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CESÁRO ERGODIC THEOREMS FOR WEAKLY Y-INTEGRABLE OPERATOR SEMIGROUPS**

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Abstract

Cesáro ergodic properties for weakly Y-integrable semigroups of operators on Banach spaces are studied and several equivalent conditions for ergodicity are examined. Results obtained considerably generalize early works on this subject by others.

§1. Introduction

Let $\{T(t): t>0\}$ be a semigroup of bounded linear operators on a complex Banach space X. One of the important subjects of studies for $T(\cdot)$ concerns its Cesàro ergodic properties. When the semigroup $T(\cdot)$ is strongly contonuous on $(0, \infty)$, remarkable results on this subject have been achieved in the past thirty years (cf. Hille and Phillips [3, Chapter 18], Dunford and Schwartz [1, VIII. 7], Masani[6], Eberlein[2], Lin et al. [5] and others). However, not every semigroup of interest is strongly continuous. For instance, the dual semigroup of a strongly continuous semigroup and the tensor product of two strongly continuous semigroups are no longer in general strongly continuous. To extend the Cesàro eagodic theory for strongly continuous semigroups to more general case, Shaw introduced a new class of semigroups which are called locally Y-integrable (see [7, 8] for details). But in Shaw's theory, he assumed that $T(t)x \to x$ as $t \to +0$ in certain topology. This is in fact an extension of C_0-semigroups.

The author and Lange[9] introduced another class of semigroups which are called weakly Y-integrable and include locally Y-integrable semigroups as a special case. In the present paper, we shall examine the Cesàro ergodic properties for the former kind of semigroups.

In what follows we shall use these notations:

$$\mathcal{N}(T) = \text{the null space of the operator } T;$$
$$\mathcal{R}(T) = \text{the range of } T;$$
$$D(T) = \text{the domain of } T.$$

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Assume that Y is a closed subspace of the dual X^*, X and Y are reciprocal, that is,

$$\|x\| = \sup \{|\langle x, y\rangle| / \|y\|: y \in Y, y \neq 0\}$$

for each $x \in X$. A semigroup $\{T(t): t>0\}$ of bounded linear operators on X is called weakly Y-integrable if it satisfies:

(W1) Y is invariant under $T(t)^*$ for each $t>0$;

(W2) $T(\cdot)x$ is $\sigma(X, Y)$ continuous on $(0, \infty)$ for each $x \in X$;

(W3) (a) for each $x \in X$ and $y \in Y$, the function $\langle T(t)x, y\rangle$ of t is L-integrable on $[0, 1]$

(b) $\int_0^1 \langle T(t)x, y\rangle\, dt$ is $\sigma(Y, X)$ continuous with respect to $y \in Y$ for each $x \in X$ and hence $\sigma(X, Y)$ continuous with respect to $x \in X$ for each $y \in Y$ by [9];

(W4) let $X_0 = \bigcup \{T(\eta)X: \eta>0\}$. Then X_0 is $\sigma(X, Y)$ dense in X and $\bigcap \{\mathcal{N}(T(\eta)): \eta>0\} = \{0\}$.

Condition (W3) seems to be a little complicated, but as shown in [9, Proposition. 3.5], if $T(\cdot)$ satisfies (W1), (W2) and there exists a nonnegative L-integrable function $\varphi(\cdot)$ on $[0, 1]$ such that

$$\|T(t)\| \leqslant \varphi(t) \quad (\text{a. e. on } [0, 1]), \tag{1.1}$$

then (W3) holds. Clearly, if $T(\cdot)$ is bounded in a neighborhood of $t=0$, then (1.1) holds. Therefore, what Shaw considered in [7, 8] was a very special case of this paper

