Chin. Ann. of Math
13B: 4(1992), 406—410.

THE EXISTENCE OF μ-HOLOMORPHIC SEPARATING FUNCTION ON BOUNDED SMOOTH DOMAINS

YE XUAN(叶 轩)* ZHANG JINHAO(张锦豪)*

Abstract

The Complex analysis of strongly pseudoconvex domains in C^n is rather well known. In this paper it is proved that for a bounded smoothly domain Ω there is a new complex structure on it under which Ω will locally become a strongly convex even though the point on $b\Omega$ is not a pseudoconvex point from the view of the original complex structure. Particularly if Ω is a weakly pseudoconvex domain, the μ can be made sufficiently close to the original complex structure. Therefore a lot of properties of strongly pseudoconvex domains will become true on weakly pseudoconvex domains, or general domains. For example, it is proved that there is a μ-holomorphic separating function which is holomorphic under the new complex structure.

§ 0. Introduction

The complex analysis of strongly pseudoconvex domains in $\mathbf{C}^n$ is rather well known. Let $\Omega\subset\mathbf{C}^n$ be a bounded smooth domain. If $p\in b\Omega$ is a strongly pseudoconvex point, then the most important elementary fact is the existence of a local holomorphic separating function at p, i. e., there exists a holomorphic function $f(z)$ defined on a neighborhood of p such that for small ε,

$$\overline{\Omega}\cap\{z;\ |z-p|<\varepsilon,\ f(z)=0\}=\{p\}.$$

Indeed if

$$\Omega=\{z\in\mathbf{C}^n,\ r(z)<0\}$$

satisfies

$$\sum_{j,k=1}^{n}\frac{\partial^2 r}{\partial z^j\partial\bar{z}^k}(p)\xi^j\bar{\xi}^k\geqslant C\|\xi\|^2,$$

where $\sum_{j=1}^{n}\frac{\partial r}{\partial z^j}(p)\xi^j=0$, then the Levi polynomial

$$Lp(z)=\sum_{j=1}^{u}\frac{\partial r}{\partial z^j}(p)(z^j-p^j)+\sum_{j,k=1}^{n}\frac{\partial^2 r}{\partial z^j\partial z^k}(p)(z^j-p^j)(z^k-p^k)$$

Manuscript received October 31, 1990.
* Department of Mathematics, Fudan University, Shanghai 200433, China.

is the local holomorphic separating function which we need. Alternatively, there is a local biholomorphic change of coordinates near p which renders $b\Omega$ strongly convex.

The local holomorphic separating function has a lot of applications (see [5]).

(1) It is a critical step in the solution of the Levi problem.

(2) It is fundamental in the construction of integral formulas using the Cauchy-Fantappie machinery.

(3) It provides important information about optimal regularity for the $\bar\partial$-problem.

(4) It is very closely related to holomorphic peaking function which is basic for function algebraic considerations.

During a long time whenlittle is known about weakly pseudoconvex domains, people wish that smoothly bounded weakly pseudeconvex domains would be locally biholomorphically equivalent to weakly convex domains. In fact the pullback of

$$\varphi(z) = \Sigma \frac{\partial r^*(p^*)}{\partial \omega^j}(\omega^j - p^{*j}),$$

where $r^*(\omega)$ is a defining function for the convex domain, would give a weak local holomorphic separating function h_p at each point of the boundary. This would mean that

$$p \in \bar{\Omega} \cap \{z: |z-p| < \varepsilon,\ h_p(z) = 0\} \subseteq b\Omega.$$

In 1973, Kohn and Nirenberg[4] destroyed this optimistic program by proving that the origin in the boundary of the smooth pseudoconvex domain

$$\Omega = \{(z, w) \in \mathbf{C}^2: \operatorname{Re} w + |z|^8 + \frac{15}{7}|z|^2 \operatorname{Re} z^6 < 0\}$$

has no local holomorphic separating function. Indeed if h is a holomorphic in a neighborhood of 0 and $h(0) = 0$, then the zero's set of h will penetrate the boundary.

For studying the property of weakly pseudoconvex domains, Kohn[3] first defined the "point of finite type" for points on the boundaries of smoothly bounded pseudoconvex domains in $\mathbf{C}^2$ in 1972. Since then there has been a lot of development in the study of the domains of finite type (see [1]).

There are few people studying the property of general domains.

In this paper, we will prove that for a bounded smoothly domain Ω there is a new complex structure on it under which Ω will locally become a strongly convex even though the point on $b\Omega$ is not a pseudoconvex point from the view of the original complex structure. We will also prove that there is a μ-holomorphic separating function which is holomorphic under the new complex structure μ. If Ω is a weakly pseudoconvex domain, we can make our μ sufficiently close to the original complex structure.

