^*, p_g^N-w)-b(u_g^N-v, p^N-p^*).$$

Hence by putting $v=\mathscr{P}_0 u_g^N$ and $w=\mathscr{L}p_g^N$, we obtain from (3.1) and (3.3)

$$|(g, u^N-u^*)|\leqslant Ch(\|u^N-u^*\|_1+\|p^N-p^*\|)(\|u_g^N\|_2+\|p_g^N\|_1),$$

by which and (4.5), (4.8), we get

$$\|P_N u-P_\delta u\|\leqslant Ch^{\bar r}(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1}).$$

Thus (4.6) follows from (3.21).

Lemma 26. *If* $(u, p)\in[A^{\bar\mu,s}]^3\times(H_p^{\bar\mu-1,s-1}(\Omega)\cap W)$ *and* $1\leqslant\bar\mu<3/2$, $s\geqslant 1$, *then*

$$\|P_\delta u\|_{A^{\bar\mu,s}}+\|P_\delta p\|_{\bar\mu-1,s-1}\leqslant C(\|u\|_{A^{\bar\mu,s}}+\|p\|_{\bar\mu-1,s-1}). \tag{4.9}$$

Proof It can be shown that $\partial_y P_\delta=P_\delta\partial_y$. Then on use of (4.4), we have

$$\|\partial_y^{s-1}P_\delta u\|_1+\|\partial_y^{s-1}P_\delta p\|\leqslant C(\|\partial_y^{s-1}u\|_1+\|\partial_y^{s-1}p\|). \tag{4.10}$$

Next, by inverse property, (4.5), (3.1) and (3.3), we obtain

$$\|P_\delta u\|_{A^{\bar\mu,1}}+\|P_\delta p\|_{0(\bar\mu-1)}\leqslant Ch^{1-\bar\mu}(\|P_\delta u-\mathscr{P}_0P_Nu\|+\|P_\delta p-\mathscr{L}P_Np\|)$$
$$+\|\mathscr{P}_0P_Nu\|_{A^{\bar\mu,1}}+\|\mathscr{L}P_Np\|_{0(\bar\mu-1)}\leqslant C(\|u\|_{A^{\bar\mu,1}}+\|p\|_{0(\bar\mu-1)}). \tag{4.11}$$

Thus (4.9) follows from (4.10) and (4.11).

Lemma 27. *If the conditions of Lemma 24 are fulfilled* $r\geqslant 3/2$, $s>3/2$, $2/r+1/s<2$. $u_j^*=(P_\delta u)^{(j)}$, $w\in V_\delta^{(1)}$ *and*

$$M_1=|E_h(P_C(u_l^*\partial_j u_l^*)w)|,\quad M_2=|E_h(P_C(u_l^*\partial_j w)u_l^*)|,$$

then

$$M_1+M_2\leqslant Ch_{\bar r}(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1})^2\|w\|_1. \tag{4.12}$$

Proof Let $u_j^N=P_Nu^{(j)}$. We have from (2.3), (3.6) and (3.24)

$$M_1=|E_h(P_C[u_l^*\partial_j u_l^*-\mathscr{P}(u_l^N\partial_j u_l^N)]w)|$$
$$\leqslant Ch\|u_l^*\partial_j u_l^*-\mathscr{P}(u_l^N\partial_j u_l^N)\|\|w\|_1\leqslant Ch(I_1+I_2)\|w\|_1,$$

where

$$I_1=\|u_l^*\partial_j(u_l^*-u_l^N)\|+\|(u_l^*-u_l^N)\partial_j u_l^N\|,$$
$$I_2=\|(I-\mathscr{P})(u_l^N\partial_j u_l^N)\|.$$

Due to (3.47), (3.48), (4.9) and (4.5), there exist $\mu>0$, $1<\bar\mu<3/2$ such that

$$I_1\leqslant\|u_l^*\|_\infty\|u_l^*-u_l^N\|_1+\|\partial_j u_l^N\|_{1/2+\mu(\mu)}\|u_l^*-u_l^N\|_1$$
$$\leqslant Ch^{\bar r-1}(\|u\|_{A^{\bar\mu,s}}+\|p\|_{\bar\mu-1,s-1})(\|u\|_{A^{\bar r,s}}+\|p\|_{\bar r-1,s-1}).$$

Since $H^{\bar r,s}(\Omega)$ is an algebra for $2/\bar r+1/s<2$, we get from (3.2)

$$I_2\leqslant Ch^{\bar r-1}|u_l^N\partial_j u_l^N|_{0(\bar r-1)}\leqslant Ch^{\bar r-1}\|u\|_{\bar r,s}^2.$$

We can estimate M_2 in the same way. Thus the proof is completed.