If $T(\cdot)$ is a weakly Y-integrable semigroup on X, the resolvent $R(\lambda)$ of $T(\cdot)$ exists for every complex number λ with $\mathrm{Re}\, \lambda > \omega_0$, where ω_0 is the type of $T(\cdot)$:

$$\omega_0 = \lim_{t\to\infty} \frac{1}{t} \log\|T(t)\|.$$

Y is invariant under $T(\cdot)^*$ and $R(\lambda)^*$, $R(\lambda)$ is injective. A_0 is defined to be the operator

$$A_0 x = Y - \lim_{t\to+0} \frac{[T(t)-I]x}{t},$$

whenever the limit on the right exists. A_0 is closable with closure A, while A satisfies $D(A) = \mathscr{R}(R(\lambda))$ and

$$AR(\lambda)x = \lambda R(\lambda)x - x \tag{1.2}$$

for each $x \in X$ and

$$R(\lambda)Ax = \lambda R(\lambda)x - x \tag{1.3}$$

for each $x \in D(A)$. A_0 and hence A is $\sigma(X, Y)$ densely defined.

Y being invariant under $T(\cdot)$, $R(\cdot)$, we denote

$$T(\cdot)' = T(\cdot)^*|Y; \; R(\cdot)' = R(\cdot)^*|Y$$

and call them for convenience the duals of $T(\cdot)$, $R(\cdot)$, respectively. The reader can find in [9] all the mentioned properties.

§ 2. Dual Semigroups of Weakly Y-Integrable Semigroups

The objective of this section is to establish the relationship between $T(\cdot)$ and its dual defined in § 1. We shall show that $T(\cdot)$ is weakly Y-integrable on X if and only if $T(\cdot)'$ is weakly X-integrable on Y. From our observation, one can easily see that the semigroup $T(\cdot)$ and its dual are on completely symmetric footing.

Theorem 2. 1. *If $T(\cdot)$ is a weakly Y-integrable semigroup on X, then $T(\cdot)'$ is a weakly X-integrable semigroup on Y.*

Conversely, assume that $\hat{T}(\cdot)$ is a weakly X-integrable semigroup on Y. Set $T(\cdot)=[\hat{T}(\cdot)]^*|X$, where X is viewed as a subspace of Y^*. Then $T(\cdot)$ is a weakly Y-integrable semigroup on X such that $\hat{T}(\cdot)$ is the dual of $T(\cdot)$ in the sense defined in § 1.

Proof Assume that $T(\cdot)$ is weakly Y-integrable. Since X, as a subspace of Y^*, is invariant under $[T(\cdot)']^*$ and $[T(\cdot)']^*|X=T(\cdot)$ by the equalities

$$\langle [T(t)']^*x, y\rangle=\langle x, T(t)'y\rangle=\langle T(t)x, y\rangle \tag{2.1}$$

for all $x\in X$ and $y\in Y$, condition (W1) for $T(\cdot)'$ is fulfilled.

From the second equality of (2.1), one can easily find that $T(\cdot)'$ is $\sigma(Y, X)$ continuous on $(0, \infty)$. Condition (W3) for $T(\cdot)'$ follows from its symmetrical character between X and Y.

Let $Y_0=\bigcup\{\mathscr{R}(T(\eta)'): \eta>0\}$. It follows from [9, Proposition 4.2] that Y_0 is $\sigma(Y, X)$ dense in Y. Now assume that $y\in Y$ satisfies $T(\eta)'y=0$ for all $\eta>0$. Then

$$\langle T(\eta)x,y\rangle=\langle x, T(\eta)'y\rangle=0$$

for all $\eta>0$ and $x\in X$. Since $X_0=\bigcup\{\mathscr{R}(T(\eta)): \eta>0\}$ is $\sigma(X, Y)$ dense in X, one has $y=0$. Therefore

$$\bigcap\{\mathscr{N}(T(\eta)'): \eta>0\}=\{0\}.$$

Condition (W4) for $T(\cdot)'$ is thus satisfied.

The second conclusion follows in a routine way, We omit the details.

Theorem 2.1 and [9, Theorem 4.6] imply the existence and injectivity of the resolvent of $T(\cdot)'$ which is clearly equal to the dual $R(\lambda)'$ of $R(\lambda)$, where λ sa-tisfies Re $\lambda>\omega_0$. Accordingly, there exists a $\sigma(Y, X)$ closed and densely defined linear operator A' such that

$$(\lambda-A')R(\lambda)'y=y \tag{2.2}$$

for all $y\in Y$ and

$$R(\lambda)'(\lambda-A')y=y \tag{2.3}$$

for all $y\in D(A')$. By definition, A' is the infinitesimal generator of $T(\cdot)'$.

