Chin. Ann. of Math.
13B: 4(1992), 396—405.

AVERAGE KOLMOGOROV N-WIDTHS (N-K WIDTH) AND OPTIMAL INTERPOLATION OF SOBOLEV CLASS IN $L_p(R)$**

CHEN DIRONG (陈迪荣)*

Abstract

The author obtains the exact values of the average n-K widths for some Sobolev classes defined by an ordinary differential operator $P(D)=\prod_{i=1}^{r}(D-t_iI)$, $t_i\in R$, in the metric $L_p(R)$, $1\leqslant p\leqslant\infty$, and identifies some optimal subspaces. Furthermore, the optimal interpolation problem for these Sobolev classes is considered by sampling the function values at some countable sets of points distributed reasonably on $\mathbf{R}$, and some exact results are obtained.

§ 1. Introduction and Some Known Results

Given $P(t)=\prod_{i=1}^{r}(t-t_i)$, $t_i\in\mathbf{R}$, $i=1, \cdots, r$, donete by I the real line $\mathbf{R}$ or a compact interval $[a, b]\subseteq\mathbf{R}$, and

$$W^r_{p,I}=\{f\,|\,f^{(r-1)} \text{ is abs. cont. on } I \text{ (locally abs. cont. on } I \text{ in case } I=\mathbf{R}),\ f,\ f^{(r)}\in L_p(I)\},$$

where $p\in[1, +\infty]$. We define a Sobolev class as follow

$$B_{p,I}(P(D))=\{f\,|\,f\in W^r_{p,I},\ \|P(D)f\|_{L_p(I)}\leqslant 1\}.$$

When $I=[a, b]$ is compact, we furthermore define the periodic classes

$$\widetilde{W}^r_{p,I}=\{f\,|\,f\in W^r_{p,I},\ f^{(i)}(a)=f^{(i)}(b),\ i=0, \cdots, r-1\},$$

$$\widetilde{B}_{p,I}(P(D))=B_{p,I}(P(D))\cap\widetilde{W}^r_{p,I},$$

where $D=d/dx$, $\|\cdot\|_{L_p(I)}$ is the usual $L_p(I)$-norm. We also use the abbreviations for simplicity

$$\|\cdot\|_p=\|\cdot\|_{L_p(R)},\ B_p=B_{p,R}(P(D)),\ W^r_{p,R}=W^r_p,$$

$$\|\cdot\|_{\widetilde{L}_p}=\|\cdot\|_{L_p,[-1,1]},\ \widetilde{B}_p=\widetilde{B}_{p,[-1,1]}(P(D)).$$

As usual, we use d_n, d^n, δ_n and b_n to denote the Kolmogorov, the Gelfand, the linear

---

Manuscript received December 20, 1990. Revised June 21, 1991.
* Department of Mathematics, Beijing Normrl University, Beijing 100875, China.
** Supported by the National Fund of Natural Sciences.

and the Bernstein n-width respectively. The symbol s_n will stand for any one of above four sombols. The quantities $s_n(\tilde{B}_p, \tilde{L}_p)$ have long been investigated by many authors (cf. [1]). Recently the author[2] has obtained the exact values of $d_{2n}(\tilde{B}_p, \tilde{L}_p)$, $d^{2n}(\tilde{B}_p, \tilde{L}_p)$ and $\delta_{2n}(\tilde{B}_p, \tilde{L}_p)$ for $p\in(1, \infty)\setminus\{2\}$, which improves the results of [3]. Although B_p is the analogy of $\tilde{B}_p$, it is meaningless to consider its n-widths because B_p is not compact in $L_p(\mathbf{R})$ (for details the reader may refer to [4]). In [5], V. M. Tikhomirov proposed the concept of average n-K width which is adequate to the non-compact case. Some results of average n-K widths have been obtained (see [4—6]). In this paper we get the exact values for the average n-K widths of B_p in $L_p(R)$ and identify some of its optimal subspaces.

It is well-known that the problem of minimization of the intrinsic error for the optimal interpolation of $\tilde{B}_p$ in $\tilde{L}_p$ over a collection of linear information operators with cardinal$\leqslant n$ is closely related to the n-width problem (cf. [7]). To treat the case under consideration, we follow the idea of [8] in which a collection of denumerable sets of sampling points with average density$\leqslant 1$ has been used. In § 4 we compute the exact values of the optimal instrinsic error for the optimal interpolation of B_p in $L_p(\mathbf{R})$ by using function values taken on denumerable sets of sampling points with average density$\leqslant 1$. Furthermore, we construct a linear optimal algorithm realizing the optimal intrinsic error.

Now we introduce some notations and known results for later use. Let $P(D)$ be given above. The Bernoulli function corresponding to $P(D)$ is

$$G(t)=\frac{1}{2}\sum_{k\in Z}{}' \frac{e^{ikt\pi}}{P(ik\pi)},\quad i=\sqrt{-1},$$

where $\sum'$ means that the term of $k=0$ is discarded when $P(0)=0$.

