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COMPARISON BETWEEN MAXIMAL AND SQUARE FUNCTIONS OF 1-FORMS AND ITS APPLICATIONS**

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Abstract.

A norm relation between nontangential square function and nontangential maximal function of 1-forms is obtained. As applications, the author gets L^p-boundedness of Littlewood-Paley-Stein g-function operator on 1-forms, and gives an analytic proof of L^p-boundedness of Riesz transform, where $1<p<\infty$.

§1. Introduction

Let M be a noncompact complete Riemannian manifold of dimension n, $\mathrm{Ric}(M)\geqslant 0$, Δ and Δ_1 the Laplace operators $(d\delta+\delta d)$ on functions and 1-forms respectively, $\{P_t\}_{t>0}$ and $\{\boldsymbol{P}_t\}_{t>0}$ the corresponding Poisson semi-groups. For C_C^∞ function f and 1-form ω on M, define its square function and maximal function as follows

$$\begin{cases} A_{P,\alpha}(\omega)(x)=\left(\int_0^\infty\int_{B_x(\alpha t)}|\nabla\boldsymbol{P}_t(\omega)(y)|^2V_x^{-1}(\alpha t)\,t\,dy\,dt\right)^{\frac{1}{2}} \\ N_{P,\alpha}(\omega)(x)=\sup\limits_{d(x,y)<\alpha t}|\boldsymbol{P}_t(\omega)(y)|, \end{cases} \quad (0<\alpha<\infty) \qquad (1)$$

$A_{P,\alpha}(f)$ and $N_{P,\alpha}(f)$ are similar, where $\nabla=(\nabla,\partial/\partial t)$, ∇ is gradient operator of M.

In [7], we developed the technique of [4] and proved (for product manifolds)

Theorem A. $\|A_{P,\alpha}(f)\|_p\leqslant C_{n,p,\alpha,\beta}\|N_{P,\beta}(f)\|_p$ *for* $0<\alpha,\ \beta,\ p<+\infty$.

In this paper, we prove

Theorem 1. $\|A_{P,\alpha}(\omega)\|_p\leqslant C_{n,p,\alpha,\beta}\|N_{P,\beta}(\omega)\|_p$ *for* $0<\alpha,\ \beta,\ p<+\infty$.

Applying thist heorem, we give an analytic proof of L^p-boundedness of Rieiz transform $\nabla(+\Delta)^{-1/2}(1<p<+\infty)$, which was originally obtained in [1, 2, 5] and proposed in [11].

In this paper, $C_{n,p,\cdots}$ denotes a positive number depending only on n, p, $\cdots$, it may be different when it appears in different places.

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§ 2. A Note on Carleson Measures

For a ball $B\subset M$, $\hat{B}=B\times[0, 2r_B]$ where r_B is radius of B. For a nonnegative Borel measure $d\mu$ on $M\times R^1_+$,

$$\|d\mu\|_{C.M.}\triangleq\sup\{\mu(\hat{B})/|B|: \text{for all ball } B\}.$$

Lemma 1. *For $0<\gamma, \beta<+\infty$, there hold*

1°. *$\forall C^\infty$ 1-form η on $M\times R^1_+$, there holds* $\left(\textit{where } \Delta_*=-\Delta+\left(\frac{\partial}{\partial t}\right)^2, \Delta=-\Delta_1+\left(\frac{\partial}{\partial t}\right)^2\right)$

$$\Delta_*|\eta|^2\geqslant 2|\nabla\eta|^2+2\langle\eta, \Delta_*\eta\rangle.$$

2°. $\exists\ \delta_0=\delta_0(\beta, n)\in(0, 1)$ *and* $K_n>0$, s. t.

a). *$\forall$ ball B, $P_t(\chi_B)\geqslant\delta_0$ on $\hat{B}$, $P_t(\chi_B)\leqslant\delta_0/2$ out of $(K_nB)^\wedge$.*

b). *$\forall$ closed set E, $\exists$ closed set $F\subset E$,* s. t.

$$|F^C|\leqslant C_{n,\gamma,\beta}|E^C|,$$
$$P_t(\chi_E)\geqslant 1-\delta_0/2 \text{ on } \bigcup_{y\in F}\Gamma_\gamma(y),$$
$$P_t(\chi_E)\leqslant 1-\delta_0 \text{ out of } \bigcup_{y\in E}\Gamma_\beta(y),$$

where $\Gamma_\alpha(y)\triangleq\{(x\in M\times R^1_+: d(x, y)<\alpha t\}$.

