Chin. Ann. of Math.
13B:3(1992), 368—373.

DIGRAPH CATEGORIES AND LINE DIGRAPH FUNCTORS

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Abstract

The authors define line digraph functors on digraph category which are full and faithful and, as a consequence of the result, determine all homomorphisms in De Bruijn-Good graph category and automorphisms of Kautz digraph category. Moreover the authors consider a type of arc-full morphisms of digraph category such that $F^n(f)$ is arc-full for each functor F^n, and succeeding paper [9] study the strong homomorphisms of de Bruijn-Good digraph.

§ 1. Introduction

Line graph of a graph was introduced by J. Krausz[2] in 1943. H. Whitney[1] proved the most important theorem about line graphs. From then on, Harary and Norman defined line digraphs for digraphs. Works on this topic can be found in the references of [3]. Now we regard line graph as a functor on digraph category and can prove that the functor is full and faithful. As special cases of the result, we discuss homomorphisms of de Bruijn-Good graphs (Lempel[4] used this kind of homomorphisms in the design of feedback shift registers) and automorphisms of Kautz digraphs[5]. We obtain all homomorphisms of de Bruijn-Good graphs and the number of them. On the other hand, the strong homomorphisms of de Bruijn-Good graphs was studied in [9], here we give a lower bound of the numbers of all strong homomorphisms of D_k^2 onto D_k^1. When $k=2$, 3 the lower bound is sharp, and the corresponding strong homomorphisms can be determined.

§ 2. Category of Digraph, Functor

A digraph D is defined to be a pair $(V(D), A(D))$, where $V(D)$ is a non-empty finite (or infinite) set of elements called vertices and A(D) is a finite (or

Manuscript received August 15, 1989. Revised August 20, 1991.
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infinite) family of ordered pairs of elements of $V(D)$ called arcs. $V(D)$ and $A(D)$ are called the vertex-set and arc-family of D. Arc(a, a) for some $a \in V(D)$ is called a self-loop. D is said to be strongly connected iff for any two vertices a and b of D there are two dipaths such that one of them is from a to b and the other is from b to a.

The line digraph $L(D)$ of a digraph D has as its vertex-set the family of arcs of D; for $e, e' \in A(D)$, (e, e') is an arc of $L(D)$ iff there are vertices a_1, a_2, a_3 in D with $e=(a_1, a_2)$ and $e'=(a_2, a_3)$. We denote the arc (e, e') of $L(D)$ by a triple (a_1, a_2, a_3). It is clear that $L(D)$ has no multi-arc and that $L(D)$ has a self-loop at vertex α iff α is a self-loop of D.

For any non-negtive integer k we can define

$$L^{k+1}(D)=L(L^k(D)),$$

where $L^0(D)=D$ and $L^1(D)=L(D)$. It is not difficult to show that $V(L^k(D))=\{(a_1, a_2, \cdots, a_{k+1}) \mid a_i \in V(D) \text{ and } (a_i, a_{i+1}) \in A(D),\ i=1, 2, \cdots, k\}$ and that there is an arc from vertex u of $L^k(D)$ to verter v iff u and v have the following forms

$$u=(a_1, a_2, \cdots, a_{k+1}) \to v=(a_2, a_3, \cdots, a_{k+1}, a_{k+2}).$$

We denote the arc $((a_1, a_2, \cdots, a_{k+1}), (a_2, \cdots, a_{k+1}, a_{k+2}))$ by a $(k+2)$-array $(a_1, a_2, \cdots, a_{k+1}, a_{k+2})$. In other words, a vertex of $L^k(D)$ is equivalent to a diwalk of D with length k and an arc of $L^k(D)$ is equivalent to a diwalk of D with length $k+1$.

The digraph category has, as its ob $\mathscr{D}$ the class of all strongly connected digraphs; for any $D_1, D_2 \in \text{ob}\mathscr{D}$, $\text{Hom}_{\mathscr{D}}(D_1, D_2)$ is defined to be the set of all graph homomorphisms of D_1 into D_2, i. e., mappings f from $V(D_1)$ to $V(D_2)$ such that for any $(a_1, a_2) \in A(D_1)$, we have $(f(a_1), f(a_2)) \in A(D_2)$.

The line digraph functor F^n on digraph category is given as follows.

(i) For any $D \in \text{ob}\mathscr{D}$, $F^n(D)=L^n(D)$. $L^n(D)$ is again strongly connected since D is strongly connected[6].