We denote by $S_{1/n}$ the periodic $\mathscr{L}$-spline subspace which is the set of function $s(t)$ such that

$$s(t)=\begin{cases}\sum\limits_{k=-n}^{n-1} c_k G(t-1/n), & \text{if } P(0)\neq 0,\\ c+\sum\limits_{k=-n}^{n-1} c_k G(t-1/n), & \sum\limits_{k=-n}^{n-1} c_k=0, \text{ if } P(0)=0\end{cases}$$

where $c, c_k\in R$, $k=-n, \cdots, n-1$.

For $p\in[1, +\infty]$, we consider the extremal problem:

$$\lambda_n:=\lambda_n(P(D), p)=\sup\{\|G*h\|_{\tilde{L}_p} \mid h\in D_n\},$$

where $(G*h)(x)=\int_{-1}^{1} G(x-t)h(t)dt$ and

$$D_n=\{h \mid h(x+1/n)=-h(x),\ h(x)\sin(n x\pi)\geqslant 0,\ \|h\|_{\tilde{L}_p}\leqslant 1\}.$$

It can be verified that

$$\lambda_n(P(D), p)=\lambda_n(P(-D), p')^{[2]},\ 1/p+1/p'=1, \tag{1.1}$$

$$\lambda_n(P(D), 2)=|P(in)|^{-1}, \tag{1.2}$$

$$\lambda(P(D), 1)=\lambda_n(P(-D), \infty)=\left\|\int_{-1}^{1} G(x-t)\operatorname{sgn}\sin(nt\pi)dt\right\|_\infty. \quad (1.3)$$

By Theorem 2.1 in [2] and an argument used in [9] we have

Theorem 1.1. *For any $p\in(1, \infty)$, $n=1, 2, \cdots$, there exists uniquely a continuous function $h_n\in D_n$ such that*

$$\|G*h_n\|_{\tilde{L}_p}=\lambda_n\|h_n\|_{\tilde{L}_p}=\lambda_n. \quad (1.4)$$

Furthermore, h_n satisfies

(1) $$\int_{-1}^{1} G(x-y)\,|(G*h_n)(x)|^{p-1}\operatorname{sgn}((G*h_n))dx$$
$$=\lambda_n^{p-1}|h_n(y)|^{p-1}\operatorname{sgn}h_n(y),\ \forall y\in[-1, 1].$$

(2) $$\varepsilon\operatorname{sgn}((G*h_n)(x))=\operatorname{sgn}\sin(n\pi(x-\alpha_n)),$$
$$\operatorname{sgn}(h_n(x))=\operatorname{sgn}(\sin(nx)),$$

where $\varepsilon=1$ or -1, and $\alpha_n\in\left[0, \frac{1}{n}\right)$ fixed.

Theorem 1.2.[2] *For every $f\in\tilde{B}_p(1<p<+\infty)$, there is just one $\tilde{s}_{\frac{1}{n}}(f, \cdot)\in\tilde{S}_{\frac{n}{n}}$ which interpolates f at $\{\alpha_n+j/n\}_{j=-n}^{n-1}$, and moreover, it holds that*

$$\sup_{f\in\tilde{B}_p}\|f-\tilde{s}_{\frac{1}{n}}(f)\|_p=\lambda_n,\ n\in Z^+. \quad (1.5)$$

Theorem 1.3[2]. *For $n=1, 2, \cdots, p\in(1, +\infty)$,*

$$d_{2n}(\tilde{B}_p, \tilde{L}_p)=d^{2n}(\tilde{B}_p, \tilde{L}_p)=\delta_{2n}(\tilde{B}_p, \tilde{L}_p)=\lambda_n\leqslant a_{2n-1}(\tilde{B}_p, \tilde{L}_p).$$

Remark 1. For $p=1, +\infty$, we know that (cf. [1]) (2) of Theorem 1.1, Theorem 1.2 and Theorem 1.3 hold when h_n is replaced by $\operatorname{sgn}(\sin(nx\pi))$.

§ 2. Upper Bound of Approximation of B_p by Cardinal $\mathscr{L}$-Spline Interpolating Operator

Lemma 2.1. *If $h_1(x)$ is given in Theorem 1.1, $1\leqslant p\leqslant\infty$, (when $p=1, \infty$, $h_1(x)=\operatorname{sgn}\sin(nx\pi)$), and $\alpha_n(P(D))$ is the unique zero of $G*h_n$ in $[0, 1/n)$, $n=1, 2, \cdots$, then*