3°. $\forall\ c<d$, $\exists\ \varphi\in C_C^\infty(R^1)$, s. t. $\varphi(t)=0$ *for* $t\leqslant c$, $\varphi(t)=1$ *for* $t\geqslant d$, $|\varphi'(t)|+|\varphi''(t)|\leqslant C\cdot\varphi^{3/4}(t)$.

1° can be obtained from Weitzenbock formula[12] because Ric $(M\times R^1_+)\geqslant 0$; 2° and 3° are from [7, Lemma 2].

Now, for $\lambda>0$, set $E_\lambda=\{x: N_{P,\beta}(\omega)(x)\leqslant\lambda\}$. By Lemma 1, there exists $\varphi_1\in C_C^\infty(R^1)$ (which depends only on γ, β and n) and $F_\lambda\subset E_\lambda$ such that

$$\begin{cases}\varphi_1(v_\lambda)=\begin{cases}0 & \text{out of } \bigcup_{y\in E_\lambda}\Gamma_\beta(y),\\ 1 & \text{on } \bigcup_{y\in F_\lambda}\Gamma_\gamma(y),\end{cases}\\ 0\leqslant\varphi_1\leqslant 1,\ |\varphi_1'(t)|+|\varphi_1''(t)|\leqslant C\cdot\varphi_1^{3/4}(t),\end{cases}\tag{2}$$

where $v_\lambda\triangleq P_t(\chi_{E_\lambda})$. We have

Lemma 2. $\|d\mu_{\omega,\lambda}\|_{C.M.}\leqslant C_{n,\gamma,\beta}\cdot\lambda^2$ *where*

$$d\mu_{\omega,\lambda}\triangleq\Delta_*(|P_t(\omega)(y)|^2)\varphi_1(v_\lambda(y, t))t\,dy\,dt.$$

Proof For a fixed ball B, set $u=P_t(\chi_B)$. By Lemma 1, there exists $\varphi_2\in C_C^\infty(R^1)$ such that

$$\begin{cases}C_{\delta_0}u^2\geqslant\varphi_2(u)=\begin{cases}1 & \text{on } \hat{B},\\ 0 & \text{out of } (K_nB)^\wedge,\end{cases}\\ 0\leqslant\varphi_2\leqslant 1,\ |\varphi_2'(t)|+|\varphi_2''(t)|\leqslant C\cdot\varphi_2^{3/4}(t).\end{cases}\tag{3}$$

Set

$$A(B)=\iint_{M\times R_+^1}\varphi_2(u)\,d\mu_{\omega,\lambda}\quad(\geqslant\mu_{\omega,\lambda}(B))\tag{4}$$

and take $h\in C_C^\infty(R^1)$ such that $h(t)=1$ for $t\leqslant 1$ and $h(t)=0$ for $t\geqslant 2$. Then (if necessary, we can replace $A(B)$ by

$$A_s(B)=\iint_{M\times R_+^1}\Delta_*(|\boldsymbol{P}_{t+s}(\omega)(y)|^2)\varphi_1(v_\lambda(y,t+s))\varphi_2(u(y,t+s))\cdot t\,dt\,dy,$$

then, a limit procedure will give our desired results)

$$\begin{aligned}A(B)&=\lim_{R\to+\infty}\iint_{M\times R_+^1}h\Big(\frac{t}{R}\Big)\varphi_2(u)\,d\mu_{\lambda,\omega}\\&=\lim_{R\to+\infty}\iint_{M\times R_+^1}\Big\{\nabla(|\boldsymbol{P}_t(\omega)|^2)\cdot\nabla(\varphi_1(v_\lambda)\varphi_2(u))\\&\quad+\Big(\frac{\partial}{\partial t}\Big)^2(|\boldsymbol{P}_t(\omega)|^2\varphi_1(v_\lambda)\varphi_2(u))-|\boldsymbol{P}_t(\omega)|^2\Big(\frac{\partial}{\partial t^2}\Big)^2(\varphi_1(v_\lambda)\varphi_2(u))\\&\quad-2\frac{\partial}{\partial t}(|\boldsymbol{P}_t(\omega)|^2)\cdot\frac{\partial}{\partial t}(\varphi_1(v_\lambda)\varphi_2(u))\Big\}h\Big(\frac{t}{R}\Big)t\,dt\,dy.\end{aligned}$$

