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INVARIANT SUBSPACES AND EIGEVALUES OF ELEMENTS IN C*-ALGEBRAS

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Abstract

Let A be a C^*-algebra and x an element in A. the following invariant subspace problem is considered: Does there exist an irreducible representation π of A such that $\pi(x)$ has a non-trivial invarint subspace? And a positive solution of the problem for finite separable matroid C^*-algebras is given. Also the eigenvalues of elements in C^*-algebras is considered. Some versions of Fredholm Alternatives are given.

§ 1.

Let H be a separable Hilbert space, $B(H)$ the C^*-algebra of all bounded linear operators on H. The invariant subspace problem on H is: Does every operator in $B(H)$ has a non-trivial invariant subspace? Let A be a non-abelian C^*-algebra and x an element in A. We will consider the following problem: Does there exist a separable irreducible representation π of A such that $\pi(x)$ has a non-trivial invariant subspace?

Lemma 1.1. *Let A be a separable simple C^*-algebra and x an element in A. Then there is a faithful, separable and irreducible representation π of A such that $\pi(x)$ has a non-trivial invariant subspace if one of the following holds.*

(1) *A is unital.*

(2) *x is not a quasi-nilpotent.*

Proof (1) We may assume that $x\neq\alpha\cdot 1$ for any scalar α. Suppose that $\lambda\in \operatorname{sp}(x)$. Then either $(\lambda-x)^*(\lambda-x)$ or $(\lambda-x)(\lambda-x)^*$ is not invertible. We first assume that $(\lambda-x)^*(\lambda-x)$ is not invertible. Let A_0 be the abelian C^*-subalgebra generated by 1 and $(\lambda-x)^*(\lambda-x)$. By Gelfand representation, there it a pure state ρ of A_0 such that

$$\rho((\lambda-x)^*(\lambda-x))=0.$$

We can extend ρ to a pure state $\tilde{\rho}$ of A. Let π be the faithful, separable and irreducible representation associated with $\tilde{\rho}$. Then there is $\xi\in H_\pi$ with $\|\xi\|=1$ such that

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$$\pi(\lambda-x)\xi=0, \text{ i. e., } \pi(x)\xi=\lambda\xi.$$

Hence $\pi(x)$ has a non-trivial invariant subspace.

Now we assume that $(\lambda-x)(\lambda-x)^*$ is not invertible. By the same argument, there is a faithful, separable and irreducible representation π of A such that

$$H_\pi \neq \ker \pi(\bar{\lambda}-x^*) \neq \{0\}.$$

Thus $[\operatorname{Ker} \pi(\bar{\lambda}-x^*)]^\perp$ is a non-trivial invariant subspace for $\pi(x)$.

(2) Suppose that there is a non-zero number $\lambda \in \operatorname{Sp}(x)$. We consider unital C^*-algebra $\widetilde{A}$. By the argument used in (1), there is a separable irreducible representation π of $\widetilde{A}$ such that

$$H_\pi \neq \ker \pi(\lambda-x) \neq \{0\} \text{ or}$$
$$H_\pi \neq \ker \pi(\bar{\lambda}-x^*) \neq \{0\}.$$

Since $\lambda \neq 0$, we see $\pi(x) \neq 0$. Thus $\pi|_A$ is a faithful, separable and irreducible representation of A and $\pi(x)$ has a non-trivial invariant subspace.

Corollary 1. 1. *Let $a \in B(H)$, where H is a separable Hilbert space. Then there is a C^*-subalgebra A of $B(H)$ containing a such that there is an infinite (but separable) dimensional irreducible representation π of A such that $\pi(a) \neq 0$ and $\pi(a)$ has a non-trivial invariant subspace.*

Proof Let K be the C^*-subalgebra of $B(H)$ consisting of compact operators. We may assume that $a \notin K$. Let ϕ: $B(H) \to B(H)/K$ be the canonical homomorphism. By [2, Proposition 7.], there is a separable simple C^*-subalgebra B of $B(H)/K$ containing $\phi(1)$ and $\phi(a)$. If B is of finite dimension, then a is polynomially compact. Thus we may assume that B is infinite dimensional. By Lemma 1.1 there is faithful separable irreducible representation π of B such that $\pi(\phi(a))$ has a non-trivial invariant subspace. Let $A=\phi^{-1}(B)$. Then $\pi\cdot\phi$ is a separable irreducible representation of A.