(1) $$\lambda_n(P(D/n))=\lambda_1(P(D)).$$

(2) $$\alpha_n(P(D/n))=n^{-1}\alpha_1(P(D)).$$

Proof Put $g_n(x)=h_1(nx)$. Then $h\perp 1$, $g_n\perp 1$, $G*h_1\perp 1$ and $G_n*g_n\perp 1$, where G_n denotes the Bernoulli function relating to $P(D/n)$. From the equality

$$P\left(\frac{D}{n}\right)(G*h_1)(nx)=(P(D)G*h_1)(nx)=h_1(nx),$$

it follows that $G_n*g_n=(G*h_1)(nx)$. If $1<p<+\infty$, then $g_n\in D_n$ and

$$\lambda_n(P(D/n))\geqslant\|G_n*g_n\|_{\tilde{L}_p}=\|G*h_1\|_{\tilde{L}_p}=\lambda_1(P(D)).$$

The opposite inequality may be proven in a similar manner. Therefore (1) holds for $p\in(1, \infty)$ and $g_n\in D_n$ is the unique continuous function which satisfies Theorem

1.1 for $G_n * g_n = (G*h_1)(nx)$, (2) holds for $p \in (1, \infty)$.

When $p=1, \infty$, the lemma is true by the Remark 1. So the proof is completed.

The Cardinal $\mathscr{L}$-spline relating to the operator $P(D)$ with knots $\{j\beta\}_{j\in Z}$, where $\beta>0$ is a fixed number, is defined as follows (cf. [10]).

$$S_\beta = S_\beta(P(D)) = \{s(x) \mid s(x) \in C^{r-2}(R),\ P(D)s(x)=0,\ \forall x \in (j\beta, (j+1)\beta),\ j \in Z\},$$

where, as usual, $f \in C^{-1}(R)$ means that f is piecewise continuous.

From Theorem 1.2, $\tilde{S}_1 \subseteq S_1$ and the theory of Cardinal $\mathscr{L}$-spline interpolation[10] we have

Lemma 2.2. *For any data $Y=\{y_j\}_{j\in Z}$ of power growth, i. e., $|y_j| = O(|j|^\nu)$ for some $\nu \geqslant 0$, there exists just one $s_1(Y, \cdot) \in S_1$ of power growth which satisfies*

$$s_1(Y, j+\alpha_1) = y_j,\ j \in Z,\ \alpha_1 = \alpha_1(P(D)).$$

Furtermore, $s_1(Y, \cdot)$ can be represented by a cardinal seriesi

$$s_1(Y, x) = \sum_{j=-\infty}^{\infty} y_j L(x-j),$$

where $L(x) \in S_1$, $L(j+\alpha_1) = \delta_{0,j}$, $j \in Z$, and $|L(x)| \leqslant Ae^{-B|x|}$, $x \in \mathbf{R}$, for some positive constants A and B.

The following lemma is proven for $P(D)=D^r$ by Li.[11] It can be easily generalized to the general case $P(D)=\prod_{j=1}^{r}(D-t_jI)$ by the method of [11]. In fact, for its proof it suffices to use the representation of $s_1(Y, \cdot)$ and the property $|L(x)| \leqslant Ae^{-B|x|}$, $x \in R$.

Lemma 2.3. (1) *For any $f \in W_p^r$,*

$$s_1(f, x) = \sum_{j\in Z} f(j+\alpha_1) L(x-j) \in L_p(R).$$

Furthermore, we have

$$\|s_1(f)\|_p \leqslant C \|\{f(j+\alpha_1)\}_{j\in Z}\|_{l_p},$$

where $C>0$ is independent of f, and $\|\cdot\|_{l_p}$ is the usual l_p-norm in l^p, $p \in [1, \infty]$.

(2) *Suppose that $Y^{(n)} = \{y_j^{(n)}\}_{j\in Z} \in l^\infty$, $n=1, 2, \cdots$, satisfy $y_j^{(n)}=0$ for $|j| \leqslant 2n$, and $|y_j^{(n)}| \leqslant M$ for $|j|>2n$, $n=1, 2, \cdots$, where M is a constant. Then*

$$\lim_{n\to\infty} \|s_1(Y^{(n)}, \cdot)\|_p = 0.$$

Theorem 2.1. *For any $f \in W_p^r$, $1 \leqslant p \leqslant \infty$, it holds that*

$$\|f - s_1(f)\|_p \leqslant \lambda_1(P(D)) \|P(D)f\|_p. \tag{2.1}$$

Proof For $p=\infty$, 1 the relation (2.1) was proven in [12] and [13] respectively. It remains for us to prove (2.1) for $p \in (1, \infty)$. Our proof follows the same lines in [11]. For every $\varepsilon>0$, there exists an integer $N>0$ such that for each $n \geqslant N$ it holds that

$$\|f - s_1(f)\|_p^p \leqslant \varepsilon + \int_{-n}^{n} |f - s_1(f)|^p dx. \tag{2.2}$$

In the following we will employ Cavaretta's technique. Take a function

$$g(x)=\begin{cases}1, & |x|\leq 1,\\ (-1)^r(x-2)^r\sum_{j=0}^{r-1}\binom{r+j-1}{k}(x-1)^j, & 1<x<2,\\ (x+2)^r\sum_{j=0}^{r-1}\binom{r+j-1}{j}(x+1)^j, & -2<x<-1,\\ 0, & |x|\geq 2.\end{cases}$$