Now, $|\nabla|\boldsymbol{P}_t(\omega)|^2|=|\nabla\langle\boldsymbol{P}_t(\omega),\boldsymbol{P}_t(\omega)\rangle|\leqslant 2|\nabla\boldsymbol{P}_t(\omega)||\boldsymbol{P}_t(\omega)|$, so

$$\begin{aligned}A(B)\leqslant C_{n,\gamma,\beta}&\iint_{M\times R_+^1}\{|\nabla\boldsymbol{P}_t(\omega)||\boldsymbol{P}_t(\omega)||\nabla(\varphi_1(v_\lambda)\varphi_2(u))|\\&+|\boldsymbol{P}_t(\omega)|^2(|\nabla v_\lambda|^2u^2+|\nabla u|^2+(|\nabla v_\lambda||\nabla u|u))\}t\,dy\,dt\\&+\lim_{R\to+\infty}\iint_{M\times R_+^1}\Big(\frac{\partial}{\partial t}\Big)^2(|\boldsymbol{P}_t(\omega)|^2\varphi_1(v_\lambda)\varphi_2(u))\cdot h\Big(\frac{t}{R}\Big)\cdot t\,dy\,dt.\end{aligned}\tag{5}$$

By Lemma 1 and the proof of [7, Lemma 11], we have

$$|\nabla\boldsymbol{P}_t(\omega)|^2\leqslant\frac{1}{2}\Delta_*(|\boldsymbol{P}_t(\omega)|^2),$$

$$\iint_{M\times R_+^1}|\nabla v_\lambda|^2u^2t\,dy\,dt\leqslant C_n\cdot|B|\cdot\lambda^2,$$

$$\iint_{M\times R_+^1}|\nabla u|^2t\,dy\,dt\leqslant C_n\cdot|B|\cdot\lambda^2;$$

on the other hand, noting that $(h(t/R)t)''_{tt}=O(1/R)$, $(h(t/R)t)'_t=O(1)$, we have the last term of (5) (without $\lim\limits_{R\to+\infty}$)

$$\begin{aligned}&=-\iint_{M\times R_+^1}\frac{\partial}{\partial t}(|\boldsymbol{P}_t(\omega)|^2\varphi_1(v_\lambda)\varphi_2(u))\cdot\frac{\partial}{\partial t}\Big(h\Big(\frac{t}{R}\Big)t\Big)dt\,dy\\&=C(1)\int_M|\boldsymbol{P}_0(\omega)|^2\varphi_1(v_\lambda)\varphi_2(u)dy\\&\quad+\iint_{M\times R_+^1}|\boldsymbol{P}_t(\omega)|^2\varphi_1(v_\lambda)\varphi_2(u)\Big(\frac{\partial}{\partial t^2}\Big)^2\Big(h\Big(\frac{t}{R}\Big)t\Big)dt\,dy\\&=O(1)\cdot\lambda^2\cdot|B|+O(1)\cdot\int_M\sup_{t>R}|u(y,t)|^2dy,\end{aligned}$$

where $\int_M \sup_{t>R} |u(y,t)|^2 dy \xrightarrow{R\to\infty} 0$. Thus

$$A(B) \leqslant C_{n,\gamma,\beta}((A(B)\cdot\lambda^2|B|)^{1/2}+\lambda^2|B|),$$

so, $A(B)\leqslant C_{n,\gamma,\beta}\lambda^2|B|$, and $\mu_{\omega,\lambda}(B)\leqslant C_{n,\gamma,\beta}\lambda^2|B|$ (by (4)), i. e., $\|d\mu_{\omega,\lambda}\|_{C.M.}\leqslant C_{n,\gamma,\beta}\cdot\lambda^2$.

§ 3. A Note on BMO(M) Functions

For an $f\in L^1_{loc}(M)$

$$\|f\|_{BMO}\triangleq\sup\left\{|B|^{-1}\int_B |f(y)-(f)_B|dy\text{:for all ball } B\right\},$$

where $(f)_B\triangleq|B|^{-1}\int_B f(y)dy$. We have

Lemma 3. *If* $\|f\|_{BMO}=1$, $\lambda>1$ *and* $\delta>\varepsilon>0$, *then*

1°. $\forall$ *ball* B

$$|\{x\in B: |f(x)-(f)_B|>\lambda\}|\leqslant C_n\cdot e^{-\lambda/C_n}\cdot|B|.$$

2°. $|\{x\in M: |f(x)|>\lambda\delta\}|\leqslant C_{n,\varepsilon}e^{-\lambda/C_n}|\{x\in M: |f(x)|>\delta\}|$.