§ 1. Definition

Let $z^1, \cdots, z^n$ be a system of complex coordinates in $\mathbf{C}^n$, $\Omega\subset\mathbf{C}^n$ be a domain with smooth boundary. Denote by $\mathbf{C}^\infty_{p,q}(\overline{\Omega})$ the space of (p, q) forms which is smooth up to the boundary. In this paper, we will use A^jB_j to represent the sum $\sum_{j=1}^{n} A^jB_j$.

Now given a tensor field μ:

$$\mu=\mu_j^k d\bar{z}^j \otimes \frac{\partial}{\partial z^k} \tag{1.1}$$

an operator ${}_\mu\bar\partial$: $\mathbf{C}^\infty_{p,q}(\overline{\Omega})\to\mathbf{C}^\infty_{p,q+1}(\overline{\Omega})$ as follow:

$${}_\mu\bar\partial=\bar\partial-\mu_j^k d\bar{z}^j\wedge\frac{\partial}{\partial z^k}. \tag{1.2}$$

Let μ satisfy following three conditions:

(I) integerability condition, namely, ${}_\mu\bar\partial\,{}_\mu\bar\partial=0$.

(II) $\det(I-\overline{M}M)\neq 0$, where M is the matrix (μ_j^k).

(III) $\mu_j^k\in\mathbf{C}^\infty(\Omega)$, $j, k=1, 2, \cdots, n$.

Any function f satisfying equation

$${}_\mu\bar\partial f=0 \tag{1.3}$$

will be called the ${}_\mu\bar\partial$-holomorphic function and any mapping F whose every component satisfies (1.3) the μ-holomorphic mapping. The concepts above are proposed by the second author in [7].

Lemma 1. 1. *The μ which satisfies conditions* (I) (II) (III) *can define a new complex structure.*

Proof Because the n equations in (1.3) and their conjugates are linearly independent, by Newlander-Nirenberg theorem (see [6] or [2]) there is a neighborhood U and a 1-1 μ-holomorphic mapping $F=(f_1, \cdots, f_n)$ defined on U. It is not difficult to verify that

$$\det J=\det(I-\overline{M}M)\cdot\det\left[\frac{\partial f_j}{\partial z^k}\right],$$

then $\det\left[\frac{\partial f_j}{\partial z^k}\right]\neq 0$ by condition (II) and $\det J\neq 0$.

Now let $\zeta_j=f_j(z)$, $j=1, \cdots, n$. Given any μ-holomorphic function, we have

$$0={}_\mu\bar\partial_j g=\frac{\partial g}{\partial\zeta_i}\cdot{}_\mu\bar\partial_j\zeta_i+\frac{\partial g}{\partial\bar\zeta_i}\,{}_\mu\bar\partial_i\bar\zeta_j=\frac{\partial g}{\partial\bar\zeta_i}(\delta_j^k-\mu_j^l\bar\mu_l^k)\partial_k\bar\zeta_i.$$

It follows from condition (II) and $\det\left[\frac{\partial\zeta_j}{\partial z_k}\right]\neq 0$ that

$$\frac{\partial g}{\partial\bar\zeta_j}=0.$$

Therefore, $(\zeta_1, \cdots, \zeta_n)$ gives a new complex coordinate system which makes g holomorphic.

§ 2. Main Result

Theorem 2. 1 (Main theorem). *Let $\Omega=\{r<0\}\subset\mathbb{C}^n$ be a bounded domain with smooth boundary. For any $p\in b\Omega$, $|dr|_p\neq 0$, there is a new complex structure μ such that p will become a strongly pseudoconvex point under the new complex structure.*

We will give the proof of the main theorem by several steps. At first, we note the condition (I) is equivalent with

$$\text{(I')}\qquad \frac{\partial\mu_j^k}{\partial\bar z^i}-\frac{\partial\mu_i^k}{\partial\bar z^j}=\sum_l\left[\mu_i^l\frac{\partial\mu_j^k}{\partial z^l}-\mu_j^l\frac{\partial\mu_i^k}{\partial z^l}\right],\quad i<j.$$

Fix $p\in b\Omega$, we may assume p is the origin and

$$\frac{\partial r}{\partial z^1}(p)=1,\quad \frac{\partial r}{\partial z^2}(p)=\cdots=\frac{\partial r}{\partial z^n}=0.$$

Lemma 2. 1. *Under the locally coordinate assumption as above, we say*

$$M=\begin{bmatrix}0&0&0&\cdots&0\\-cz_2&0&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\-cz_n&0&0&\cdots&0\end{bmatrix}$$

satisfies condition (I')(II)(III), and define a new complex structure, where c is a positive constant.

The proof is easy by Lemma 1.1.