It is easily verified that $g\in C^{r-1}(\mathbf{R})$, $\|g^{(k)}\|_\infty<+\infty$, $k=0,1,\cdots,r$, and $0\leq g(x)\leq 1$, $x\in\mathbf{R}$. Put

$$F_n(x)=(2n)^{1/p}f(2nx)g(2x).$$

Then $F_n\in C^{r-1}(\mathbf{R})$, $F_n(x)=0$, for $|x|\geq 1$. Denote by $\widetilde{F}_n$ the 2-periodic extention of F_n from $[-1,1]$ to $\mathbf{R}$. By Leibnitz rule

$$P(D/2n)\widetilde{F}_n(x)=(2n)^{1/p}\sum_{j=0}^{r}\left(\frac{1}{2n}\right)^j\{P^{(j)}(D/2n)[f(2nx)]\}2^jg^{(j)}(2x)$$

$$=(2n)^{1/p}\sum_{j=0}^{r}(1/n)^jg^{(j)}(2x)P^{(j)}(D/2n)[f(2nx)],$$

where $P^{(p)}(\cdot)$ is the j-th derivative of $P(\cdot)$. It follows from Stein's inequality[14] that

$$\|P(D/(2n))\widetilde{F}_n\|_{\widetilde{L}_p}\leq\|P(D)f\|_p+cn^{-1},$$

where c is a constant independent of n. By Theorem 1.2, there exists a unique function $\tilde{s}_{1/(2n)}(\widetilde{F}_n,\cdot)\in\widetilde{S}_{1/(2n)}(P(D/(2n)))$ which interpolates $\widetilde{F}_n(\cdot)$ at $\{j/2n+\alpha_{2n}(P(D/(2n)))\}_{j=-n}^{n-1}$. Furthermore,

$$\|\widetilde{F}_n-\tilde{s}_{1/2n}(\widetilde{F}_n)\|_{\widetilde{L}_p}\leq\lambda_{2n}(P(D/2n))(\|P(D)f\|_p+cn^{-1})$$

$$=\lambda_1(P(D))(\|P(D)f\|_p+cn^{-1}). \tag{2.3}$$

Since $\alpha_{2n}(P(D/(2n)))=(2n)^{-1}\alpha_1(P(D))$, $(2n)^{-1/p}\,\tilde{s}_{1/(2n)}(\widetilde{F}_n,x/(2n))\in S_1(P(D))$ and $(2n)^{-1/p}\tilde{s}_{1/(2n)}(\widetilde{F}_n,x/(2n))$ interpolates $\widetilde{E}_n(x):=(2n)^{-1/p}\widetilde{F}_n(x/(2n))$ at $\{\alpha_1+j\}_{j\in Z}$, we have, from the uniqueness of the interpolator,

$$s_1(\widetilde{E}_n,x)=(2n)^{1/p}\tilde{s}_{1/(2n)}(\widetilde{F}_n,x/(2n)).$$

(2.3) reads

$$\int_{-2n}^{2n}|\widetilde{E}_n(x)-s_1(\widetilde{F}_n,x)|^pdx\leq\lambda_1^p(P(D))(\|P(D)f\|_p+cn^{-1})^p.$$

Therefore

$$\|f-s_1(f)\|_{L_p(I_n)}=\|\widetilde{E}_n-s_1(f)\|_{L_p(I_n)}$$

$$\leq\|\widetilde{E}_n-s_1(\widetilde{E}_n)\|_{L_p(I_n)}+\|s_1(\widetilde{E}_n)-s_1(E_n)\|_{L_p(I_n)}$$

$$+\|s_1(f)-s_1(E_n)_n\|_{L_p(I_n)},$$

where $E_n(x)=(2n)^{-1/p}F_n(x/(2n))$, $I_n=[-n,n]$. by Lemma 2.3 we have

$$\lim_{n\to\infty}\|s_1(\widetilde{E}_n,x)-s_1(E_n,x)\|_{L_p(I_n)}=0,$$

$$\lim_{n\to\infty}\|s_1(f,x)-s_1(E_n,x)\|_{L_p(I_n)}=0. \tag{2.3}$$

So there is an integer $N_1 \geqslant N$ such that for $n \geqslant N_1$

$$\|f-s_1(f)\|_p^p \leqslant \varepsilon + (\lambda_1(P(D)))^p(\|P(D)f\|_p+\varepsilon)^p.$$

The proof is completed by the arbitrariness of ε.

By changing of scale we have

Corollary 2.1. *For any $f \in W_p^r$, $\beta>0$, there exists uniquely a $s_\beta(f, \cdot) \in S_\beta(P(D))$ which interpolates f at $\{j\beta+\alpha_1(P(D/\beta))\beta\}_{j\in Z}$. Furthermore,*

$$\|f-s_\beta(f)\|_p \leqslant \lambda_1(P(D/\beta))\|P(D)f\|_p.$$

Remark 2. From Corollary 2.1 it follows that, for $\beta>0$, $p\in[1, +\infty)$, $\alpha\in[0, \beta)$,

$$\sup\{\|f\|_p \mid f\in B_p, f(j\beta+\alpha)=0, j\in Z\} \leqslant \lambda_1(P(D/\beta)). \tag{2.4}$$

When $p=2$, (2.4) was established by the author in [4] in a different way.