Remark 2. If P_δ is replaced by RP_δ $(\gamma\geqslant s)$ in (4.4), (4.6), (4.9) and (4.12), then the conclusions still hold due to (3.25).

§ 5. The Generalized Stability and Convergence

Now we consider the generalized stability[2] of the scheme (2.8). Suppose that u_δ^k, p_δ^k and the right terms in (2.8) have errors $(\tilde u^k,\tilde p^k)\in V_\delta\times W_\delta$ and $\tilde f_l^k$ $(l=1,2)$ respectively. Then the errors satisfy the following equations

$$\begin{cases}\tilde L_1(\tilde u^k,u_\delta^k,\tilde p^k,v)\equiv(\tilde u_t^k,v)+J_\delta(R\tilde u^{k+\theta},Ru_\delta^k+R\tilde u^k,Rv)+J_\delta(Ru_\delta^{k+\theta},R\tilde u^k,Rv)\\ \qquad a(\tilde u^{k+\sigma},v)-b(v,\tilde p^{k+\alpha})=(\tilde f_1^k,v),\quad\forall v\in V_\delta,\ \tau\geqslant 0, & (5.1)\\ \tilde L_2(\tilde u^k,\tilde p^k,w)\equiv\beta(\tilde p_t^k,w)+b(\tilde u^{k+\alpha},w)=(\tilde f_2^k,w),\quad\forall w\in W_\delta,\ k\geqslant 0. & (5.2)\end{cases}$$

By taking $v=2\tilde u^{k+\alpha}$ in (5.1), and using the following inequalities

$$2|(\tilde u^k,\tilde f_1^k)|\leqslant\varepsilon\nu|\tilde u^k|_1^2+\frac{C}{\varepsilon\nu}\|\tilde f_1^k\|_{-1}^2,\quad\varepsilon>0,$$

$$2\alpha\tau|(\tilde u_t^k,\tilde f_1^k)|\leqslant\varepsilon\nu\tau^2|\tilde u_t^k|_1^2+\frac{C\alpha^2}{\varepsilon\nu}\|\tilde f_1^k\|_{-1}^2,$$

we obtain

$$(\|\tilde{u}^k\|^2)_t+\tau(2\alpha-1)\|\tilde{u}_t^k\|^2+\nu(2-\varepsilon)|\tilde{u}^k|_1^2+\nu\tau(\sigma+\alpha)(|\tilde{u}^k|_1^2)_t$$
$$+\nu\tau^2(2\alpha\sigma-\sigma-\alpha-\varepsilon)|\tilde{u}_t^k|_1^2+\sum_{l=1}^{3}F_l^k-2b(\tilde{u}^{k+\alpha},\tilde{p}^{k+\alpha})\leqslant\frac{C(1+\alpha^2)}{\varepsilon\nu}\|\tilde{f}_1^k\|_{-1}^2, \tag{5.3}$$

where

$$F_1^k=2J_\delta(Ru_\delta^{k+\theta},R\tilde{u}^k,R\tilde{u}^k),$$
$$F_2^k=2\alpha\tau J_\delta(Ru_\delta^{k+\theta},R\tilde{u}^k,R\tilde{u}_t^k)+2\tau(\alpha-\theta)J_\delta(R\tilde{u}^k,Ru_\delta^k,R\tilde{u}_t^k),$$
$$F_3^k=2\tau(\alpha-\theta)J_\delta(R\tilde{u}^k,R\tilde{u}^k,R\tilde{u}_t^k).$$

By taking $w^k=2\tilde{p}^{k+\alpha}\tau$ in (5.2) and arguing as above, we have

$$\beta(\|\tilde{p}^k\|^2)_t+\beta\tau(2\alpha-1-\varepsilon)\|\tilde{p}_t^k\|^2+2b(\tilde{u}^{k+\alpha},\tilde{p}^{k+\alpha})$$
$$\leqslant\beta\|\tilde{p}^k\|^2+\left(\frac{1}{\beta}+\frac{\tau\alpha^2}{\beta\varepsilon}\right)\|\tilde{f}_2^k\|^2. \tag{5.4}$$

