Chin. Ann. of Math.
13B: (1992). 385—395.

REGULARITY OF HARMONIC MAPS INTO POSITIVELY CURVED MANIFOLDS**

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Abstract

Let M be a compact Riemannian manifold of dimension m, N a complete simply connected δ-pinched Riemannian manifold of dimension n. There exists a constant $d(n)$. It is proved that if $m \leqslant d(n)$, then every minimizing map from M into N is smooth in the interior of M. If $m = d(n)+1$, such a map has at most discrete singular set and in general the Hausdorff dimension of the singular set is at most $m - d(n) - 1$.

§ 1. Introduction

One of the methods to have existence for harmonic maps is the direct method of the calculus of variations. The main ingredient of this method is the regularity of generalized harmonic maps. Schoen-Uhlenbeck developed a regularity theory for minimizing harmonic maps in their remarkable paper [9]. They showed that such maps are always regular in the interior outside a closed set of Hausdorff codimension at least three. Precisely, they proved the following:

Let $\phi \in \mathscr{L}_1^2(M, N)$ be an energy minimizing map. Suppose N has the property that there exists an integer $l \leqslant 3$ such that any minimizing tangent map from $\mathbf{R}^j$ into N is constant for $3 \leqslant j < l$. Then the Hausdorff dimension of the singular set of the map ϕ is at most $m - l - 1$. If $m = l + 1$, then the singular set is discrete. If $m < l + 1$, then ϕ is smooth.

Using the above criterion Schoen-Uhlenbeck analyzed singular set when the target manifold is Euclidean sphere in [10]. The key point is to derive stability inequality. In the case when the target manifold is sphere, the variational cross-sections are conformal vector fields along the image of the map.

When the Euclidean sphere is embedded canonically in the Euclidean space, the conformal vector fields in the sphere can be expressed as the tangential components of the parallel vector fields in ambient space. Thus, the results in [10] can be generalized to the submanifolds in the Euclidean space [12]. On the other

Manuscript received October 31, 1991.

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** Research at MSRI supported in part by NSF Grant DMS-850550, in part by NNSFC and SFECC.

hand, conformal vector fields in the sphere can also be viewed as the gradient vector fields of the eigenfunctions for the first non-zero eigenvalue of the Laplace operator in the sphere. Consequently, the results in [10] can be generalized to the case when the target manifold is a compact irreducible homogeneous space[13].

In the present paper we consider the case when the target manifold is a δ-pinched Riemannian manifold where there is the Ruh's construction which enables us to have the average process for second variation.

In fact, by using this technique Okayasu[6] proved the instability of harmonic maps into sufficiently pinched manifolds. Our main result of this paper is the following:

Let M be a compact Riemannian manifold of dimension m, N a complete simply connected δ-pinched Riemannian manifold of dimension n. There exists a constant $k(n, \delta)$ depending on n and δ whose precise formulation will be given latter. Set

$$d(n)=\begin{cases} 2 & \text{when } \dfrac{2n}{2n+(1+\delta)k(n,\delta)}<1, \\ 5 & \text{when } \dfrac{2n}{2n+(1+\delta)k(n,\delta)}=5, \\ \left[\min\left(1+\dfrac{2n}{2n+(1+\delta)k(n,\delta)},\ 6\right)\right] & \text{in other cases,} \end{cases}$$

where $[\cdot]$ denotes the greatest integer in a number.

Main Theorem. *If $m\leqslant d(n)$, then every minimizing map from M into N is smooth in the interior of M. If $m=d(n)+1$, such a map has at most discrete singular set and in general the Hausdorff dimension of the singular set is at most $m-d(n)-1$.*

When N is the sphere $\delta\to 1$ and $k(n, \delta)\to 2-n$. Consequently, the above result is a generalization of a result of Schoen-Uhlenbeck in [10].

In this paper we also consider some related Liouville type theorems of harmonic maps from complete manifolds into δ-pinched manifolds.