Remark. (i) If $T(\cdot)$ is weakly Y-integrable on X, then we may define the operator A_0' for $T(\cdot)'$ in Y by

$$A_0'y = X\text{-}\lim_{t\to+0}\frac{T(t)'y-y}{t},$$

whenever the limit on the right exists. A_0' is linear and closable with closure A' ([9, Theorem 4.6]).

(ii) It is clear that A' is the dual of A and vice versa.

For the purpose of latter use, we need the following Theorem.

Theorem 2.2. (i) *If $T(\cdot)$ satisfies conditions $(W1)$, $(W2)$ and $(W3)$, then*

(a) *for each $t>0$, the equality*

$$\langle S(t)x,\ y\rangle = \int_0^t \langle T(s)x,\ y\rangle ds$$

defines a bounded linear operator $S(t)$ on X such that Y is invariant under $S(t)^*$, and hence $S(t)$ is $\sigma(X, Y)$ continuous for each $t>0$;

(b) *$S(\cdot)$ is continuous on $(0, \infty)$ in the uniform operator topology and $\sigma(X, Y)$ continuous at $t=0$;*

(ii) *if $T(\cdot)$ is weakly Y-integrable, then for each $t>0$ and $x\in X$, $S(t)x\in D(A)$ and*

$$AS(t)x = (T(t)-I)x. \tag{2.4}$$

If $x\in D(A)$, then

$$S(t)Ax = AS(t)x. \tag{2.5}$$

Proof (i, a) follows from [9, Lemmas 2.3, 2.4] and the sufficiency of [9, Proposition 3.4].

(i, b) The continuity of $S(\cdot)$ in the uniform operator topology on $(0, \infty)$ follows from the boundedness of $T(\cdot)$ on every closed subinterval $[a, b]$ of $(0, \infty)$, and the $\sigma(X, Y)$ continuity of $S(\cdot)$ at $t=0$ follows from the L-integrability of $\langle T(\cdot)x,\ y\rangle$ on $[0, 1]$ for each $x\in X$ and $y\in Y$.

(ii) Assume that $T(\cdot)$ is weakly Y-integrable. Let x be in $D(A_0)$. Then

$$\begin{aligned}\langle S(t)A_0x,\ y\rangle &= \int_0^t \langle T(s)A_0x,\ y\rangle ds\\ &= \int_0^t \frac{d}{ds}\langle T(s)x,\ y\rangle ds = \langle (T(t)-I)x,\ y\rangle\end{aligned} \tag{2.6}$$

for each $y\in Y$. On the other hand, for each $x\in D(A_0)$, $y\in D(A')$,

$$\begin{aligned}\langle S(t)A_0x,\ y\rangle &= \int_0^t \langle A_0T(s)x, y\rangle ds\\ &= \int_0^t \langle T(s)x,\ A'y\rangle ds = \langle S(t)x,\ A'y\rangle.\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) gives for $x\in D(A_0)$, $y\in D(A')$

$$\langle S(t)x,\ A'y\rangle = \langle (T(t)-I)x,\ y\rangle, \tag{2.8}$$

which asserts that $\langle S(t)x,\ A'y\rangle$ is a $\sigma(Y, X)$ continuous linear functional of

$y \in D(A')$ for each fixed $x \in D(A_0)$. Since A is the dual of A' by Remark (ii) of Theorem 2.2, one has $S(t)\, x \in D(A)$ and

$$\langle AS(t)x, y \rangle = \langle S(t)x, A'y \rangle. \tag{2.9}$$

Applying [9, Theorem 4.6] to $T(\cdot)'$ we assert that $D(A')$ is $\sigma(Y, X)$ dense in Y and hence separates points of X. So we have from (2.8) and (2.9)

$$AS(t)x = [T(t) - I]x \tag{2.10}$$

for each $x \in D(A_0)$.