When $p=1, \infty$, we can prove more results. First we cite the duality theorem proved in [4].

Lemma 2.4.[4] *For any $g\in L_p(\mathbf{R})$, $p\in[1, \infty)$,*

$$\inf_{s\in S_\beta}\|g-s\|_p = \sup\left\{\int_{-\infty}^{\infty} g(x)\,(P(-D)f)(x)dx \mid f\in B_q(P(-D)),\ f(j\beta)=0,\ j\in Z\right\},$$

where $1/p+1/q=1$.

Theorem 2.2. *For every polynomial $P_m(t)$ with only real zeros such that $P(t)$ is a factor of $P_m(t)$, we have*

$$E(B_p(P(D)), S_\beta(P_m(D)), L_p) \leqslant \lambda_1(P(D/\beta)),\ p=1, +\infty,$$

where $E(A, B, L_p) := \sup_{f\in A}\inf_{g\in B}\|f-g\|_p$, $A, B\subseteq L_p(\mathbf{R})$.

Proof Set $\hat{P}(t)=P_m(t)/P(t)$. Noticing for $p<\infty$, $f\in W_p^r$, $\lim_{|x|\to\infty} f^{(j)}=0$, $j=0, \cdots, r-1$, by Lemma 2.4 and integrating by part we get

$$E(B_p(P(D)), S_\beta(P_m(D)), L_p)$$
$$= \sup\{\|\hat{P}(-D)f\|_q \mid f\in B_q(P_m(-D)), f(j\beta)=0, j\in Z\}.$$

From Remark 2 we see that, if $f\in B_q(P_m(-D))$, $f(j\beta)=0$, $j\in Z$, then for $q=1, \infty$,

$$\|f\|_q \leqslant \lambda_1(_m(-D/\beta), q) = \|(G_\beta(m, u)*\operatorname{sgn}\sin\pi u)(\cdot)\|_\infty,$$

where $G_\beta(m, t)$ is the Bernoulli function relating to $P_m(-D/\beta)$, and the equality holds by (1.1) and (1.3). The proof is completed by employing the Landau-Kolmogorov type inequality for the differential operator $P_m(-D/t)$, $q=1, \infty$ (cf.* [15]).

§ 3. Average n-k Width of B_p in L_p

Let $A\subseteq L_p(\mathbf{R})$ be a linear subspace. Given a, $\varepsilon\in R_+$, set

$$A_a = \{f|_{[-a,a]} \mid f\in A,\ \|f\|_p\leqslant 1\},$$

where $f|_{[-a,a]}$ is the restriction of f to $[-a, a]$, and

$$K(\varepsilon, a, A) = \min\{\dim L \mid L\subseteq L_p[-a, a],\ E(A_a, L, L_p[-a, a])<\varepsilon\}.$$

Definition 3.1.[5] *Let $A\subseteq L_p(\mathbf{R})$ be a linear subspace. If there exists an $\varepsilon_0>0$ such that*

$$\lim_{a\to\infty}\frac{K(\varepsilon,\ a,\ A)}{2a}=n<\infty$$

for all $\varepsilon\in(0,\ \varepsilon_1)$ and n is independent of ε, then A is said to be average dimension n. In this case we use the symbol $\overline{\dim}\ A=n$. Note that n is not necessarily an integer.

Definition 3.2.[5] *Let $F\subseteq L_p(R)$ be symmetric with respect to the origin. The average n–K width of F in L_p is defined by*

$$\bar{d}_n(F,\ L_p)=\inf\{E(F,\ A_n,\ L_p)\mid \overline{\dim}\ A_n\leqslant n\}.$$

A linear subspace $A_n^*\subseteq L_p(\mathbf{R})$ is said to be optimal for $\bar{d}_n(F,\ L_p)$ if $\overline{\dim}\ A_n^*\leqslant n$, and $E(F,\ A_n^*,\ L_p(\mathbf{R}))=\bar{d}_n(F,\ L_p)$.

Theorem 3.1. *For any $n\in\mathbf{R}_+$, denote $S_{1/n,p}(P(D))=\{S_{1/n}(P(D))\}\cap L_p(\mathbf{R})$. Then*

(1) $\bar{d}_n(B_p,\ L_p)=\lambda_1(P(nD))$.