§ 2. Preliminaries

Let M and N be Riemannian manifolds of dimension m and n respectively, ϕ: $M\to N$ a smooth map. We agree the following range of indices

$$1\leqslant i, j, \cdots\leqslant m,$$

$$1\leqslant \alpha, \beta, \cdots\leqslant n.$$

Choosing local orthonormal frame field $\{e_i\}$ in M, the energy density of the map φ is defined by

$$e(\phi)=\frac{1}{2}\langle\phi_* e_i, \phi_* e_i\rangle. \tag{2.1}$$

Here and in the sequel we use the summation convension. The energy functional is

$$E(\phi)=\int_M e(\phi)*1. \tag{2.2}$$

ϕ is harmonic if it is a critical point of the above energy functional.

In the regularity theory the tangent maps play an important role which is special harmonic maps from $\mathbf{R}^m\setminus\{0\}$ with the same image on each ray issuing from the origin

Any harmonic map as a solution to variational problem possesses its index form as follows[1]:

$$I(v, v)=\int_M \langle -\nabla^2 v-R^N(\phi_* e_i, v)\phi_* e_i, v\rangle *1 \tag{2.3}$$

far $v\in\Gamma(\phi^{-1}TN)$ with compact support, where ∇^2 stands for trace Laplace operator on vector bundle $\phi^{-1}TN$ over M. If, in addition, ϕ has zero index, it is called the stable harmonic map.

In this paper we also need the following Bochner type formula[1]

$$\Delta e(\phi)=|\nabla d\phi|^2+\langle\phi_* \operatorname{Ric} e_i, \phi_* e_i\rangle-\langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_i, \phi_* e_j\rangle, \tag{2.4}$$

where ∇ denotes Riemannian connection on vector bundle $T^*M\otimes\phi^{-1}TN$. For simplicity we use ∇ to denote natural connections on various vector bundles. Its precise meaning can be seen from the context.

A Riemannian manifold whose sectional curvature K satisfies the condition $\delta<K\leqslant 1$ (δ is a positive constant) is called δ-pinched manifold. Let N be a complete simply connected δ-pinched n-dimensional Riemannian manifold. In order to obtain diffeomorphic sphere theorem Ruh introduced the following structure[7].

Consider a Riemannian vector bundle $E=TN\oplus\varepsilon(N)$ over N, where TN is the tangent bundle of N and $\varepsilon(N)$ is a trivial line bundle with fiber metric. Define a Riemannian connection ∇'' in E as follows

$$\nabla''_X=\nabla_X Y-\langle X, Y\rangle e, \tag{2.5}$$

$$\nabla''_X e=X, \tag{2.6}$$

where $X, Y\in\Gamma(TN)$, ∇ is Levi-Civita connection on N and e is the unit cross-section in $\varepsilon(N)$. It is easy to see that the curvature of ∇'' is small provided δ is close to one. Furthemore, by using ∇'' a flat connection ∇' on E can be constructed.

In their subsequent papers [3, 4] the difference of connections ∇' and ∇'' has been estimated. By multiplication with $\frac{1}{2}(1+\delta)$ we have normolized δ-pinched metric on N with sectional curvature in the interval $\left(\frac{2\delta}{1+\delta}, \frac{2}{1+\delta}\right)$.

Since $\nabla'-\nabla''$ is simply a skew-symmetric linear map. By using the Finsler norm on $o(n+1)$ define

$$\|\nabla'-\nabla''\| \overset{\text{def.}}{=\!=} \max\{|\nabla'_X Y-\nabla''_X Y|;\ x\in\Gamma(TN),\ |X|=1,\ Y\in\Gamma(E),\ |Y|=1\}. \tag{2.7}$$

Thus

$$\|\nabla'-\nabla''\|\leqslant k_3(\delta), \tag{2.8}$$

where

$$k_1(\delta)=\frac{4}{3}(1-\delta)\delta^{-1}\left[1+\left(\delta^{\frac{1}{2}}\cdot\sin\frac{\pi}{2}\,\delta^{-\frac{1}{2}}\right)\right]^{-1}\Big], \tag{2.9}$$

$$k_2(\delta)=\left[\frac{1}{2}(1+\delta)\right]^{-1}k_1(\delta), \tag{2.10}$$

and

$$k_3(\delta)=k_2(\delta)\left\{1+\left[1-\frac{1}{24}\pi^2(k_1(\delta))^2\right]^{-2}\right\}^{\frac{1}{2}}. \tag{2.11}$$

§ 3. Stability Inequality of Harmonic Maps into δ–Pinched Manifolds

In what follows we consider a harmonic map ϕ from Riemannian manifold M of dimension m into N which is a complete simply connected δ–pinched Riemannion manifold of dimension n. The idea is the same as that in a previous paper [12] where the target manifold is a submanifold in the Euclidean space provided the Ruh's construction has been employed instead of tangent bundle of the ambient Euclidean space along the submanifolds.