Now let $x \in X$ be arbitrary. Then there exists a net $\{x_\alpha\} \subset D(A_0)$ such that $\{x_\alpha\} \to x$. Passing to the limit in the following equality

$$AS(t)x_\alpha = [T(t) - I]x_\alpha,$$

we see that (2.10) remains valid for each $x \in X$ by the $\sigma(X, Y)$ closedness of A and the $\sigma(X, Y)$continuity of $S(t)$, $T(t)$ for each $t > 0$.

Finally, (2.6) and the properties of A, $S(t)$, $T(t)$ assert that

$$S(t)Ax = [T(t) - I]x$$

holds for each $x \in D(A)$. The proof of the theorem is complete.

§ 3. Ergodic Properties for Weakly Y-Integrable Semigroups

This section is devoted to Cesàro ergodic properties of weakly Y-integrable semigroups. Our results considerably generalize those of [2, 5, 6, 7] and will be used in the forthcoming papers by the author.

Let $T(\cdot)$ be a weakly Y-integrable semigroup of operators on X. It has been shown in § 2 that the dual $T(\cdot)'$ is a weakly X-integrable semigroup of operators on Y.

Under conditions(W1)—(W4)on $T(\cdot)$, the linear operator $S(t)$ has remarkable properties listed in Theorem 2.3. The Cesàro average of $T(\cdot)$ over $(0, t]$ is suitably defined to be the operator $t^{-1}S(t)$ and what we are interested in are the Cesàro ergodic properties of $T(\cdot)$, that is, the convergence of $t^{-1}S(t)$ in certain topology as $t \to \infty$. Let P_S be the operator defined by $P_S\, x = S - \lim_{t\to\infty} t^{-1}S(t)x$ with the domain $D(P_S)$ consisting of all x for which the limit exists in the strong topology of X. Also let P_W and P_Y be operators similarly defined with the limit replaced by the weak limit W-lim and the $\sigma(X, Y)$ limit Y-lim.

To prove the main theorem of this section, we begin with two lemmas. It is well known that $\mathscr{R}(\lambda R(\lambda) - I)$ and $\mathscr{N}(\lambda R(\lambda) - I)$ are independent of the choice of λ with Re $\lambda > \omega_0$.

Lemma 3.1. *For the weakly Y-integrable semigroup $T(\cdot)$, the following* *assertions hold:*

(i) $\mathcal{N}(A)=\bigcap\{\mathcal{N}(T(t)-I):\ t>0\}=\mathcal{N}(\lambda R(\lambda)-I)$;

(ii) $\mathcal{R}(A)=\mathcal{R}(\lambda R(\lambda)-I)$;

(iii) $\overline{\mathcal{R}(A)}^{Y}=\overline{\bigcup\{\mathcal{R}(T(t)-I):\ t>0\}}^{Y}$, *where* $\overline{E}^{Y}$ *is the closure of the set* E *in the* $\sigma(X, Y)$ *topology.*

Proof We only claim (i). The proof of others will be omitted. Let $x\in\bigcap\{\mathcal{N}(T(t)-I):\ t>0\}$. Then $T(t)x-x=0$ for each $t>0$ and hence $A_0x=0$ by the definition of A_0. A being the closure of A_0, one has $Ax=0$. So $\mathcal{N}(T(t)-I)\subset\mathcal{N}(A)$. Conversely, assume that $x\in\mathcal{N}(A)$. It follows from Theorem 2.2 (ii) that

$$[T(t)-I]x=S(t)Ax=0$$

for each $t>0$. Consequently,

$$\mathcal{N}(A)=\bigcap\{\mathcal{N}(T(t)-I):\ t>0\}. \tag{3.1}$$

Next, assume that $x\in\bigcap\{\mathcal{N}(T(t)-I):t>0\}$. Since the equality

$$\langle\lambda R(\lambda)x-x,\ y\rangle=\int_0^\infty \lambda e^{-\lambda t}\langle T(t)x-x,\ y\rangle\, dt=0$$

holds for each $y\in Y$, one has $\lambda R(\lambda)x-x=0$ and hence $x\in\mathcal{N}(\lambda R(\lambda)-I)$. Thus the inclusion

$$\bigcap\{\mathcal{N}(T(t)-I):\ t>0\}\subset\mathcal{N}(\lambda R(\lambda)-I) \tag{3.2}$$

holds.