(ii) For any $D_1, D_2 \in \text{ob}\mathscr{D}$ and $f \in \text{Hom}_{\mathscr{D}}(D_1, D_2)$, $F^n(f)$ is defined as follows

$$\forall (a_1, a_2, \cdots, a_{n+1}) \in V(F^n(D_1)),\ F^n(f)(a_1, a_2, \cdots, a_{n+1})$$
$$=(f(a_1), f(a_2), \cdots, f(a_{n+1})) \in V(F^n(D_2)).$$

Obviously, $F^n(f) \in \text{Hom}_{\mathscr{D}}(F^n(D_1), F^n(D_2))$. In fact, if $u, v \in V(F^n(D_1))$ and $(u, v) \in A(F^n(D_1))$, then u, v have the following forms

$$u=(a_1, a_2, \cdots, a_{n+1}),\ v=(a_2, \cdots, a_{n+1}, a_{n+2}).$$

Thus $F^n(f)(u)=(f(a_1), f(a_2), \cdots, f(a_{n+1}))$ and

$$F^n(f)(v)=(f(a_2), f(a_3), \cdots, f(a_{n+1}), f(a_{n+2})).$$

This implies $(F^n(f)(u), F^n(f)(v)) \in A(F^n(D_2))$ because $f \in \text{Hom}_{\mathscr{D}}(D_1, D_2)$.

The following two facts are immediate.

C1. $F^n(gf)=F^n(g)F^n(f)$ for $f \in \text{Hom}_{\mathscr{D}}(D_1, D_2)$ and $g \in \text{Hom}_{\mathscr{D}}(D_2, D_3)$, where $D_1, D_2, D_3 \in \text{ob}\mathscr{D}$.

C2. $F^n(I_D)=I_{F^n(D)}$ for the identity $I_D \in \text{Hom}_{\mathscr{D}}(D, D)$.

Hence F^n is indeed a functor on the digraph category[7].

Note that if we use $\mathscr{D}'$, the category containing all digraphs, instead of $\mathscr{D}$, then all the above facts are true. But, we have an interesting result for $\mathscr{D}$.

Theorem 1. *The line digraph functor F^n on digraph category $\mathscr{D}$ is full and faithful.*

The faithfulness is obvious. For the fullness, we only need to prove that for any $g^n \in \mathrm{Hom}_{\mathscr{D}}(F^n(D_1), F^n(D_2))$ there exists an $f \in \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$ such that $g^n = F^n(f)$. We notice that for any $(a_1, a_2, \cdots, a_n) \in V(F^{n-1}(D_1))$, there are $a_0, a_{n+1} \in V(D_1)$ such that $((a_0, a_1, \cdots, a_n), (a_1, a_2, \cdots, a_n, a_{n+1})) \in A(F^n(D_1))$ and $(g^n(a_0, a_1, \cdots, a_n), g^n(a_1, \cdots, a_n, a_{n+1})) \in A(F^n(D_2))$. Thus we can write

$$g^n(a_0, a_1, \cdots, a_n) = (b_0, b_1, \cdots, b_n),$$
$$g^n(a_1, a_2, \cdots, a_{n+1}) = (b_1, b_2, \cdots, b_n, b_{n+1})$$

and define $g^{n-1}(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$. In this way we can accomplish our proof by induction on n. We ommit the details.

§ 3. Application to de Bruijn-Good Graph Category

Lempel first investigated homomorphisms of de Bruijn-Good graphs and used them to design feedback shift registers. Wan Zhexian and Liu Mulan[8] determined all $2-1$ homomorphisms of de Bruijn-Good graph D_2^n to D_2^{n-1}. Zhang Fuji and Lin Guoning discussed a special kind of homomorphisms, i. e., strong homormorphisms of de Bruijn-Good graph D_k^n into D_k^{n-1}[9] (paper [10] corrected a result of [9]). Now, we can determine all homomorphisms of de Bruijn-Good graph D_M^{n+j} to D_N^n and give the number of them. These improve the main result of [11] and completely solve the problem of homomorphisms in de Bruijn-Good graph category.

Let $n \geqslant 1$ be an integer and M a set. The de Bruijn-Good graph D_M^n is defined as follows

$$V(D_M^n) = \{(a_1, a_2, \cdots, a_n) : a_i \in M,\ i = 1, 2, \cdots, n\};$$

for any $u, v \in V(D_M^n)$, $(u, v) \in A(D_M^n)$ iff u, v have the following forms

$$u = (a_1, a_2, \cdots, a_n),\ v = (a_2, \cdots, a_n, a_{n+1}),$$

where $a_i \in M (i = 1, 2, \cdots, n+1)$. We denote such an arc (u, v) by an $(n+1)$-array $(a_1, a_2, \cdots, a_n, a_{n+1})$.