Proof 1° is a easy corollary of [9, Proposition 1]. To prove 2°, set $E=\{x\in M: |f(x)|>\delta\}$, and suppose $|E|<+\infty$. Fix $x_0\in M$ and $R>0$. Take a numerable dense subset of $E-E_0$, $\{x_j\}_j$, where E_0 is the set of nondensity points of E(i. e., $x\in E-E_0\Leftrightarrow x\in E$ and $\lim_{r\to 0} V_x^{-1}(r)|B_x(r)\cap E|\neq 1$). Without loss of generality, we assume $E_0=\varnothing$. Let

$$B_{i,j}=B_{x_j}(2^i),\ i=0,\ \pm1,\ \pm2,\ \cdots.$$

B^j is the maximal ball of $\{B_{i,j}\}_i$ which satisfies

$$|E\cap B^j|\geqslant|B^j|/2 \text{ and } |E\cap B^j/2|\geqslant|B^j/2|/2,$$

$$\mathscr{F}=\{B^j\}_j,$$

$$\mathscr{F}_R=\{B^j\in\mathscr{F}: \text{center of } B^j\in B_{x_0}(R)\}.$$

By density of x_j, B^j is well-defined; and, by the denseness of $\{x_j\}_j$ in E and the fact $|E|<+\infty$, we see that

$$i_R\triangleq\max\{i: \mathscr{F}_R\cap\{B_{i,j}\}_j \text{ is nonempty}\}<+\infty.$$

$\mathscr{F}_R$ is an open covering of $E\cap B_{x_0}(R)$.

Now, select a subfamily of $\mathscr{F}_R$, $\mathscr{F}^R\triangleq\bigcup_{i\leqslant i_R}\{B^i_l\}_l$, as follows. When $i=i_R$, $\{B^{i_R}_l\}_l$ is chosen to be a maximal subfamily of $\mathscr{F}_R\cap\{B_{i_R,j}\}_j$ (which is a finite family) such that any ball in which does not contain centers of other balls. If we have chosen $\{B^i_l\}_l$ $(i^*\leqslant i\leqslant i_R)$, then choose $\{B^{i^*-1}_l\}_l$ to be a maximal subfamily of $\mathscr{F}_R\cap\{B_{i^*-1,j}\}_j$ (which is also a finite family) such that any ball in which does not contain centers of others and its own center is not in $\bigcup_{i^*\leqslant i\leqslant i_R}(\bigcup_l B^i_l)$. By induction, we have got $\mathscr{F}^R$ which satisfies a°. $\sum_{B\in\mathscr{F}^R}|B|\leqslant C_n\cdot|E|$,

b°. $\bigcup_{B\in\mathscr{F}^R}2B\supset E\cap B_{x_0}(R)$,

c°. $|E^c\cap 2B|\geqslant|2B|/2$, $(\forall B\in\mathscr{F}^R)$.

Proof of a° By definition of $\mathscr{F}^R$, $\{B/2\}_{B\in\mathscr{F}^R}$ is disjoint mutually, and by definition of $\mathscr{F}$ (note that $\mathscr{F}^R\subset\mathscr{F}$), $|B/2|\leqslant 2|E\cap B/2|$ for any $B\in\mathscr{F}^R$, so

$$\sum_{B\in\mathscr{F}^R}|B|\leqslant 2^n\cdot\sum_{B\in\mathscr{F}^R}|B/2|\leqslant 2^{n+1}\cdot\sum_{B\in\mathscr{F}^R}|E\cap B/2|\leqslant 2^{n+1}|E|.$$

Proof of b° Bo maximality of $\{B_l^i\}_1$

$$\bigcup_{B\in\mathscr{F}^R\cap\{B_{i,j}\}_j}B\subset\bigcup_{i\leqslant s\leqslant i_B}(\bigcup_l 2B_l^s).$$

Thus

$$E\cap B_{x_0}(R)\subset\bigcup_{B\in\mathscr{F}^R}B=\bigcup_{-\infty<i\leqslant i_B}(\bigcup_{B\in\mathscr{F}^R\cap\{B_{i,j}\}_j}B)$$
$$\subset\bigcup_{i\leqslant i_B}\bigcup_{i\leqslant s\leqslant i_B}\bigcup_l 2B_l^s=\bigcup_{B\in\mathscr{F}^R}2B.$$

Proof of c° $\forall B\in=\mathscr{F}^R$, by definition of $\mathscr{F}$, $|2B\subset E|\leqslant\frac{1}{2}|2B|$, so, $|2B\cap E^c|\geqslant\frac{1}{2}|2B|$, Now by c°, wes get $(\forall B\in\mathscr{F}^R)$

$$|(f)_{2B}|\leqslant 2(|f|)_{2B}\cdot|E^c\cap 2B|/|2B|$$
$$\leqslant 2\left\{|2B|^{-1}\int_{2B\cap E^c}|f(x)|dx-|2B|^{-1}\int_{2B\cap E^c}|f(x)-(|f|)_{2B}|dx\right\}$$
$$\leqslant 2(\delta+\||f|\|_{BMO})\leqslant 2\delta+4.$$