To prove the opposite inclusion of (3.2), let x be in $\mathcal{N}(\lambda R(\lambda)-I)$. The equalities $\lambda R(\lambda)x=x$ and $D(A)=\mathcal{R}(R(\lambda))$ imply that $x\in D(A)$. From (1.2), one has

$$R(\lambda)Ax=[\lambda R(\lambda)-I]x=0.$$

The injectivity of $R(\cdot)$ concludes that $Ax=0$ or equivalently,

$$\mathcal{N}(\lambda R(\lambda)-I)\subset\mathcal{N}(A). \tag{3.3}$$

(3.1)—(3.3) complete the proof of (i).

Since $\mathcal{N}(A)$, $\bigcap\{\mathcal{N}(T(t)-I):\ t>0\}$ and $\mathcal{N}(\lambda R(\lambda)-I)$ are the same, we shall use the notation $\mathcal{N}$ to denote them.

The following lemma is similar to [7, Lemma 3.1].

Lemma 3. 2[7]. (i) *The operators* P_S, P_W, P_Y *defined at the beginning of this section are projections with* $\mathcal{R}(P_S)=\mathcal{R}(P_W)=\mathcal{R}(P_Y)=\mathcal{N}$;

(ii) $\mathcal{N}(P_S)\subset\overline{\mathcal{R}(A)}$, *where* "$\overline{E}$" *stands of the normal closure of the set* E.

Proof We only sketch out the proof of (ii), that of (i) will be omitted. From Theorem 2.3, $S(t)$ is bounded on every closed subinterval $[a,\ b]$ of $[0,\ \infty)$ and continuous on $(0,\ \infty)$ in the uniform operator topology, so for each $x\in X$ and $t>0$, the Boohner integral on the right of

$$F(t)x=t^{-1}\int_0^t S(u)\,x du \tag{3.4}$$

exists and $F(t)$ is a bounded linear operator on X for each $t>0$. Moreover, one can show that

$$AF(t)x=t^{-1}S(t)x-x. \tag{3.5}$$

If x is in $\mathcal{N}(P_S)$, then $S-\lim_{t\to\infty} t^{-1}S(t)x=P_Sx=0$. Thus (3.5) implies

$$x=S-\lim_{t\to\infty}[t^{-1}S(t)x-AF(t)x]\in\overline{\mathcal{R}(A)}. \tag{3.6}$$

(ii) holds.

Theorem 3. 3. *For the weakly Y-integrable semigroup $T(\cdot)$, if*

$$\overline{\lim_{t\to\infty}}\, t^{-1}\|S(t)\|<\infty \tag{3.7}$$

holds, then the following statements are equivalent:

(i) *for each $x\in X$ and $u>0$,*

$$S-\lim_{t\to\infty} t^{-1}T(t)S(u)x=0; \tag{3.8}$$

(ii) $\mathcal{N}(P_S)=\overline{\mathcal{R}(A)}$;

(iii) *for each $x\in D(A)$,*

$$S-\lim_{t\to\infty} t^{-1}T(t)x=0. \tag{3.9}$$

Proof (i)⇒(ii). Assume that (3.8) holds. To prove (ii), it suffices to show the opposite inclusion of Lemma 3.2 (ii). For each $x\in X$, $y\in Y$, $t>0$ and λ with $\operatorname{Re}\lambda>\max\{2\omega_0, 0\}$, one has

$$\begin{aligned}&\langle t^{-1}S(t)[\lambda R(\lambda)-I]x, y\rangle\\&=\int_0^\infty \lambda e^{-\lambda u}\langle t^{-1}S(t)[T(u)-I]x, y\rangle du\\&=\left[\int_0^N+\int_N^\infty\right]\lambda e^{-\lambda u}\langle t^{-1}S(t)[T(u)-I]x, y\rangle du.\end{aligned}$$

We may choose $N>0$ such that

$$\|T(u)-I\|\leqslant e^{\lambda u/2}$$

whenever $u\geqslant N$. (3.7) implies the existence of $M>0$ such that $t^{-1}\|S(t)\|\leqslant M$ whenever t is sufficiently large. Therefore we may choose N such that

$$\begin{aligned}&\left|\int_N^\infty \lambda e^{-\lambda u}\langle t^{-1}S(t)[T(u)-I]x, y\rangle du\right|\\&\leqslant M\left(\int_N^\infty \lambda e^{-\lambda u/2}du\right)\|x\|\,\|y\|=2Me^{-\lambda N/2}\|x\|\,\|y\|<\varepsilon\|x\|\,\|y\|,\end{aligned} \tag{3.10}$$

where $\varepsilon>0$ is given.