(2) *$S_{1/n,p}(P(D))$ is an optimal subspace of $\bar{d}_n(B_p,\ L_p)$.*

(3) *when $p=1,\ \infty$, any $S_{1/n,p}(P_m(D))$ given in Theorem 2.2 is optimal for $\bar{d}_n(B_p,\ L_p)$.*

Proof First we establish, for any $P(t)$,

$$\overline{\dim}\ S_{1/n,p}(P(D))=n. \tag{3.1}$$

Given any $a,\ \varepsilon>0$, since every $s(\cdot)\in S_{1/n,p}(P(D))$ can be represented as a linear combination of B-spline[10], we have

$$\dim S_{1/n,p}(P(D))\,|_{[-a,a]}\leqslant 2[na]+\deg P.$$

On the other hand, if denote $S_{1/n,p}(P(D))_0=\{s\in S_{1/n,p}(P(D))\,|\operatorname{supp}s\subseteq[-a,\ a]\}$, then

$$\dim(S_{1/n,p}(P(D))_0)\geqslant 2[na]-\deg P.$$

There fore, by the definition of $K(\varepsilon,\ a,\ A)$ and Theorem 1.5 of Chapter II in [1], we have, for $\varepsilon\in(0,\ 1)$,

$$2[na]-\deg P\leqslant K(\varepsilon,\ a,\ S_{1/n,p}(P(D)))\leqslant 2[na]+\deg P.$$

This means that (3.1) holds. On account of Corollary 2.1, Theorem 2.2 and (3.1) we need only to prove $\bar{d}_n(B_p,\ L_p)\leqslant\lambda_1(P(nD))$.

Suppose that A is a subspace with $\overline{\dim}\ A\leqslant n$, i. e., there exists an $\varepsilon_0>0$ such that

$$\lim_{a\to\infty}\frac{K(\varepsilon,\ a,\ A)}{2a}\leqslant n,\ \varepsilon\in(0,\ \varepsilon_0). \tag{3.2}$$

Put $a_N=N\beta$, $N=1,\ 2,\ \cdots,\ \beta\in(0,\ 1/n)$ fixed. By (3.2) there exists a sufficiently large N such that

$$E(A_{N\beta},\ L_N,\ L_p(I_N))<\varepsilon, \tag{3.3}$$

where $A_{N\beta}$ denotes the restriction of $\{x\,|\,x\in A,\ \|x\|_p\leqslant 1\}$ on $I_N=[-N\beta,\ N\beta]$, and

L_N is some subspace in $L_p(I_N)$ with dim $L_N \leqslant 2N-r-1$.

From the definition of b_N (i. e., the Bernstein n-width) and Theorem 1.3 we have, by changing of scale,

$$\begin{aligned} &b_{2N-r-1}(B_{p,I_N}(P(D))_0,\ L_p(I_N)) \\ &\geqslant b_{2N-1}(\tilde{B}_{p,I_N}(P(D)),\ L_p(I_N)) \\ &= b_{2N-1}\Big(\tilde{B}_{p,[-1,1]}\Big(P\Big(\frac{D}{N\beta}\Big)\Big),\ \tilde{L}_p\Big) \geqslant \lambda_N\Big(P\Big(\frac{D}{N\beta}\Big)\Big) = \lambda_1\Big(P\frac{D}{\beta}\Big)\Big), \end{aligned}$$

where $B_{p,I_N}(P(D))_0 = \{f \mid f \in B_p,\ \operatorname{supp} f \subseteq I_N\}$. Therefore, by the proof of Theorem 1.5 of ([1], p. 12), for any $\lambda \in (0, \lambda_1(P(D/\beta)))$, there is an $f \in B_{pI_N}(P(D))_0$ such that

$$\|f\|_p = \|f\|_{L_p(I_N)} = \inf_{g \in L_N} \|f-g\|_{L_p(I_N)} = \lambda,$$

which, together with (3.3), gives

$$\begin{aligned} \inf_{\varphi \in A} \|f-\varphi\|_p &= \inf_{\substack{\varphi \in A \\ \|\varphi\|_p \leqslant 2\|f\|_p}} \|f-\varphi\|_p \geqslant \inf_{\substack{\varphi \in A \\ \|\varphi\|_p \leqslant 2\|f\|_p}} \|f-\varphi\|_{L_p(I_N)} \\ &\geqslant \inf_{g \in L_N} \|f-g\|_{L_p(I_N)} - 2\|f\|_p \varepsilon = \lambda(1-2\varepsilon). \end{aligned}$$

So we have $E(B_p, A, L_p) \leqslant \lambda(1-2\varepsilon)$. Letting $\lambda \uparrow \lambda_1(P(D/\beta))$, $\beta \uparrow 1/n$, $\varepsilon \downarrow 0$ orderly, we get $E(B_p, A, L_p) \geqslant \lambda_1(P(nD))$. Since A is arbitrary, the proof is completed.

Remark 3. Li[16] proposed another type of infinite-dimensional Kolmogorov width. We point out that Theorem 3.1 holds true in the sense of [16].