Thus from a°, b° and 1°, we have $\left(\text{without loss of generality, we may assume }\lambda>2-\frac{4}{\delta}\right)$

$$|\{x\in B_{x_0}(R):\ |f(x)|>\lambda\delta\}|\leqslant\sum_{B\in\mathscr{F}^R}|\{x\in 2B:\ |f(x)|>\lambda\delta\}|$$
$$\leqslant\sum_{B\in\mathscr{F}^R}|\{x\in 2B:\ |f(x)-(f)_{2B}|>\lambda\delta-2\delta-4\}|$$
$$\leqslant C_n\cdot\sum_{B\in\mathscr{F}^R}e^{-\lambda/C_n}|2B|\leqslant C_n\cdot e^{-\lambda/C_n}\cdot|E|.$$

Finally, letting $R\to+\infty$, we get 2°.

§ 4. Proof of Theorm 1

Take $\psi\in C_c^\infty(R^1)$ such that

$$\psi(t)=\begin{cases}1, & t\leqslant 1\\ 0, & t\geqslant 2\end{cases}\quad\text{and } 0\leqslant\psi\leqslant 1.$$

Then, for $\alpha>0$, let

$$\Psi_t^\alpha(x,y)=\psi\left(\frac{d(x,y)}{\alpha t}\right)\Big/\int_M\psi\left(\frac{d(x,y)}{\alpha t}\right)dy. \tag{6}$$

We have

Lemma 4. *For* $f\in H^1(M)$, $0<\alpha<+\infty$, $\|N_\Psi(f)\|_1\leqslant C_{n,\psi,\alpha}\|f\|_{H^1}$, where

$$N_\Psi(f)(x) \triangleq \sup_{d(x,y)<t} \left| \int_M \Psi_t^\alpha(x, y) f(y) dy \right|.$$

Proof By [6, Theorem 1], it is enough to prove

$$\|N_\Psi(a)\|_1 \leqslant C_{n,\psi,\alpha}$$

for any H^1 -atom a. Suppose a is an H^1-atom, i. e., supp $a \subset B = B_{x_0}(r_0)$, $\|a\|_2 \leqslant |B|^{-1/2}$, $\int_M a(x)dx = 0$. For $z \overline{\in} 2B$ and $d(y, z) \leqslant t$, if $x \in B$, then (note that $|\nabla d(x, y)| = 1$) $\exists \xi_x \in \overline{xx_0}$, such that

$$|\Psi_t^\alpha(x, y) - \Psi_t^\alpha(x_0, y)| \leqslant \left|\psi'\left(\frac{d(\xi_x, y)}{\alpha t}\right)\right| \frac{d(x_0, x)}{\alpha t} \Big/ \int_M \psi\left(\frac{d(\xi_x, y)}{\alpha t}\right) dy$$

$$+ \psi\left(\frac{d(\xi_x, y)}{\alpha t}\right) \int_M \left|\psi'\left(\frac{d(\xi_x, y)}{\alpha t}\right)\right| \frac{dy}{\alpha t} d(x_0, x) \Big/ \left(\int_M \psi\left(\frac{d(\xi_x, y)}{\alpha t}\right) dy\right)^2$$

$$\leqslant C_{n,\psi} \cdot \frac{r_0}{\alpha t} (V_y(\alpha t) + V_{\xi_x}(\alpha t))^{-1} \chi_{d(\xi_x, y) \leqslant 2\alpha t} \triangleq C_{n,\psi} \cdot r_0 \cdot \Phi_t^\alpha(\xi_x, y),$$

because

$$\int_M \psi\left(\frac{d(\xi_x, y)}{\alpha t}\right) dy \geqslant \int_{B_{\xi_x}(\alpha t)} dy = V_{\xi_x}(\alpha t) \geqslant C_n \cdot V_y(\alpha t),$$

$$\int_M \left|\psi'\left(\frac{d(\xi_x, y)}{\alpha t}\right)\right| dy \leqslant \int_{B_{\xi_x}(\alpha t)} \|\psi'\|_\infty dy \leqslant C_{n,\psi} \cdot V_y(\alpha t).$$