Next, we consider the integral over $[0, N]$. At first, the equality

$$S(t)[T(u)-I]=[T(t)-I]S(u)$$

shown in [7, Lemma 2.3] and condition (i) give

$$\begin{aligned}&S-\lim_{t\to\infty} t^{-1}S(t)[T(u)-I]x\\&=S-\lim_{t\to\infty} t^{-1}[T(t)-I]S(u)x=0\end{aligned} \tag{3.11}$$

for each $u\in[0, N]$ and $x\in X$. Secondly, the equality

$$t^{-1}T(t)S(u)=t^{-1}[S(t+u)-S(t)]$$

and (3.7) imply

$$\|t^{-1}T(t)S(u)\|\leqslant t^{-1}[\|S(t+u)\|+\|S(t)\|]$$
$$\leqslant t^{-1}(t+u)M+M\leqslant 3M \tag{3.12}$$

whenever t is sufficiently large. (3.12) shows that the Lebesgue's dominated convergence theorem is applicable to the integral over $[0, N]$ and leads to the following inequality by (3.11):

$$\left|\int_0^N \lambda e^{-\lambda u}\langle t^{-1}S(t)[T(u)-I]x, y\rangle du\right|\leqslant \varepsilon\|y\| \tag{3.13}$$

for each fixed $x\in X$ whenever t is suffuciently large.

(3.10) and (3.13) assert that

$$|t^{-1}\langle S(t)[\lambda R(\lambda)-I]x, y\rangle|\leqslant\varepsilon(\|x\|+1)\|y\|$$

and hence

$$\|t^{-1}S(t)[\lambda R(\lambda)-I]x\|\leqslant\varepsilon(\|x\|+1) \tag{3.14}$$

for each fixed $x\in X$ whenever t is sufficiently large. (3.14) implies the following inclusion

$$\overline{\mathscr{R}[\lambda R(\lambda)-I]}\subset\mathscr{N}(P_S),$$

which together with Lemma 3.2 (ii) and Lemma 3.1 (ii) completes the argument (i)⇒(ii).

(ii)⇒(iii). Let x be in $D(A)$. It follows from the equality $\mathscr{N}(P_S)=\overline{\mathscr{R}(A)}$ that

$$0=P(Ax)=S-\lim_{t\to\infty}t^{-1}S(t)Ax=S-\lim_{t\to\infty}t^{-1}[T(t)-I]x. \tag{3.15}$$

Therefore

$$S-\lim_{t\to\infty}t^{-1}T(t)x=0.$$

Inclusion (iii)⇒(i) is evident since $\mathscr{R}(S(u))\subset D(A)$.

Corollary 3.1. *Let $T(\cdot)$ be as in Theorem 3.3. Then $D(P_S)$ is the direct sum of $\mathscr{N}(A)$ and $\overline{\mathscr{R}(A)}$ or equivalently, $\mathscr{R}(P_S)=\mathscr{N}(A)$, $\mathscr{N}(P_S)=\overline{\mathscr{R}(A)}$ if and only if one of (3.8) and (3.9) holds.*

Proof It suffices to verify the "only if" part. If $D(P_S)$ is the direct sum of $\mathscr{N}(A)$ and $\overline{\mathscr{R}(A)}$, then Lemma 3.2 together with the following equality

$$\mathscr{N}(A)\oplus\overline{\mathscr{R}(A)}=\mathscr{R}(P_S)\oplus\mathscr{N}(P_S)$$

concludes that $\mathscr{N}(P_S)=\overline{\mathscr{R}(A)}$. Hence (3.8) and (3.9) hold by Theorem 3.3.