Lemma 1. 2. *Let A be a separable C^*-algebra, f a state of A and $\bar{f}$ the normal extension of f to A^{**}. Suppose that x_n is in A, y_n and z_n are in A^{**} and $x_n=y_n+z_n$. If furthermore $\bar{f}(y_n^*y_n)=\bar{f}(z_nz_n^*)=0$ and each $a_n=y_n^*y_n+z_nz_n^*$ is a Borel affine function on Q, the quasi-state space, then there is a pure state p of A such that*

$$p(x_n)=0 \quad \text{for all } n.$$

Proof Since A is separable, the quasi-state space Q is metrizable. By the Choquet theorem ([1, Corollary 1.49]), there is a positive probability mersure v on Q concentrated on the pure state space $P(A)$ such that

$$f(a)=\int a dv \text{ for all a in } A_{s.a.}.$$

Since each a_n is a bounded Borel affine function on Q,

$$\bar{f}(a_n)=\int a_n dv=0 \text{ for all } n.$$

Hence there are $E_n\subset P(A)$ such that $v(E_n)=0$ and $\tilde{p}(a_n)=0$ for all p in $P(A)\setminus E_n$. Let $E=\bigcup_{n=1} E_n$. Then $v(E)=0$. Thus if p is in $P(A)\setminus E$, $\tilde{p}(a_n)=0$ for all n. By the Cauchy-Schwarz inequality, it is evident that $p(x_n)=0$ for all n, if p is $P(A)\setminus E$.

Recall that a C^*-algebra A is called matroid if for every $\varepsilon>0$ and $a_1, a_2, \cdots, a_n\in A$, there exists a C^*-subalgebra B of A, which is isomorphic to a finite dimensional matrix algebra and $x_1, x_2, \cdots, x_n\in B$ such that

$$\|a_i-x_i\|<\varepsilon,\ i=1, 2, \cdots, n.$$

Dixmier showed [3] that if A is a separable matroid C^*-algebra, then there are integers

$$0<q_1\leqslant r_1\leqslant q_2\leqslant r_2\leqslant\cdots$$

such that A is the norm closure of the following inductive limit:

$$M_{q_1}\xrightarrow{g_{p_1r_1}} M_{r_1}\xrightarrow{f_{r_1q_2}} M_{q_2}\xrightarrow{g_{q_2r_2}} M_{r_2}\xrightarrow{f_{r_2q_3}} M_{q_3}\cdots,$$

where $q_n|r_n$ and f_{mn} are homomorphisms consisting in adding $n-m$ rows and columns of zeros to each matrix in M_m, $g_{mn}=1\otimes 1_p$ and are specified:

$$g_{mn}(x)=\begin{bmatrix} x & & 0 \\ & x & \\ 0 & & \ddots \\ & & & x\end{bmatrix}_{p\times p},$$

where $p=n/m$.

A separable matroid C^*-algebra A is called finite if A has a finite trace, or equivalently $\prod_{i=1}^{\infty}\frac{r_i}{q_{i+1}}>0$ [3].

Theorem 1.4. *Let A be a separable finite matroid C^*-algebra and x an element in A. Then there is a faithful, separable irreducible representation π of A such that $\pi(x)$ has a non-trivial invariant subspace.*

Proof Since A is simple [3], by Lemma 1.1, we may assume that A has no identity. Since A is finite, we may also assume that

$$1<q_1<r_1<q_2<r_2<\cdots.\quad (\text{Thus } r_1/q_1>1.)$$

We will identify M_{r_n}, M_{q_n} with the inductive limits of M_{r_n} and M_{q_n} in A.

Fix $x\in A$, by Lemma 1.1, we may assume that x is a quasi-nilpotent. There are $x_n\in M_{r_n}$ such that

$$\lim_{n\to\infty}\|x_n-x\|=0.$$

For each n, there is a unitary element $U_n\in M_{r_n}$ such that $U_n^*x_nU_n$ is an upper triangular matrix. Let e_n be the identity for M_{r_n}, $n=1, 2, \cdots$. Notice that $e_1\in M_{r_n}$ for all n. Let $U_n^*e_1U_n=(a_{ij}^{(n)})$, an $r_n\times r_n$ matrix. Since

$$\frac{1}{r_n}\operatorname{Tr}(e_1)=\prod_{i=1}^{n-1} r_i/q_{i+1}\geqslant\prod_{i=1}^{\infty} r_i/q_{i+1}>0,$$

there is an integer $i(n)$ such that

$$a^{(n)}_{i(n)i(n)} \geq \prod_{i=1}^{\infty} r_i/q_{i+1}.$$

Take an $r_n \times r_n$ matrix $e^{(1)}_n = (b_{ij})$ with $b_{i(n)i(n)} = 1$ and other $b_{ij} = 0$. Let $P^{(1)}_n = U_n e^{(1)}_n U^*_n$. Then $P^{(1)}_n$ is a minimal projection in M_{r_n}. For every $a \in M_{r_n}$, $P^{(1)}_n\, a\, P^{(1)}_n = \lambda_n(a) P^{(1)}_n$, where $\lambda_n(a)$ is a constant. Moreover, $a \to \lambda_n(a)$ is a state of M_{r_n}. We also notice that $\lambda_n(e_1) = a^{(n)}_{i(n)i(n)} \geq \prod_{i=1}^{\infty} r_i/q_{i+1}$. Since x is a quasi-nilpotent, we may further assumi that the diagonals of the matrices $U^*_n x^i_n U_n$ are zeros for all i. So $\lambda_n(x^i_n) = 0$, for all i.