Combining (5.3) with (5.4) leads to

$$(\|\tilde{u}^k\|^2+\beta\|\tilde{p}^k\|^2)_t+\tau(2\alpha-1-\varepsilon)(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2)+\nu(2-\varepsilon)|\tilde{u}^k|_1^2$$
$$+\nu\tau(\sigma+\alpha)(|\tilde{u}^k|_1^2)_t+\nu\tau^2(2\alpha\sigma-\sigma-\alpha-\varepsilon)|\tilde{u}_t^k|_1^2+\sum_{l=1}^{3}F_l^k\leqslant\beta\|\tilde{p}^k\|^2+\|\tilde{f}^k\|^2, \tag{5.5}$$

where

$$\|\tilde{f}^k\|^2=\frac{C_1}{\nu\varepsilon}\|\tilde{f}_1^k\|_{-1}^2+\frac{C_2}{\beta}\|\tilde{f}_2^k\|^2. \tag{5.6}$$

We now estimate $|F_l^k|$. From (3.57)–(3.59) and (3.45), we get

$$|F_1^k|\leqslant\varepsilon\nu|\tilde{v}^k|_1^2+\frac{C}{\nu\varepsilon}\|u_\delta\|_{\bar{\mu},s}^2\|\tilde{u}^k\|^2,\quad 1\leqslant\bar{\mu}\leqslant3/2,\ \bar{s}>3/2,$$
$$|F_2^k|\leqslant\varepsilon\nu\tau^2|\tilde{u}_t^k|_1^2+\frac{C}{\nu\varepsilon}(\alpha^2+(\alpha-\theta)^2)\|u_\delta\|_{\bar{\mu},s}^2\|u^k\|^2,$$
$$|F_3^k|\leqslant\varepsilon\tau\|\tilde{u}_t^k\|^2+\frac{\tau NC(\alpha-\theta)^2}{\varepsilon h^2}\|\tilde{u}^k\|^2|\tilde{u}^k|_1^2.$$

By substituting the above estimates into (5.5), we obtain

$$(\|\tilde{u}^k\|^2+\beta\|\tilde{p}^k\|^2)_t+\tau(2\alpha-1-\varepsilon)(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2)+\nu|\tilde{u}^k|_1^2+\nu\tau(\sigma+\alpha)(|\tilde{u}^k|_1^2)_t$$
$$+\nu\tau^2(2\alpha\sigma-\sigma-\alpha-2\varepsilon)|\tilde{u}_t^k|_1^2\leqslant\|\tilde{f}^k\|^2+M(\|\tilde{u}^k\|^2+\beta\|\tilde{p}^k\|^2)+B(\tilde{u}^k)|\tilde{u}^k|_1^2, \tag{5.7}$$

where

$$M=\frac{C_3}{\nu\varepsilon}(1+\alpha^2+(\alpha-\theta)^2)\|u_\delta\|_{\bar{\mu},s}^{2s}$$
$$B(\tilde{u}^k)=-\nu(1-2\varepsilon)+\frac{C_r\tau N}{\varepsilon h^2}(\alpha-\theta)^2\|\tilde{u}^k\|^2.$$

Now assume that

$$\sigma>\frac{1}{2},\quad\text{or}\quad\lambda<\frac{2}{\nu C_I(1-2\sigma)}. \tag{5.8}$$

Let ε be suitably small, $d_0>0$, and consider the two cases as follows

(i) $$2\alpha\sigma\geqslant\alpha+\sigma+2\varepsilon,\quad 2\alpha\geqslant1+\varepsilon+d_0, \tag{5.9}$$

(ii) $$2\alpha\sigma\leqslant\alpha+\sigma+2\varepsilon,\quad 2\alpha\geqslant\frac{1+\varepsilon+d_0+\nu\lambda C_I(\sigma+2\varepsilon)}{1-\nu\lambda C_I(1/2-\sigma)}. \tag{5.10}$$

In the case (i), it follows from (5.7) that

$$(\|\tilde{u}^k\|^2+\beta\|\tilde{p}^k\|^2)_t+d_0\tau(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2)+\nu|\tilde{u}^k|_1^2+\nu\tau(\sigma+\alpha)(|\tilde{u}^k|_1^2)_t$$
$$\leqslant\|\tilde{f}^k\|^2+M(\|\tilde{u}^k\|^2+\beta\|\tilde{p}^k\|^2)+B(\tilde{u}^k)|\tilde{u}^k|_1^2. \qquad (5.11)$$

In the case (ii), we get from (3.46)

$$\tau(2\alpha-1-\varepsilon)(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2)+\nu\tau^2(2\alpha\sigma-\sigma-\alpha-2\varepsilon)|\tilde{u}_t^k|_1^2\geqslant d_0\tau(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2).$$

Thus (5.11) holds also. Let

$$E^n=\|\tilde{u}^n\|^2+\beta\|\tilde{p}^n\|^2+\tau\sum_{k=0}^{n-1}[\nu\|\tilde{u}^k\|_1^2+d_0\tau(\|\tilde{u}_t^k\|^2+\beta\|\tilde{p}_t^k\|^2)],$$

$$\rho^n=C\|\tilde{u}^0\|^2+\beta\|\tilde{p}^0\|^2+\tau\sum_{k=0}^{n-1}\|\tilde{f}^k\|^2.$$

Then we have from (5.11)

$$E^n\leqslant\rho^n+\tau\sum_{k=0}^{n-1}\{ME^k+B(\tilde{u}^k)|\tilde{u}^k|_1^2\}. \qquad (5.12)$$

Finally, we employ the lemma 4.16 of [2] to obtain the following result.