Corollary 3.2. *Let $T(\cdot)$ be as in Theorem 3.3. If one of (i)—(iii) of Theorem 3.3 holds, then $P_S=P_W$.*

Proof From Lemma 3.2 (i) and the evident inclusion $\mathscr{N}(P_S)\subset\mathscr{N}(P_W)$, it suffices to show that $\mathscr{N}(P_W)\subset\mathscr{N}(P_S)$. Let x be in $\mathscr{N}(P_W)$. Then

$$x = W-\lim_{t\to\infty}[x - t^{-1}S(t)x]$$
$$= W-\lim_{t\to\infty}[-AF(t)x] \in \overline{\mathscr{R}(A)} = \mathscr{N}(P_S).$$

In the following we consider the locally integrable case and operator P_W. Under the condition (3.7) on $S(t)$, it is easily seen that (3.10) and (3.12) remain valid for this case. Let $x \in \mathscr{N}(P_W)$. Then (3.6) becomes

$$x = W-\lim_{t\to\infty}[t^{-1}S(t)x - AF(t)x] \in \overline{\mathscr{R}(A)},$$

and Lemma 3.2 (ii) becomes

$$\mathscr{N}(P_W) \subset \overline{\mathscr{R}(A)}. \tag{3.16}$$

As regards the replacement of condition (3.8), we shall use

$$W-\lim_{t\to\infty} t^{-1}T(t)S(u)x = 0 \tag{3.17}$$

for each $x \in X$. Then (3.11) becomes

$$W-\lim_{t\to\infty} t^{-1}S(t)[T(u)-I]x = 0$$

for each $u \in [0, N]$ and $x \in X$. (3.13) thus becomes

$$\left|\int_0^N \lambda e^{-\lambda u}\langle t^{-1}S(t)[T(u)-I]x, y\rangle\, du\right| \leqslant \varepsilon \tag{3.18}$$

for fixed $x \in X$ and $y \in X^*$ whenever t is sufficiently large. Therefore, one has from (3.10) and (3.18)

$$|\langle t^{-1}S(t)[\lambda R(\lambda) - I]x, y\rangle| \leqslant \varepsilon(\|x\|\|y\| + 1) \tag{3.19}$$

for fixed $x \in X$ and $y \in X^*$ whenever t is sufficiently large. (3.19) implies

$$\overline{\mathscr{R}[\lambda R(\lambda) - I]} \subset \mathscr{N}(P_W),$$

which together with (3.16) and Lemma 3.1 (ii) gives

$$\mathscr{N}(P_W) = \overline{\mathscr{R}(A)}. \tag{3.20}$$

Next, assume that (3.20) holds. Then the following analogue of (3.15) is clear:

$$0 = P_W(Ax) = W-\lim_{t\to\infty} t^{-1}[T(t) - I]x = W-\lim_{t\to\infty} t^{-1}T(t)x \tag{3.21}$$

for each $x \in D(A)$. Finally, (3.21) evidently implies (3.17). We summarize the above observation in

Theorem 3.4. *For the locally integrable semi-group $T(\cdot)$, if (3.7) holds, then the following statements are equivalent:*

(i) *for each $x \in X$ and $u > 0$,*

$$W-\lim_{t\to\infty} t^{-1}T(t)S(u)x = 0;$$

(ii) $\mathscr{N}(P_W) = \overline{\mathscr{R}(A)}$;

(iii) *for each $x \in D(A)$,*

$$W-\lim_{t\to\infty} t^{-1}T(t)x = 0. \tag{3.22}$$

We say that the semigroup $T(\cdot)$ is strongly (resp. weakly) Cesàro ergodic, if $D(P_S)$ (resp. $D(P_W)$) $= X$. Let (a), (b), (c) stand for (3.7), (3.8), (3.9), respectively, and let (b′), (c′) stand for (3.17), (3.22), respectively. Let (d) denote the condition that $\mathcal{N}(A)$ separates $\mathcal{R}(A)^\perp$, that is, $\mathcal{N}(A)^\perp \cap \mathcal{R}(A)^\perp = \{0\}$, where the notation $E^\perp$ denotes the annihilator of $E \subset X$ in the dual X^*. Let (e) stand for the condition that for each $x \in X$, there exists a sequence $\{t_n\} \to \infty$ such that $W-\lim\limits_{n\to\infty} t_n^{-1} S(t_n) x$ exists.