Let $\Theta=\{V\in\Gamma(E),\ \nabla'_X V=0$ for any $X\in\Gamma(TN)\}$. Certainly, Θ is a $(n+1)$–dimensional Euclidean space. Denote V^T to be projection of V into TN. Take cross–section in the vector bundle $\phi^{-1}TN$ over M

$$v=uV^T,$$

where u is any function on M with compact support. For any local orthonormal frame field $\{e_i\}$ on M

$$\nabla_{e_i}v=(\nabla_{e_i}u)V^T+u\nabla_{e_i}V^T, \tag{3.1}$$

and

$$|\nabla_{e_i}v|^2=|\nabla u|^2\langle V^T,V^T\rangle+u\nabla_{e_i}u\langle V^T,\nabla_{e_i}V^T\rangle+u^2\langle\nabla_{e_i}V^T,\nabla_{e_i}V^T\rangle. \tag{3.2}$$

Substituting (3.2) into the index form (2.3) we have

$$I(v,v)=\int_M[|\nabla u|^2\langle V^T,V^T\rangle+u\nabla_{e_i}u\langle V^T,\nabla_{e_i}V^T\rangle$$
$$+u^2(\langle\nabla_{e_i}V^T,\nabla_{e_i}V^T\rangle-\langle R^N(\phi_*e_i,V^T)\phi_*e_i,V^T\rangle)]*1 \tag{3.3}$$

which is a quadratic form on Θ.

At each point $x\in M$ choose orthonormal basis ε_α at $\phi(x)$ and unit vector e in the fiber of $\varepsilon(N)$ over $\phi(x)$. By the parallel translation in E with respect to the flat connection ∇' which was described in the previous section we have orthonormal basis $\{\hat\varepsilon_\alpha,\ \hat e\}$ in Θ. Thus,

$$\operatorname{trace}\langle V^T, V^T\rangle_\alpha=\sum_{\alpha=1}^{n}\langle \varepsilon_\alpha, \varepsilon_\alpha\rangle=n. \tag{3.4}$$

Since

$$\langle V^T, \nabla_{e_i}V^T\rangle=\frac{1}{2}\nabla_{e_i}\langle V, V\rangle-\langle V, e\rangle\nabla_{e_i}\langle V, e\rangle,$$

we have

$$\begin{aligned}\operatorname{trace}\langle V^T, \nabla_{e_i}V^T\rangle_\alpha&=\sum_\alpha\langle \hat{\varepsilon}_\alpha^T, \nabla_{e_i}\hat{\varepsilon}_\alpha^T\rangle_\alpha+\langle \hat{e}^T, \nabla_{e_i}\hat{e}^T\rangle_\alpha\\&=\sum_\alpha\langle \hat{\varepsilon}_\alpha^T, \nabla_{e_i}\hat{\varepsilon}_\alpha^T\rangle\\&=\frac{1}{2}\nabla_{e_i}\langle \varepsilon_\alpha, \varepsilon_\alpha\rangle=0.\end{aligned} \tag{3.5}$$

From (2.5) it follows that

$$\begin{aligned}\nabla_{e_i}V^T&=\nabla''_{\phi_*e_i}V^T+\langle V, \phi_*e_i\rangle e=\nabla''_{\phi_*e_i}(V-\langle V, e\rangle e)+\langle V, \phi_*e_i\rangle e\\&=\nabla''_{\phi_*e_i}V-\langle V, e\rangle\phi_*e_i.\end{aligned} \tag{3.6}$$