§ 4. Optimal Interpolation of B_p in L_p

In this section we denote by $T = \{t_i\}_{i \in Z}$ a biinfinite number sequence. For ease of exposition, we also view T as a set of points. We say $T \subset \Theta$, if

(1) $t_i < t_{i+1},\ i \in Z$;

(2) $\varliminf_{a \to \infty} \dfrac{\operatorname{card}\{T \cap [-a, a]\}}{2a} \leqslant 1$;

where card$A$ denotes the cardinality of set A. The number

$$\underline{D}(T) := \varliminf_{a \to \infty} \frac{\operatorname{card}\{T \cap [-a, a]\}}{2a}$$

may be termed as the lower average density of T in R. So Θ is the collection of all denumerable subsets T with $\underline{D}(T) \leqslant 1$. For a $T \in \Theta$, we can determine a mapping I_T: $B_p \to R^Z$, by setting $I_T f = \{f(t_i)\}_{i \in Z}$ which is said to be a method of sampling or an information operator. Any mapping A: $I_T(B_p) \to L_p(R)$ may be taken to be an approximating formula (algorithm) for calculating function of B_p. Following [7], the intrinsic error of optimal interpolation problem (B_p, T, L_p) is

$$E(B_p, T, L_p) = \inf_A \sup_{f \in B_p} \|f - A(I_T f)\|_p.$$

The optimal intrinsic error is defined by

$$E(B_p, L_p)=\inf_{T\in\Theta} E(B_p, T, L_p),$$

which is the minimization of $E(B_p, T, L_p)$ when T runs over the whole collection Θ. We call some A^*, $T^*\in\Theta$ optimal algorithm and optimal set of sampling points respectively, if

$$E(B_p, L_p)=\sup_{f\in B_p}\|f-A^*(I_{T^*}f)\|_p.$$

Put $e(B_p, T, L_p)=\sup\{\|f\|_p \mid f\in B_p, I_T f=0\}$ and $e(B_p, L_p)=\inf_{T\in\Theta} e(B_p, T, L_p)$. It is known from [9] that

$$E(B_p, L_p)\geqslant e(B_p, L_p). \tag{4.1}$$

The problem to be considered includes to obtain the exact expression for $E(B_p, L_p)$, to construct optimal algorithm and to identify optimal set of sampling points. The linear optimal algorithm is especially interesting. Some spacial cases of this problem were thoroughly investigated ($p=1$[13]; $p=2$[4,18]; $p=\infty$[8,19]; $1<p<\infty$, $P(D)=D^r$[11]; etc.). By the method of [11] or [18] we have

Theorem 4.1. (1) $E(B_p, L_p)=\lambda_1(P(D))$.

(2) $T^*=\{j+\alpha_1(P(D))\}_{j\in Z}$ *is an optimal set of points of sampling.*

(3) $A^*I_{T^*}f=S_1(f, x)=\sum_{j\in Z} f(j+\alpha_1(P(D)))L_1(x-j)$ *is a linear optimal algorithm.*

Proof By Corollary 2.1, we have

$$E(B_p, L_p)\leqslant\sup_{f\in B_x}\|f-S_1(f)\|_p\leqslant\lambda_1(P(D)).$$

Therefore, we have only to prove $e(B_p, L_p)\geqslant\lambda_1(P(D))$. Given any $T\in\Theta$, we first assume that $\underline{D}(T)=1$. Then there exists a sequence of intervals $J_k=[-a_k, a_k]$, $k=1, 2, \cdots$, such that $a_k\to+\infty$ and $\lim_{k\to\infty}\frac{n_k}{2a_k}=1$, where $n_k=\text{card}\,\{T\cap J_k\}$. Put

$$S^*_{T,J_k}=\{s(t)\mid s(t)\in C^{r-2}(J_k), P(-D)s(t)=0,$$
$$\forall t\in(t_j, t_{j+1}), \text{ if } (t_j, t_{j+1})\cap J_k\neq\phi\}.$$

Then $\dim S^*_{T,J_k}\leqslant n_k+r$. By the method of [13] we can prove

$$\{P(D)f\mid f\in B_{p,J_k}(P(D))_0, I_T f=0\}=\{h\mid h\perp S^*_{T,J_k}, \|h\|_{L_p(J_k)}\leqslant 1\},$$

where $h\perp S^*_{T,J_k}$ means that $\int_{J_k} h(t)dt=0$ for all $s(t)\in S^*_{T,J_k}$ and $B_{p,J_k}(P(D))_0$ is defined in the proof of Theorem 3.1. Thus from the duality theorem of best approximation by linear subspace it follows that

$$\begin{aligned}&E(B_{p',J_k}(P(-D)), L_{p'}(J_k))\\&=\sup_{g\in B_{p',J_k}(P(-D))}\Big[\sup_{\substack{f\in B_p(P(D))_0\\ I_Tf=0}}\Big(\int_{J_k} g(t)(P(D)f)(t)dt\Big)\Big]\\&=\sup_{\substack{f\in B_p(P(D))_0\\ I_Tf=0}}\Big[\sup_{g\in B_{p',J_k}(P(-D))}\Big(\int_{J_k} f(t)(P(-D)g)(t)\,dt\Big)\Big]\\&=\sup\{\|f\|_p\mid f\in B_p, J_k(P(D))_0, I_Tf=0\}, 1/p+1/p'=1.\end{aligned}$$