Theorem 1. Assume that (5.8), and (5.9) or (5.10) hold, and that

$$\rho^{[T/\tau]}e^{MT}\leqslant\frac{\nu\varepsilon(1-2\varepsilon)}{C_4(\alpha-\theta)^2\tau N}.$$

Then for all $n\tau\leqslant T$, we have $E^n\leqslant\rho^n e^{Mn\tau}$.

We next consider the convergence of scheme (2.8). Let $(u_*^k, p_*^k)=P_\delta(U^k, P^k)$ be defined in (4.1)—(4.2). We have from (2.1) and (2.8)

$$\begin{cases} L_1(u_*^k, p_*^k, v)=(RP_C\Pi^{m+1}f^k, v)+g_1^k(v), & \forall v\in V_\delta, \\ L_2(u_*^k, p_*^k, w)=(g_2^k, w), & \forall w\in W_\delta, \end{cases} \qquad (5.13)$$

where

$$g_1^k(v)=(u_{*t}^k-\partial_t U^k, v)+J_\delta(Ru_*^k, Ru_*^k, Rv)-J(U^k, U^k, v)+\theta\tau J_\delta(Ru_{*t}^k, Ru_*^k, Rv)$$
$$+\tau\sigma a(u_{*t}^k, v)-\tau\alpha b(v, p_{*t}^k)+(f^k-RP_C\Pi^{m+1}f^k, v),$$

$$g_2^k=\beta P_t^k.$$

Letting $\underset{\sim}{y}^k=u_*^k-u_\delta^k$, $\underset{\sim}{p}^k=p_*^k-p_\delta^k$ and substracting (2.8) from (5.13), we get

$$\begin{cases} \tilde{\tilde{L}}_1(\underset{\sim}{y}^k, u_*^k, \underset{\sim}{p}^k, v)=g_1^k(v), & \forall v\in V_\delta, \\ L_2(\underset{\sim}{y}^k, \underset{\sim}{p}^k, w)=(g_2^k, w), & \forall w\in W_\delta. \end{cases} \qquad (5.14)$$

Now we estimate $\tau\sum_{k=0}^{n-1}\|g_1^k\|_{-1}^2$, where

$$|g_1^k(v)|\leqslant\|g_1^k\|_{-1}\|v\|_1, \quad \forall v\in V_\delta.$$

By (4.6), we have

$$\tau\sum_{k=0}^{n-1}\|u_{*t}^k-U_t^k\|^2\leqslant C\tau(h^{2\bar{r}}+N^{-2s})\sum_{k=0}^{n-1}(\|U_t^k\|_{\bar{r},s}^2+\|P_t^k\|_{\bar{r}-1,s-1}^2)$$
$$\leqslant C(h^{2\bar{r}}+N^{-2s})(\|\partial_t U\|_{L^2(0,T;[H^{\bar{r},s}(\Omega)]^2)}^2+\|\partial_t P\|_{L^2(0,T;H^{\bar{r}-1,s-1}(\Omega))}).$$

According to

$$U_t^k-\partial_t U^k=\frac{1}{\tau}\int_{k\tau}^{k\tau+\tau}(k\tau+\tau-t)\partial_t^2 U(t)dt,$$

we have

$$\tau\sum_{k=0}^{n-1}\|U_t^k-\partial_t U^k\|_{-1}^2\leqslant C\tau^2\|\partial_t U\|_{L^2(0,T;[H^{-1}(\Omega)]^2)}^2.$$

Let $r\geqslant 3/2$, $s>3/2$ and $2/r+1/s<2$. It follows from (4.12), (3.61) and (4.9) that

$$|J_\delta(Ru_*^k, Ru_*^k, Rv)-J(Ru_*^k, Ru_*^k, v)|$$
$$\leqslant Ch^{\bar r}(\|U^k\|_{A^{\bar r,s}}+\|P^k\|_{\bar r-1,s-1})^2\|v\|_1+CN^{-s}(\|U^k\|_{A^{\bar r,s}}+\|P^k\|_{\bar r-1,s-1})^2\|v\|_1.$$