(2.7) and (3.6) yield

$$\begin{aligned}\langle \nabla_{e_i}V^T, \nabla_{e_i}V^T\rangle&\leqslant(1+c)\langle\nabla''_{\phi_*e_i}V, \nabla''_{\phi_*e_i}V\rangle+\left(1+\frac{1}{c}\right)\langle V, e\rangle\langle V, e\rangle|d\phi|^2\\&=(1+c)\langle(\nabla''_{\phi_*e_i}-\nabla'_{\phi_*e_i})V, (\nabla''_{\phi_*e_i}-\nabla'_{\phi_*e_i})V\rangle\\&\quad+\left(1+\frac{1}{c}\right)\langle V, e\rangle\langle V, e\rangle|d\phi|^2\\&\leqslant\left[\frac{1}{4}(1+c)k_3^2(\delta)\langle V, V\rangle\left(+1+\frac{1}{c}\right)\langle V, e\rangle\langle V, e\rangle\right]|d\phi|^2,\end{aligned} \tag{3.7}$$

which gives

$$\operatorname{trace}\langle\nabla_{e_i}V^T, \nabla_{e_i}V^T\rangle_\alpha\leqslant\left[\frac{(n+1)(1+c)}{4}k_3^2(\delta)+\left(1+\frac{1}{c}\right)\right]|d\phi|^2, \tag{3.8}$$

where c is a constant chosen later. From the bounds of normalized sectional curvature on N we have

$$\langle R^N(\phi_*e_i, V^T)\phi^*e_i, V^T\rangle\geqslant\frac{2\delta}{1+\delta}(\langle V^T, V^T\rangle|d\phi\|^2-\langle\phi_*e_i, V^T\rangle\langle\phi_*, V^T\rangle),$$

which gives

$$\operatorname{trace}\langle R^N(\phi_*e_i, V^T)\phi_*e_i, V^T\rangle_\alpha\geqslant\frac{2\delta}{1+\delta}(n-1)|d\phi|^2. \tag{3.9}$$

Substituting (3.4), (3.5), (3.8) and (3.9) into (3.3) we obtain

$$\begin{aligned}Q=\operatorname{trace}I\leqslant\int_M\Big\{n|\nabla u|^2+&\frac{(n+1)(1+c)}{4}k_3^2(\delta)+\left(1+\frac{1}{c}\right)\\&-\frac{2\delta}{1+\delta}(n-1)\Big]u^2|d\phi|^2\Big\}*1.\end{aligned} \tag{3.10}$$

Taking $c=\dfrac{2}{k_3(\delta)\sqrt{n+1}}$ we obtain the following stability inequality

$$\int_M\left(|\nabla u|^2+\frac{k(n, \delta)}{n}|d\phi|^2u^2\right)*1\geqslant0, \tag{3.11}$$

where

$$k(n,\delta)=\frac{n+1}{4}k_3^2(\delta)+\sqrt{n+1}\,k_3(\delta)+1-\frac{2\delta}{1+\delta}(n-1). \qquad (3.12)$$

§ 4. A Regularity Result

Now we are in a position to prove the following result.

Therem 4.1. *Let M be a compact m-dimensional Riemannian manifold and ϕ: $M\to N$ an energy minimizing map. If $m\leqslant d(n)$, ϕ is smooth in the interior of M. If $m=d(n)+1$, ϕ has at most discrete singular set and in general the Hausdorff dimension of the singular set is at most $m-d(n)-1$, where $d(n)$ has been defined in Section 1 of this paper.*

Proof By the Schoen–Uhlenbeck regularity theorem it suffices to prove any minimizing tangent map f: $\mathbf{R}^l\setminus\{0\}\to N$ is constant for $3\leqslant l\leqslant d(n)$. For any tangent map f there is an associate harmonic map f_1: $S^{l-1}\to N$. Thus (3.11) becomes in this case

$$\int_{S^{l-1}\times(0,\infty)}\left(-\bar{\Delta}u+\frac{k(n,\delta)}{n}u|df_1|^2-\gamma\frac{\gamma^2\partial^2u}{\partial\gamma^2}-\gamma(l-1)\frac{\partial u}{\partial\gamma}\right)\gamma^{l-3}u*1\geqslant 0, \qquad (4.1)$$

where $\bar{\Delta}$ denotes Laplacian on S^{l-1}. Now we consider a strongly elliptic operator on S^{l-1}

$$L_1=\Delta-\frac{k(n,\delta)}{n}|df_1|^2$$

and an ordinary differential operator on $(0,\infty)$

$$L_2=\gamma^2\frac{d^2}{d\gamma^2}+(l-1)\gamma\frac{d}{d\gamma}.$$

The eigenvalues of L_1 and L_2 are

$$\mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_i\leqslant\cdots\to\infty$$