By the definition of d_n (i. e., Kolmogorov n-width) we have

$$\begin{aligned}&\sup\{\|f\|_p \mid f\in B_{p,J_k}(P(D))_0,\ I_T f=0\}\\&\geqslant d_{n_k+r}(\widetilde{B}_{p',J_k}(P(-D)),\ L_{p'}(J_k))\geqslant d_{n_k+r}(\widetilde{B}_{p',J_k}(P(-D)),\ L_{p'}(J_k))\\&=d_{n_k+r}(\widetilde{B}_{p'}(P(-D/a_k),\ \widetilde{L}_{p'})\\&\geqslant\lambda_{1+\left[\frac{n_k+r}{2}\right]}(P(-D/a_k),\ p')=\lambda_1(P(([(n_k+r)/2]+1)D/a_k),\ p).\end{aligned}$$

Letting $k\to\infty$, we get

$$\lambda_1(P(D)\leqslant\sup\{\|f\|_p|f\in B_{p,J_k}(P(D))_0,\ I_T f=0\}\leqslant e(B_p,\ T,\ L_p).$$

If $\underline{D}(T)<1$, we may choose $T'\in\Theta$ such that $T\subseteq T'$ and $\underline{D}(T')=1$. Noticing $e(B_p,\ T,\ L_p)\geqslant e(B_p,\ T',\ L_p)$, we also have $e(B_p,\ T,\ L_p)\geqslant\lambda_1(P(D))$. The proof is completed by the arbitrariness of $T\in\Theta$.

Remark 4. Instead of Θ we may take Θ_h which is the collcetion of all T with $\underline{D}(T)\leqslant h$, $h>0$ fixed. By changing of scale properly we get all conclusions similar to Theorem 4.1.

The author thanks his advisor Prof. Sun Yongsheng for his valuable guidance.

References

[1] Pinkus, A., N-widths in approximation theory, Springer-verlag, 1985.

[2] Chen Dirong, On the n-width of some classes of periodic functions *Scientia Sincia*), **35**:1 (1992), 42—54.

[3] Chen Hanlin, Li Chun, Exact and asymptotic estimation for n-widths of some classes of periodic functions (*Codstructive Approx*). **8**:3 (1992), 287—307.

[4] Chen Dirong, Optimal recovery and average n-K widths for some function classes in $L_2(\mathbf{R})$ ATA, **7**:3 (1991), 45—62.

[5] Tikhomirov, V. M., Extremal problem and approximation theory, *Advance in Math.*, **19**: 4 (1990), 499—451.

[6] Magaril-Iljaev, G. G., φ-average widths of function classes in the real axis (in Russian), *YMH*, **45**: 2 (1990), 211—212.

[7] Traub, J. F. & Wozniakowski, H., A general theory of optimal algorithms, Acad. Press, N. Y., 1980.

[8] Sun Yongsheng, On optimal interpolation for a differentiable function class (I), *ATA*, **2**:4 (1986), 49—54.

[9] Pinkus, A., n-widths of Sobolev space, *Constr. Approx.*, **1**:1 (1985), 15—62.

[10] Micchelli, C. A., Cardinal $\mathscr{L}$-spline, Studies in spline functions and approximation, Acad. Press, N. Y., 1977.

[11] Li Chun, Infinite-dimensinoal widths of function classes (II) *J. Approx. Theory*, **69**: 1 (1992), 15—34.

[12] Tang Xuhui, Approximation of some classes of differentiable functions with Cardinal $\mathscr{L}$-spline on **R**, *ATA*, **3**: 2 (1987), 1—17.

[13] Li Chun, $L_1(\mathbf{R})$-optimal recovery on some differentiable function classes, Approx., Optimization and Computing, Proc. of ISAOC, Elsevier/North Holland. 1990,

[14] Stein, E. M., Functions of exponential type, *Annals of Math.*, **65**: 3 (1957), 582—592.

[15] Sun Yongsheng, Some extremal problems for the class of differentiable functions, *J. of Beijing Normvl Univ. (Natural Sciences)*, **4** (1984), 11—26

[16] Li Chun, Infinite-dimensional widths of function classes (I), *KeXue TongBao*, **35**: 3 (1990), 163—166.

[17] Micchelli, C. A., & Rivlin, T. J., A survery of optimal recovery, Ostimal Estimation in Approx. Plenum Press, N. Y., 1977.

[18] Sun Yongsheng & Li Chun, Optimal recovery for $W^r(\mathbf{R})$ in $L_2(\mathbf{R})$ *Acta Math. Sinica*, **7**:4 (1991), 309∼323.

[19] Sun Yongsheng & Huang Daren, Optimal interpolation on some class of differentiable functions, *ATA*, **4**: 3 (1988), 13—22.