Now, when $d(x_0, z)/3 \leqslant d(\xi_x, y) \leqslant 3d(x_0, z)$

$$\Phi_t^\alpha(\xi_x, y) \leqslant \frac{2}{d(\xi_x, y)} V_y^{-1}\left(\frac{d(\xi_x, y)}{2}\right) \leqslant C_n \cdot d^{-1}(x_0, z) V_{x_0}^{-1}(d(x_0, z));$$

when $d(\xi_x, y) \geqslant 3d(z, x_0)$

$$\Phi_t^\alpha(\xi_x, y) \leqslant \frac{2}{d(\xi_x, y)} V_{\xi_x}^{-1}\left(\frac{d(\xi_x, y)}{2}\right) \leqslant C_n \cdot d^{-1}(x_0, z) V_{\xi_x}^{-1}\left(\frac{d(z, x_0)}{2}\right)$$

$$\leqslant C_n \cdot d^{-1}(x_0, z) V_{x_0}^{-1}\left(\frac{d(z, x_0)}{2}\right);$$

when $d(\xi_x, y) \leqslant d(z, x_0)/3$, $d(x_0, z)/2 \leqslant d(\xi_x, z) \leqslant d(\xi_x, y) + d(y, z) \leqslant (1+\alpha)t$

$$\Phi_t^\alpha(\xi_x, y) \leqslant C_n \cdot \left(\frac{1+\alpha}{\alpha}\right)^{n+1} \cdot d^{-1}(x_0, z) V_{x_0}^{-1}(d(x_0, z)).$$

Thus, for $z \overline{\in} 2B$, $d(y,z) \leqslant t$ and $x \in B$

$$|\Psi_t^\alpha(x, y) - \Psi_t^\alpha(x_0, y)| \leqslant C_{n,\psi} \cdot \left(\frac{1+\alpha}{\alpha}\right)^{n+1} \cdot \frac{r_0}{d(x_0, z)} V_{x_0}^{-1}(d(x_0, z)).$$

So (note that $\|a\|_1 \leqslant 1$)

$$\sup_{d(y,z)<t} \left| \int_M \Psi_t^\alpha(y, x) a(x) dx \right| = \sup_{d(y,z)<t} \left| \int_M (\Psi_t(y, x) - \Psi_t(y, x_0)) a(x) dx \right|$$

$$\leqslant C_{n,\psi,\alpha} \cdot \frac{r_0}{d(x_0, z)} V_{x_0}^{-1}(d(x_0, z)),$$

$$\|N_\Psi(a)\|_{L^1((2B)^c)} \leqslant C_{n,\psi,\alpha} \cdot \int_{d(x_0,z) \geqslant 2r_0} \frac{r_0\, dz}{d(x_0, z) V_{x_0}(d(x_0, z))} \leqslant C_{n,\psi,\alpha}. \tag{7}$$

On the other hand, $N_\Psi(a) \leqslant C_{\psi,\alpha} M_0(a)$ (Hardy-Littlewood maximal function of a), so

$$\|N_\Psi(a)\|_{L^1(2B)} \leqslant |2B|^{\frac{1}{2}} \|N_\Psi(a)\|_2 \leqslant C_{\psi,n,\alpha} \|a\|_2 \cdot |B|^{\frac{1}{2}} \leqslant C_{n,\psi,\alpha}.$$

Combining this with (7), we get $\|N_\Psi(a)\|_1 \leqslant C_{n,\psi,\alpha}$. Lemma 4 is proved.

Now, let E_λ, F_λ and $d\mu_{\omega,\lambda}$ as in section 2. Set

$$A_{P,\psi,\alpha}(\omega)(x)=\Big(\iint_{M\times R_+^1}\Psi_t^\alpha(y,x)\Delta_*(|P_t(\omega)(y)|^2)t\,dt\,dy\Big)^{\frac{1}{2}},$$

$$f_{\psi,\alpha,\beta,\gamma,\lambda}(x)=\iint_{M\times R_+^1}\Psi_t^\alpha(y,x)d\mu_{\omega,\lambda}.$$

Then, by duality of $H^1(M)$ and $BMO(M)$[6], and Carleson inequality (see [7, Lemma 8])

$$\|f_{\psi,\alpha,\beta,\gamma,\lambda}\|_{BMO}=\sup_{\|g\|_{H^1}<1}\Big|\int_M f_{\psi,\alpha,\beta,\gamma,\lambda}(x)g(x)dx\Big|$$
$$=\sup_{\|g\|_{H^1}<1}\Big|\iint_{M\times R_+^1}\Big(\int_M\Psi_t^\alpha(y,x)g(x)dx\Big)d\mu_{\omega,\lambda}\Big|$$
$$\leqslant\sup_{\|g\|_{H^1}<1}\|N_\Psi(g)\|_1\|d\mu_{\omega,\lambda}\|_{CM}\leqslant C_{\psi,\alpha,\beta,\gamma,n}\cdot\lambda^2.$$

Thus, by Lemma 3, we get (for $\delta\geqslant 1$)

$$|\{x: f_{\psi,\alpha,\beta,\gamma,\lambda}(x)>\delta^2\lambda^2\}|\leqslant C_{n,\alpha,\beta,\gamma,\psi}e^{-\delta^2/Cn}|\{x: f_{\psi,\alpha,\beta,\gamma,\lambda}(x)>\lambda^2\}|.$$