and

$$\delta_1\leqslant\delta_2\leqslant\cdots\leqslant\delta_i\leqslant\cdots\to\infty$$

respectively. By the way done by Simons in [11], the stability condition (4.1) becomes

$$\mu_1+\delta_1\geqslant 0. \qquad (4.2)$$

By a direct computation

$$\delta_1=\frac{(l-2)^2}{4}. \qquad (4.3)$$

Let us estimate μ_1. By using (2.4) and noting the curvature conditions of S^{l-1} and N we have

$$\frac{1}{2}\bar{\Delta}|df_1|^2\geqslant|\nabla df_1|^2+(l-2)|df_1|^2-\frac{2(l-2)}{(l+1)(1+\delta)}|df_1|^4. \qquad (4.4)$$

Since

$$\frac{1}{2}\bar{\Delta}|df_1|^2=|df_1|\bar{\Delta}|df_1|+|\nabla|df_1||^2\leqslant|df_1|\bar{\Delta}|df_1|+|\nabla df_1|^2,$$

(4.4) becomes

$$|df_1|\bar{\Delta}|df_1|\geqslant(l-2)|df_1|^2-\frac{2(l-2)}{(l-1)(1+\delta)}|df_1|^4. \tag{4.5}$$

Let

$$\phi_\varepsilon=(|df_1|^2+\varepsilon)^{1/2}$$

for $\varepsilon>0$. From (4.5) it follows that

$$\phi_\varepsilon\bar{\Delta}\phi_\varepsilon\geqslant(l-2)|df_1|^2-\frac{2(l-2)}{(l-1)(1+\delta)}|df_1|^4$$

and

$$-\phi_\varepsilon\bar{\Delta}\phi_\varepsilon-\frac{2(l-2)}{(l-1)(1+\delta)}\phi_\varepsilon^2|df_1|^2\leqslant-(l-2)|df_1|^2. \tag{4.6}$$

If $l\leqslant d(n)$, namely

$$\frac{2(l-1)}{(l-1)(1+\delta)}<\frac{-k(n,\delta)}{n},$$

then from (4.6) we obtain

$$\mu_1\leqslant\inf\frac{\int_{S^{l-1}}\left(-\bar{\Delta}\phi_\varepsilon+\frac{k(n,\delta)}{n}|df_1|^2\phi_\varepsilon\right)\phi_\varepsilon*1}{\int_{S^{l-1}}\phi_\varepsilon^2*1}$$

$$<\inf\frac{\int_{S^{l-1}}\left(-\bar{\Delta}\phi_\varepsilon-\frac{2(l-2)}{(l-1)(1+\delta)}|df_1|^2\phi_\varepsilon\right)\phi_\varepsilon*1}{\int_{S^{l-1}}\phi_\varepsilon^2*1}\leqslant2-l, \tag{4.7}$$

provided $|df_1|\neq0$. But (4.2), (4.3) and (4.7) imply $l>6$ which contradicts $l\leqslant d(n)$. Thus f_1 has to be constant, so is f.

§ 5. Some Related Liouville Type Results

A Riemannian manifold is called strongly parabolic if it admits no non-constant positive superharmonic function. As is known, $\mathbf{R}^2$ is strongly parabolic while $\mathbf{R}^n(n\geqslant3)$ is not. In fact, there is the following criterion. Karp[5] introduced that a complete non-compact Riemannian manifold M has moderate volume growth if there is $F\in\mathscr{F}$ such that $\limsup_{\gamma\to\infty}\frac{1}{\gamma^2F(\gamma)}\operatorname{vol}B_\gamma(x_0)<\infty$ for some $x_0\in M$, where $\mathscr{F}=\{F:(0,\infty)\to(0,\infty);\ F$ is increasing on $(0,\infty)$ and $\int_1^\infty\frac{d\gamma}{\gamma F(\gamma)}=\infty\}$ and $B_\gamma(x_0)$ is a geodesic ball of radius γ and centered at $x_0\in M$. He proved that if M has moderate volume growth then it is strongly parabolic.

From the stability inequality (3.11) we have the following generalization of that in [6] to the case when the domain manifolds are not necessarily compact.