If M is the ring of residues modulo k, D_M^n is denoted by D_k^n. In paper [9], we proved that the line digraph of D_k^{n-1} is D_k^n. By the same reason, we can prove that the line digraph of D_M^{n-1} is D_M^n. Hence, $D_M^n = L^{n-1}(L_M^1)$.

Theorem 2. *Let M, N be two sets, j a non-negative integer. Then* $|\mathrm{Hom}_{\mathscr{D}}(D_M^{j+1}, D_N^1)| = |N|^{|M|^{j+1}}$

Proof By Theorem 1, we take $D_1 = D_M^{j+1}$ and $D_2 = D_N^1$. Then there is a bijection between $\mathrm{Hom}_{\mathscr{D}}(F^{n-1}(D_1), F^{n-1}(D_2))$ and $\mathrm{Hon}_{\mathscr{D}}(D_1, D_2)$. Because $F^{n-1}(D_1) = L^{n-1}(D_N^{j+1}) = L^{n-1+j}(D_M^1) = D_M^{n+j}$ and $F^{n-1}(D_2) = L^{n-1}(D_N^1) = D_N^n$, we have a bijection between $\mathrm{Hom}_{\mathscr{D}}(D_M^{n+j}, D_N^n)$ and $\mathrm{Hom}_{\mathscr{D}}(D_M^{j+1} D_N^1)$. Since D_N^1 is a complete digraph with N the vertex-set, every mapping of $V(D_M^{j+1})$ into N is a homomorphism of D_M^{j+1} into D_N^1. Thus, we have $|\mathrm{Hom}_{\mathscr{D}}(D_M^{n+j}, D_N^n| = |\mathrm{Hom}_{\mathscr{D}}(D_M^{j+1}, D_N^1)|$ = the number of all mappings of $V(D_M^{j+1})$ into N. Because $|V(D_M^{j+1})| = |M|^{j+1}$, the number of all mappings of $V(D_M^{j+1})$ into N is $|N|^{|M|^{j+1}}$. Our proof is completed.

Remark 2. Because $V(D_M^{j+1}) = M^{j+1}$, F^{n-1} is the bijection between the set of all mappings of M^{j+1} into N and the set $\mathrm{Hom}_{\mathscr{D}}(D_M^{n+j}, D_N^n)$. Moreover, F^{n-1} maps f into $F^{n-1}(f): (a_1, a_2, \cdots, a_{n+j}) \to (f(a_1, a_2, \cdots, a_{j+1}), f(a_2, \cdots, a_{j+2}), \cdots, f(a_n, \cdots, a_{n+j}))$. When $j=0$, we can obtain all the results of [11].

Remark 3. If we take $M=N$, then

$$|\mathrm{Hom}_{\mathscr{D}}(D_M^{n+j}, D_M^n| = |M|^{|M|^{j+1}}.$$

By Theorem 1 we can determine all homomorphisms of D_M^{n+j} into D_N^n. For the case $j=1$ and $|M|=2$, paper [4] used this result to build maximal cycles of D_2^{n+1} from maximal cycles of D_2^n. Such a method might be generalized to the case $|M|>2$ by the natural way.

§ 4. Application to Kautz Digraph Category

Let K_M be the complete digraph without multi-arcs and self-loops, where M is the vertex-set of K_M, Kautz digraph K_M^n is defined as $L^{n-1}(K_M)$. Obviously, $V(K_M^n) = \{(a_1, a_2, \cdots, a_n) | a_i \in M$ and $a_i \neq a_{i+1}, i=1, 2, \cdots, n\}$ and therefore $|V(K_M^n)| = |M|(|M|-1)^{n-1}$.

Note that $K_M^1 = K_M$ is a complete digraph. The following result is immediate.

Theorem 3. *Let M be a finite set. Then we have* $|\mathrm{Aut}(K_M^n)| = |M|!$.

§ 5. A Lower Bound of the Number of All Strong Homomorphisms of D_k^2 onto D_k^1

In this section we consider a type of morphisms of digraph category, the set $\mathscr{S}$ of all graph homomorphisms of D_1 onto D_2 which is arc-full and for each functor F^n, $F^n(f)$ is also arc-full, namely, for any $(b_1, b_2, \cdots, b_{n+2}) \in A(F^n(D_2))$, there exists $(a_1, a_2, \cdots, a_{n+2}) \in A(F^n(D))$ such that $F^n(f)(a_1, a_2, \cdots, a_{n+2}) = (b_1, b_2, \cdots, b_{n+2})$. If above conditions are satisfied, we call f a strong homomorphism. Now as the proof of Theorem 1 we can prove the following