By (3.60), (4.9) and (4.6) we get

$$|J(Ru_*^k, Ru_*^k, v)-J(U^k, U^k, v)|\leqslant C(\|U^k\|_{\bar r,s}+\|u_*^k\|_{\bar r,s})\|Ru_*^k-U^k\|\|v\|_1$$
$$\leqslant C(h^{\bar r}+N^{-s})(\|U^k\|_{A^{\bar r,s}}+\|P^k\|_{\bar r-1,s-1})^2\|v\|_1.$$

Furthermore, by (3.58), we get

$$|\theta\tau J_\delta(Ru_{*t}^k, Ru_*^k, Rv)|\leqslant C\tau\|u_*\|_{\bar r,s}\|u_{*t}^k\|\|v\|_1.$$

It follows from (4.9) that

$$C\tau^3\sum_{k=0}^{n-1}\|u_*\|_{\bar r,s}^2\|u_{*t}^k\|^2$$
$$\leqslant C\tau^2(\|U\|_{\bar r,s}+\|P\|_{\bar r-1,s-1})^2(\|\partial_t U\|_{L^2(0,T;[H^1(\Omega)]^3)}+\|\partial_t P\|_{L^2(0,T;L^2(\Omega))})^2.$$

We can estimate $|\tau\sigma a(u_{*t}^k, v)|$ and $|\tau\alpha b(v, p_{*t}^k)|$ in the same way. Using (3.25) and (3.29) we have also

$$\|RP_c\Pi^{m+1}f^k-f^k\|\leqslant C(h^{\bar r}+N^{-s})\|f\|_{\bar r,s}.$$

Let b_j $(j=1-6)$ denote the constants depending on the norms of U, P and f appeared above. Then we have

$$\tau\sum_{k=0}^{n-1}\|g_1^k\|_{-1}^2\leqslant b_1(\tau^2+h^{2\bar r}+N^{-2s}),$$
$$\tau\sum_{k=0}^{n-1}\|g_2^k\|^2\leqslant b_2\beta^2.$$

On the other hand, (3.29), (4.6), and (4.9) lead to

$$\|\tilde u^0\|\leqslant C(h^{\bar r}+N^{-s})(\|U_0\|_{\bar r,s}+\|P^0\|_{\bar r-1,s-1}),$$
$$\|\tilde p^0\|\leqslant C(1+\|U_0\|_1+\|P^0\|).$$

Finally, applying Theorem 1 to (5.14), we obtain the following result from the above estimates and (4.6).

Theorem 2. Assume that (5.8), and (5.9) or (5.10) hold, $r\geqslant 3/2$, $s>3/2$ and that

$$U\in C(0, T; [A^{r,s}]^3)\cap H^1(0,T; [H_p^{r,s}(\Omega)]^3)\cap H^2(0, T; [H^{-1}(\Omega)]^3),$$
$$P\in C(0, T; H^{r-1,s-1}(\Omega))\cap H^1(0, T; L^2(\Omega)),\quad f\in C(0, T; [H_p^{r,s}(\Omega)]^3).$$

Then there exist constants b_3-b_6 such that when

$$b_3e^{b_4T}(\beta+\tau^2+h^{2\bar r}+N^{-2s})\leqslant\frac{\nu\varepsilon(1-2\varepsilon)h^2}{C_4(\alpha-\theta)^2\tau N^3},$$

we have for all $n\tau\leqslant T$,

$$\|u_\delta^n-U^n\|^2\leqslant b_5e^{b_6n\tau}(\beta+\tau^2+h^{2\bar r}+N^{-2s}).\quad(5.15)$$

§ 6. Discussion

The parameter γ in the control operator $R(\gamma)$ must be chosen suitably. In the analysis of errors, we used Lemma 9 with $\mu=-1$. So we know that the convergence order would be lowered if $\gamma<s+1$. But if γ is too large, then the constant C in the bound of (3.26) is large (see [20]). Therefore, the precision of calculation may be cut down. If $\beta+\tau^2+h^{2r}=O(N^{-2d})$, then by (5.15) the convergence order is $O(N^{-\min(d,s)})$. Hence we should choose $\gamma=\min(d, s)+1$. Moreover, because the smoothness properties of $U^{(l)}$ may be different from each other, it is more reasonable to define

$$RU=(R(\gamma_1)U^{(1)}, R(\gamma_2)U^{(2)}, R(\gamma_3)U^{(3)}).$$

In that case, the conclusions of Theorem 1 and Theorem 2 still hold provided that γ_l are suitably large.

The numerical results [20-21] show the advanteges of the combined scheme. The method used in this paper is also applicable to other nonlinear problems.

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