Theorem 5.1. *If M is a compact or complete non-compact strongly parabolic Riemannian manifold and $k(n, \delta)<0$, then any stable harmonic map ϕ: $M\to N$ is constant.*

Proof If M is compact, by choosing $u\equiv 1$ in the stability inequality (3.11) ϕ is constant. It is the result of Okayasu in [6].

If M is complete and non-comapct, we consider a strongly elliptic operator

$$L=\Delta-\frac{k(n,\delta)}{n}|d\phi|^2 \tag{5.1}$$

on any domain $D\subset M$ with $\bar{D}$ compact. Let λ be the first eigenvalue of L in D with Dirichlet boundary condition. From the stability inequality (3.11) it follows that

$$\lambda=\inf\frac{\int_D -uLu*1}{\int_D u^2*1}=\inf\frac{\int_D\left(-u\Delta u+\frac{k(n,\delta)}{n}|d\phi|^2u^2\right)*1}{\int_D u^2*1}\geqslant 0. \tag{5.2}$$

By using a theorem in [2] there is a positive solution $u>0$ to

$$Lu=\Delta u-\frac{k(n,\delta)}{n}|d\phi|^2u=0,$$

which impiies

$$\Delta u=\frac{k(n,\delta)}{n}|d\phi|^2u\leqslant 0, \tag{5.3}$$

namely, u is a positive superharmonic function on M. Since M is strongly parabolic, u is constant. Thus (5.3) implies $|d\phi|\equiv 0$.

Theorem 5.2. *If $k(n, \delta)<0$ and $m\leqslant d(n)$ then any minimizing map ϕ: $\mathbf{R}^m\to N$ is constant.*

Proof Choosing a cut off function

$$u=\begin{cases}1, \text{ in } B_R(0),\\ 0, \text{ out of } B_{2R}(0)\end{cases}$$

in the stability inequality (4.11) we have

$$R^{2-n}\int_{B_R(0)}|d\phi|^2*1\leqslant c, \tag{5.4}$$

where c is a constant depending on n and δ.

Then from (5.4) the proof follows as in [10]. We wiil not repeat them here.

We also can do L^p estimate for the energy density of harmonic maps into δ-pinched manifold N as was done in [8], [10].

Theorem 5.3. *Let M be a complete m-dimensional Riemannian manifold with Ricci curvature bounded below by a non-positive constant $-A$ $(A\geqslant 0)$ and ϕ: $M\to N$ a stable harmonic map with rank $\phi\leqslant\gamma$. If $k(n, \delta)<0$ and $\gamma\leqslant\dfrac{2n}{2n+(1+\delta)k(n,\delta)}$ then for any non-negative function u with compact support in M the following inequality is* *valid*

$$\int_M |d\phi|^4 u^4 *1 \leqslant c\int_M (A^2 u^4 + |\nabla u|^4) *1, \tag{5.5}$$

where c is a constant depending on m, n and δ.

Proof Choosing $u=|d\phi|v$, where v is any function with compact support in M, in the stability inequality (3.11) we have

$$-\frac{k(n,\delta)}{n}\int_M |d\phi|^4 v^2 *1 \leqslant \int_M [v^2|\nabla|d\phi||^2 + |d\phi|^2|\nabla v|^2 + 2v|d\phi|\langle\nabla v, \nabla|d\phi|\rangle] *1. \tag{5.6}$$

From the Bochner type formula for energy density (2.4) it follows that

$$\frac{1}{2}\Delta|d\phi|^2 \geqslant |\nabla d\phi|^2 - A|d\phi|^2 - \frac{2(\gamma-1)}{\gamma(1+\delta)}|d\phi|^4$$

and

$$|d\phi|\Delta|d\phi| \geqslant |\nabla d\phi|^2 - |\nabla|d\phi||^2 - A|d\phi|^2 - \frac{2(\gamma-1)}{\gamma(1+\delta)}|d\phi|^4. \tag{5.7}$$

By Schoen-Uhlenbeck's estimate[10]

$$|\nabla d\phi|^2 - |\nabla|d\phi||^2 \geqslant \frac{1}{2mn}|\nabla|d\phi||^2 \tag{5.8}$$