Finally, if we take γ such that $\gamma\leqslant\alpha$, then

$$A^2_{P,\psi,\alpha}(\omega)(x)=f_{\psi,\alpha,\beta,\gamma}(x)\ \text{on}\ F_\lambda,$$
$$A^2_{P,\psi,\alpha}(\omega)(x)\geqslant f_{\psi,\alpha,\beta,\gamma}(x)\ \text{on}\ M.$$

so

$$|\{x: A_{P,\psi,\alpha}(\omega)(x)>\delta\lambda\}|$$
$$\leqslant|F_\lambda^c|+|\{x\in F_\lambda: A_{,\psi,\alpha}(\omega)(x)>\lambda\delta\}|$$
$$\leqslant C_{n,\alpha,\beta,\psi}\cdot|E_\lambda^c|+|\{x: f_{\psi,\alpha,\beta,\gamma,\lambda}(x)>\delta^2\lambda^2\}|)$$
$$\leqslant C_{n,\alpha,\beta,\psi}(|E_\lambda^c|+e^{-\delta^2/Cn}|\{x: f_{\psi,\alpha,\beta,\gamma,\lambda}(x)>\lambda^2\}|)$$
$$\leqslant C_{n,\alpha,\beta,\psi}(|E_\lambda^c|+e^{-\delta^2/Cn}|\{x: A^2_{P,\psi,\alpha}(\omega)(x)>\lambda^2\}|),$$

i.e.,

Lemma 5. *For* $0<\alpha,\ \beta<\infty,\ \delta>1$

$$\{x: A_{P,\psi,\alpha}(\omega)(x)>\delta\lambda\}|$$
$$\leqslant C_{n,\alpha,\beta,\psi}(|x: N_{P,\beta}(\omega)(x)>\lambda\}|+e^{-\delta^2/Cn}|\{x: A_{P,\psi,\alpha}(\omega)(x)>\lambda\}|).$$

From this lemma and the fact

$$\|g\|_p=p\int_0^\infty\lambda^{p-1}|\{x: |g(x)|>\lambda\}|d\lambda\quad(0<p<\infty),$$

we can easily get

$$\|A_{P,\psi,\alpha}(\omega)\|_p\leqslant C_{n,\alpha,\beta,p}\|N_{P,\beta}(\omega)\|_p\quad(0<p<\infty).$$

Finally, noting that $A_{P,\alpha}(\omega)\leqslant A_{P,\psi,\alpha}(\omega)$, we get Theorem 1.

§ 5. Applications

As an application of Theorem 1, we have

Theorem 2. *Riesz transform $\nabla(+\Delta)^{-1/2}$ is L^p-bounded for $1<p<+\infty$.*

Proof Let $p'=p/(p-1)$,

$$\|\nabla(+\Delta)^{-\frac{1}{2}}(f)\|_p=\sup_{\|\omega\|_{p'}<1}|\langle d(+\Delta)^{-\frac{1}{2}}(f),\ \omega\rangle_{L^2}| \quad (\omega \text{ is 1-form})$$

$$=\sup_{\|\omega\|_{p'}<1}\left|\int_0^\infty\left\langle\frac{\partial}{\partial t}\boldsymbol{P}_t(d(+\Delta)^{-\frac{1}{2}}(f)),\ \frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right\rangle_{L^2} t\,dt\right|$$

(by Polarization identity)

$$=\sup_{\|\omega_p\|'<1}\left|\int_0^\infty\left\langle dP_t(f),\ \frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right\rangle_{L^2} t\,dt\right|$$

$$\left(\text{because } \frac{\partial}{\partial t}\boldsymbol{P}_t(d(+\Delta)^{-1/2})=dP_t\right)$$

$$\leqslant\sup_{\|\omega\|_{p'}<1}\iint_{M\times\mathbf{R}_+^1}|dP_t(f))|\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right|t\,dt\,dx$$

$$=\sup_{\|\omega\|_{p'}<1}\iint_{M\times\mathbf{R}_+^1}\left(\int_{d(x,y)<t}|\nabla P_t(f)(y)|\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)(y)\right|V_y^{-1}(t)\,dy\right)t\,dt\,dx$$

$$\leqslant C_n\sup_{\|\omega\|_{p'}<1}\int_M A_{P,1}(\omega)(x)A_{P,1}(f)(x)\,dx$$

(by Hölder inequality and the fact that $V_y^{-1}(t)\leqslant C_nV_x^{-1}(t)$ for $d(x,y)\leqslant t$)

$$\leqslant C_n\|A_{P,1}(f)\|_p\cdot\sup_{\|\omega\|_{p'}<1}\|A_{P,1}(\omega)\|_{p'}\leqslant C_n\|f\|_p,$$

by Theorem A and Theorem 1. Theorem 2 is proved.