From Theorems 3.3, and 3.4, we may deduce the following theorem.

Theorem 3.5. *Let $T(\cdot)$ be a weakly Y-integrable (resp. locally integrable) semigroup. Then the following statements are equivalent:*

(i) *$T(\cdot)$ is strongly (resp. weakly) Cesàro ergodic;*

(ii) (a), *one of* (b) *and* (c) (*resp. one of* (b′) *and* (c′)), (d) *hold;*

(iii) (a), *one of* (b) *and* (c) (*resp. one of* (b′) *and* (c′)), (e) *hold.*

Proof We complete the proof by showing the equivalences: (i) ⇔ (ii); (i) ⇔ (iii). Only considered is the strong case.

(i) ⇒ (ii). Condition (a) follows from the uniform boundedness principle. (b) and hence (c) follows from the following calculation, Lemma 3.2 and Theorem 3.3:

$$S-\lim_{t\to\infty} t^{-1} T(t) S(u) x = S-\lim_{t\to\infty} [T(u)-I] t^{-1} S(t) x = [T(u)-I] P_S x = 0.$$

(d) follows from Lemma 3.2, Theorem 3.3 (ii) and $D(P_S) = X$.

(ii) ⇒ (i). (a) implies that P_S is bounded. (b) or (c) and Lemma 3.2 imply that

$$\mathcal{R}(P_S) = \mathcal{N}(A), \quad \mathcal{N}(P_S) = \overline{\mathcal{R}(A)} \tag{3.23}$$

and hence one has from (d) that

$$[\mathcal{R}(P_S) \oplus \mathcal{N}(P_S)]^\perp = [\mathcal{N}(A) \oplus \mathcal{R}(A)]^\perp,$$
$$\mathcal{N}(A)^\perp \cap \mathcal{R}(A)^\perp = \{0\}.$$

Therefore, $\mathcal{R}(P_S) \oplus \mathcal{N}(P_S) = X$, that is, $D(P_S) = X$.

(i) ⇒ (iii). Evident.

(iii) ⇒ (i). Let $x \in X$ be fixed and let $x_1 = W-\lim\limits_{n\to\infty} t_n^{-1} S(t_n) x$. Then

$$\begin{aligned}(T(u)-I)x_1 &= W-\lim_{n\to\infty} t_n^{-1} [T(u)-I] S(t_n) x \\ &= W-\lim_{n\to\infty} t_n^{-1} [T(t_n)-I] S(u) x = 0\end{aligned}$$

and hence $x_1 \in \mathcal{N} = \mathcal{N}(A) = \mathcal{R}(P_S)$ by Lemma 3.1 and (3.23). On the other hand, (3.5) implies that

$$x - x_1 = W-\lim_{n\to\infty} [-A\, F(t_n) x] \in \overline{\mathcal{R}(A)},$$

or equivalently, $x - x_1 \in \mathcal{N}(P_S)$. Therefore,

$$x = (x - x_1) + x_1 \in \mathcal{N}(P_S) \oplus \mathcal{R}(P_S) = D(P_S).$$

Since x is arbitrary, one has $X = D(P_S)$.

Remark. (i) If one compares Theorems 3.3, 3.4, 3.5 with the corresponding results of [2, 5, 6, 7, etc.], it is easily seen that our results are considerable generalizations of theirs. For instance, what we have obtained are equivalent conditions which, perhaps, first appear here.

(ii) For the weakly Y-integrable semigroup $T(\cdot)$, it has been shown in Theorem 2.1 that $T(\cdot)'$ is a weakly X-integrable semigroup on Y, Therefore, similar results of Theorems 3.3, 3.4, 3.5 hold for $T(\cdot)'$. Moreover, we may also consider the Cesàro ergodic properties for semigroups when X is reflexive.

(iii) Left open is the question whether the equality $P_S = P_W$ still holds under the conditions of Theorem 3.4.

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