(5.7) and (5.8) gives

$$|d\phi|\Delta|d\phi| \geqslant \frac{1}{2mn}|\nabla|d\phi||^2 - A|d\phi|^2 - \frac{2(\gamma-1)}{\gamma(1+\delta)}|d\phi|^4. \tag{5.9}$$

Multiplying (5.9) by v^2 and then integrating it over M we obtain

$$\begin{aligned}\frac{1}{2mn}\int_M v^2|\nabla|d\phi||^2 *1 &\leqslant \int_M\Big[-v^2|\nabla|d\phi||^2 + A|d\phi|^2 v^2 \\ &\quad + \frac{2(\gamma-1)}{\gamma(1+\delta)}|d\phi^4|v^2 - 2v|d\phi|\langle\nabla v, \nabla|d\phi|\rangle\Big] *1.\end{aligned} \tag{5.10}$$

By adding (5.6) and (5.10) we have

$$\begin{aligned}\frac{1}{2mn}\int_M v^2|\nabla|d\phi||^2 *1 &\leqslant \int_M (|d\phi|^2|\nabla v|^2 + A|d\phi|^2 v^2) *1 \\ &\quad + \int_M\left(\frac{2(\gamma-1)}{\gamma(1+\delta)} + \frac{k(n,\delta)}{n}\right)|d\phi|^4 v^2 *1.\end{aligned} \tag{5.11}$$

Since $\gamma \leqslant \frac{2n}{2n+(1+\delta)k(n,\delta)}$, the second term of the right-hand side of (5.11) is non-positive. Thus, we have

$$\frac{1}{2mn}\int_M v^2|\nabla|d\phi||^2 *1 \leqslant \int_M (|d\phi|^2|\nabla v|^2 + A|d\phi|^2 v^2) *1. \tag{5.12}$$

By using the Cauchy inequality for any $\varepsilon>0$

$$2v|d\phi|\langle\nabla v, \nabla|d\phi|\rangle \leqslant \varepsilon v^2|\nabla|d\phi||^2 + \varepsilon^{-1}|d\phi|^2|\nabla v|^2$$

(5.6) becomes

$$-\frac{k(n,\delta)}{n}\int_M |d\phi|^4 v^2 *1 \leqslant (1+\varepsilon)\int_M v^2|\nabla|d\phi||^2+(1+\varepsilon^{-1})\int_M |d\phi|^2|\nabla v|^2*1. \tag{5.13}$$

Substituting (5.12) in (5.13) and then replacing v by u^2 we have

$$-\frac{k(n,\delta)}{n}\int_M |d\phi|^4 u^4 *1 \leqslant c_1\int_M (4|d\phi|^2u^2|\nabla u|^2+A|d\phi|^2u^4)*1, \tag{5.14}$$

where c_1 is dependent on m and n. By using Cauchy inequality again for any $\varepsilon>0$ we have

$$|d\phi|^2u^2|\nabla u|^2 \leqslant \frac{\varepsilon}{2}u^4|d\phi|^4+\frac{\varepsilon^{-1}}{2}|\nabla u|^4 \tag{5.15}$$

and

$$A|d\phi|^2 \leqslant \frac{\varepsilon}{2}|d\phi|^4+\frac{\varepsilon^{-1}}{2}A^2. \tag{5.16}$$

Consequently the desired inequality (5.5) follows from (5.14), (5.15) and (5.16) immediately.

Corollary 5.3. *Let M be a complete m-dimensional Riemannian manifold with nonnegative Ricci curvature and*

$$\liminf_{R\to\infty}\frac{\operatorname{vol} B_R(x_0)}{R^4}=0$$

and ϕ: $M\to N$ a stable harmonic map with rank $\phi\leqslant\gamma$. If $k(n,\delta)<0$ and $\gamma\leqslant\dfrac{2n}{2n+(1+\delta)k(n,\delta)}$, then ϕ has to be constant. So does ϕ in the case when M is Euclidean space of dimension 4.

The prorf is the same as that in [13]. We will not repeat them here.

Remark. Corollary 5.3 is valid for $M=\mathbf{R}^m$, where $m\leqslant4$ and $m\leqslant\dfrac{2n}{2n+(1+\delta)k(n,\delta)}$, as well as for certain strongly parabolic manifolds.

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