Now, let

$$g_P(f)(x)=\left(\int_0^\infty\left|\frac{\partial}{\partial t}P_t(f)(x)\right|^2 t\,dt\right)^{\frac{1}{2}},$$

$$g_P(\omega)(x)=\left(\int_0^\infty\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)(x)\right|^2 t\,dt\right)^{\frac{1}{2}}.$$

In [5], we proved (by CZO-method and estimates of kernel of g_P)

Theorem B. *g_P is weak-type (1,1) bounded and BMO bounded, so it is L^p-bounded for $1<p<\infty$.*

But, for g_P, any gradient estimates of its kernel will involve more curvature properties of the manifold, so, CZO-method fails now. On the other hand, we have no "integral" of forms (such as, 1-forms), the method developed by E. M. Stein[10] also fails.

Now, as Lemma 1 (1°), we have

$$\Delta_*\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right|^2\geqslant 2\left|\nabla\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right|^2\geqslant 0$$

i.e., $\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)\right|^2$ is subharmonic on $M\times R_+^1$, so (see the proof of [7, Lemma 3])

$$\left|\frac{\partial}{\partial t}\boldsymbol{P}_t(\omega)(x)\right|^2\leqslant C_{n,a}t^{-1}V_x^{-1}\left(\frac{at}{3}\right)\int_{B_x(\frac{at}{3})\times[\frac{t}{2},\frac{3}{2}t]}\left|\frac{\partial}{\partial s}\boldsymbol{P}_s(\omega)(y)\right|^2 dy\,ds,$$

And

$$g_P(\omega)(x) \leqslant C_{n,\alpha}\int_0^\infty \left(V_x^{-1}\left(\frac{\alpha t}{3}\right)\int_{B_x(\frac{\alpha t}{3})\times[\frac{t}{2},\frac{3}{2}t]} \left|\frac{\partial}{\partial s} P_s(\omega)(y)\right|^2 dy\, ds\right)dt$$

$$\leqslant C_{n,\alpha}\int_0^\infty\left(\int_{B_x(\frac{\alpha t}{3})\times[\frac{t}{2},\frac{3}{2}t]} \left|\frac{\partial}{\partial s} P_s(\omega)(y)\right|^2 V_x^{-1}(s)\, ds\, dy\right)dt$$

$$\leqslant C_{n,\alpha}\int_0^\infty \int_{B_x(\frac{\alpha s}{2})} \left|\frac{\partial}{\partial s} P_s(\omega)\right|^2 V_x^{-1}\left(\frac{\alpha s}{2}\right) s\, ds\, dy$$

$$\leqslant C_{n,\alpha} A_{P,x,\frac{\alpha}{2}}(\omega)(x) \leqslant C_{n,\alpha} A_{P,\alpha}(\omega)(x).$$

By Lemma 5 (α by $\alpha/2$), we get

Theorem 3. *For C_c^∞ 1-form ω, there holds*

$$|\{x: g_P(\omega)(x) > \lambda\}| \leqslant C_{n,\alpha}|\{x: N_{P,\alpha}(\omega)(x) > \lambda\}| + C_{n,\alpha}|\{x: A_{P,\alpha}(\omega)(x) > \lambda\}|;$$

especially, g_P is L^p-bounded for $1<p<\infty$ and weak-type $(1, 1)$ bounded and $\|g_P(\omega)\|_p \leqslant C_{n,p,\alpha}\|N_{P,\alpha}(\omega)\|_p$ for $0<p,\ \alpha<+\infty$.

The L^p-boundedness of g_P for $1<p\leqslant 2$ has been obtained by D. Bakry[2] (probabilistic method).

As corollories of Theorem 2, we have

Corollary 1. $C_{c,0}^\infty(M) \triangleq \left\{f\in C_c^\infty(M): \int_M f(x)dx=0\right\}$ *is dense in $L^p(M)$ $(1<p<+\infty)$ when M is noncompact.*

It is obviously not true for $p=1$.

Corollary 2. *For any $\omega\in L^p_{(1,0)}(M) \triangleq \{$all L^p-integrable 1-forms$\}$, there exist $f_j\in C_c^\infty(M)$, $j=1, 2, \cdots$, such that $\omega = L^p\text{-}\lim\limits_{j\to+\infty} d(f_j)$, when M is noncompact.*

We do not give details of them here. We also notice that Theorem 1 holds for positively curved product manifolds.

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