Theorem 4. *Let $g^n \in \mathrm{Hom}_{\mathscr{D}}(F^n(D_1), F^n(D_2))$ and g^n is arc-full, then there exists an arc-full homomorphism f of D_1 onto D_2 such that $g^n = F^n(f)$.*

From the theorem above we know that to determine the strong homomorphism we need only to find f, the arc-full homomorphism of D_1 onto D_2, such that for any n $F^n(f)$ is also arc-full. It is clear that to exhaust all strong homomorphisms is a difficult problem. In the following we will deal with a special case. Now we attempt to consider the set S of all strong homomorphisms of D_k^2 onto D_k^1. A lower bound of $|S|$ is obtained here.

Theorem 5. *For de Bruijn-Good graph category we have*

$$|S|\geq 2(k!)^k-k!(k-1)!\cdots 2!.$$

Proof For $f\in \mathrm{Hom}_{\mathscr{D}}(D_k^2, D_k^1)$ and $i\in V(D_k^1)=\{0, 1, \cdots, k-1\}$ we define

$$a(f^{-1}(i))=\{b\mid (a, b)\in f^{-1}(i)\},$$

$$b(f^{-1}(i))=\{a\mid (a, b)\in f^{-1}(i)\}.$$

Now we will show that if (1) $a(f^{-1}(i))=\{0, 1, \cdots, k-1\}$ $(i=0, 1, \cdots, k-1)$ or (2) $b(f^{-1}(i))=\{0, 1, \cdots, k-1\}$ $(i=0, 1, \cdots, k-1)$, then $f\in S$. At first, we mention that $\forall (a_1, a_2, \cdots, a_{n+1})\in V(F^{n-1}(D_k^2))$, $F^n(f)(a_1, a_2, \cdots, a_{n+1})=(f(a_1, a_2), f(a_2, a_3), \cdots, f(a_n, a_{n+1}))\in V(F^{n-1}(D_k^1))$. If condition (1) is fulfiled, then $\forall (a_0', a_1', \cdots, a_n')\in A(F^{n-1}(D_k^1))$ $(n\geq 1)$ we can find $(a_n, a_{n+1})\in f^{-1}(a_n')$ and $(a_{i-1}, a_i)\in f^{-1}(a_{i-1}')$ $(i=1, 2, \cdots, n)$ which means that $F^{n-1}(f)(a_0, a_1, \cdots, a_{n+1})=(f(a_0, a_1), f(a_1, a_2), \cdots, f(a_n, a_{n+1}))=(a_0', a_1', \cdots, a_n')$. Hence $F^{n-1}(f)$ is arc-full and $f\in S$. A similar reason shows that if condition (2) is fulfiled, the conclusion is also true.

Now we consider the following array

$$\begin{array}{l}(0, 0), (1, 0), \cdots, (k-1, 0)\\ (0, 1), (1, 1), \cdots, (k-1, 1)\\ \cdots\cdots\\ (0, k-1), (1, k-1), \cdots, (k-1, k-1).\end{array}$$

When f fulfils condition (1), $\forall i\in\{0, 1, \cdots, k-1\}$, $f^{-1}(i)$ takes just one element in each row. Thus

$$|\{f\mid f \text{ fulfils condstion (1)}\}|=(k!)^k.$$

Similarly,

$$|\{f\mid f \text{ fulfils condition (2)}\}|=(k!)^k,$$

When f fulfils both condition (1) and condition (2), $\forall i\in\{0, 1, \cdots, k-1\}$, $f^{-1}(i)$ takes just one element in each row and column. Thus

$$|\{f\mid f \text{ fulfils conditions (1) and (2)}\}|=(k!)(k-1)!\cdots 2!.$$

By the principle of inclusion and exclusion we come to the required conclusion.

When $k=2$, $|S|=8-2=6$, all strong homomorphisms were exhausted in [9]. When $k=3$, our lower bound also gives the number of all strong homomorphisms of D_k^2 onto D_k^1. In fact we have proved the following

Theorem 6. *When $k=3$, the lower bound in Theorem 5 is sharp, namely,* $|S|=2(3!)^3-3!2!=420.$

By using the array in the proof of Theorem 5, we can exhaust 420 strong homomorphisms of D_3^2 onto D_3^1. Furthermore we can obtain 420 strong homomorphisms of D_3^n onto D_3^{n-1} by functor F^{n-2}.

We end this paper by proposing an open prablem: When $k=4$ whether or not the lower bound in Theorem 5 is the number of strong homomorphisms of D_k^n onto D_k^{n-1}.

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