The second author has proved in [7] if there exists a μ on a bounded smooth domain which satisfies (I)(II)(III)(2.1)

$$\Big\{\frac{\partial^2 r}{\partial z^i\partial\bar z^j}-\sum\mu_j^k\frac{\partial^2 r}{\partial z^i\partial z^k}-\sum\bar\mu_i^l\frac{\partial^2 r}{\partial\bar z^l\partial\bar z^j}+\sum\bar\mu_i^l\mu_j^k\frac{\partial^2 r}{\partial\bar z^l\partial z^k}$$
$$-({}_\mu\bar\partial_l r)({}_\mu\bar\partial_j\bar\mu_i^t)\bar a_t^l-({}_\mu\partial_l r)({}_\mu\bar\partial_i\mu_j^t)a_t^l\Big\}\xi^i\bar\xi^j\geq c'\|\xi\|^2. \tag{2.1}$$

$${}_\mu\partial_j r(p)\xi^j=0,$$

where (a_t^l) is the inverse matrix of $(I-\bar MM)$, then $p\in b\Omega$ will become a strongly pseudoconvex point under the new complex structure. Now we will prove that the μ in Lemma 2.1 will satisfy (2.1) in a neighborhood of p. We denote by ${}_\mu\mathscr{L}_{ij}$ the $\{\cdots\}$ in (2.1).

Lemma 2. 2. *μ satisfies condition (2.1), i.e., ${}_\mu\mathscr{L}_{ij}(p)\xi^i\bar\xi^j\geq c'\|\xi\|^2$, where ${}_\mu\partial_j r(p)\xi^j=0$, c' is a positive constant.*

Proof. At point p, $(a_t^l)=(I-M\bar M)^{-1}=I$, the unit matrix,

$${}_\mu\mathscr{L}_{ij}(p)=\frac{\partial^2 r}{\partial z^i\partial\bar z_j}-({}_\mu\bar\partial_1 r)({}_\mu\bar\partial_j\bar\mu_i^1)-({}_\mu\partial_1 r)({}_\mu\bar\partial_i\mu_j^1)=\frac{\partial^2 r}{\partial z^i\partial\bar z^j}+2c\cdot\delta_{ij}^*,$$

$$\delta_{ij}^*=\begin{cases}0, & i=j=1,\\ \delta_{ij}, & \text{others}.\end{cases}$$

By assumption, ${}_\mu\partial_1 r(p)=1$, ${}_\mu\partial_j r(p)=0$, $j=2,\cdots,n$, ${}_\mu\partial_j r(p)\xi^j=0\Rightarrow\xi^1=0,$

we have

$$ {}_{\mu}\mathscr{L}_{ij}\xi^i\bar{\xi}^j=\frac{\partial^2 r}{\partial z^i\,\partial \bar{z}^j}\,\xi^i\bar{\xi}^j+2c\|\xi\|^2. \tag{2.2}$$

Because $\frac{\partial^2 r}{\partial z^i\,\partial \bar{z}^j}$ is bounded, there exists $c_1>0$, $\frac{\partial^2 r}{\partial z^i\,\partial \bar{z}^j}(p)\xi^i\bar{\xi}^j\leqslant c_1\|\xi\|^2$.

We can choose c so that $2c>c_1$. Set $c'=2c-c_1>0$, then ${}_{\mu}\mathscr{L}_{ij}\xi^i\bar{\xi}^j\geqslant c'\|\xi\|^2$.

Proof of Main Theorem 2.1 Let $F=(f_1,\ \cdots,\ f_n)$ be a $1-1$ μ-holomorphic mapping defined on a neighborhood of $p\in b\Omega$ as in Lemma 1.1. Then $F(p)$ is a strongly pseudoconvex point by Lemma 1.1, Lemma 2.1 and Theorem 3 in [7].

If we notice that the first term of the right of (2.2) in the proof of Lemma 2.2 is Levi form, we can easily obtain

Theorem 2. 2. *If $p\in b\Omega$ is a weakly pseudoconvex point, we can find a new complex structure which is sufficiently close to the original one and p will become a strongly pseudoconvex p int.*

§ 3. μ-Holomorphic Separating Function

By Main Theorem we can easily obtain

Corollary 3. 1. *Assume the coordinate. f_j and μ as before. Then*

$$L_p(z)=\Sigma_{\mu}\partial_j r(p)(f_j(z)-f_j(p_j))+\Sigma_{\mu}\partial_{j\mu}\partial_k r(p)(f_j(z)-f_j(p_j))(f_k(z)-f_k(p_k))$$

is the locally μ-holomorphic separating function at p.

By noting that a complex structure is invariant under biholomorphic transformation, proceeding as in the $\bar{\partial}$-theory we have

Corollary 3. 2. *There is a locally μ-biholomorphic change of coordinates near p which render $b\Omega$ strongly conve.:.*

Remark. All we have done is locally. Is there new complex structure which can make Ω a global strongly pseudoconvex domain? It is a very interesting problem.

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