For each n, let $m(n) = r_{n+1}/q_n$, $l(n) = q_n - r_n$. In M_{r_1}, $P^{(1)}_1$ has the form

$$\underbrace{P^{(1)}_1 \oplus P^{(1)}_1 \oplus \cdots \oplus P^{(1)}_1}_{m(1)} \oplus 0_{l(1)}.$$

We take $p^{(2)}_1 = p^{(1)}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus 0_{l(1)}$. If $p^{(k)}_1$ is taken from M_{r_k}, we take

$$p^{(k+1)}_1 = \underbrace{p^1_{(k)} \oplus 0 \oplus \cdots \oplus 0}_{m(k)} \oplus 0_{l(k)}.$$

Thus $\{p^{(k)}_1\}$ is a sequence of decreasing projections and each $p^{(k)}_1$ is a minimal projection in M_{r_k}. Let $p^{(k)}_1 \downarrow p_1 \in (A^{**})_+$. Then p_1 is an upper semi-continuous function on the quasi-state space Q of A. It follows by a standard compactness argument that p_1 is of norm 1. Hence p_1 is a non-zero projection in A^{**}. Since each $p^{(k)}_1$ is a minimal projection in M_{r_k} and $\bigcup_k M_{r_k}$ is dense in A, one can easily check that p_1 is a minimal projection in A^{**}.

Now each λ_n gives an irreducible representation π_n of M_{r_n}. Let ξ: $M_{r_n} \to H_{\pi_n}$ be the GNS construction. Then $\xi_{p^{(1)}_n} \perp \xi_{x^i_n}$. There is a projection $q^{(1)}_n$ in M_{r_n} such that

$$\pi_n(q^{(1)}_n)\xi_{p^{(1)}_n} = 0 \text{ and } \pi_n(q^{(1)}_n)\xi_{x^i_n} = \xi_{x^i_n}.$$

Thus $\lambda_n(q^{(1)}_n) = 0$ and

$$\lambda_n[(x^i_n - q^{(1)}_n x^i_n)^*(x^i_n - q^{(1)}_n x^i_n)] = 0.$$

For every k, let $p^{(k)}_1\, a\, p^{(k)}_2 = \eta^{(k)}_1(a)\, p^{(k)}_2$. We see that $\eta^{(k)}_1(a)$ is a pure state for M_{r_k}. Moreover, $\eta^{(k)}_1(b) = \lambda_1(b)$ if $b \in M_{r_1}$. In M_{r_1}, $q^{(1)}_1$ has the form

$$\underbrace{q^{(1)}_1 \oplus q^{(1)}_1 \oplus \cdots \oplus q^{(1)}_1}_{m(1)} \oplus 0_{l(k)}.$$

Let us take $q^{(2)}_1 = q^{(1)}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus 0_{l(1)}$. If $q^{(k)}_1$ is taken in M_{r_k}, we take

$$q^{(k+1)}_1 = \underbrace{q^{(k)}_1 \oplus 0 \oplus \cdots \oplus 0}_{m(k)} \oplus 0_{l(k)}.$$

Then it is easy to see that $\{q^{(k)}_1\}$ is a sequence of decreasing projections such that

$$\eta^{(k)}_1(q^{(j)}_1) = 0 \text{ and}$$

$$\eta^{(k)}_1[(x^i_1 - q^{(j)}_1 x^i_1)^*(x^i_1 - q^{(j)}_1 x^i_1)] = 0$$

for all $j \leq k$ and $i = 1, 2, \cdots$. Moreover, as the above, $\{q^{(k)}_1\}$ converges strongly to a non-zero projection q_1 in A^{**}. Let $f_1(a)$ be a state defined by $p_1ap_1=f_1(a)\ p_1$. It is easy to see that $\eta_1^{(k)}(a)=f_1(a)$ for all $a\in M_{r_k}$. We conclude that $f_1(q_1)=0$ and

$$f_1[(x_1^i-q_1x_1^i)^*(x_1^i-q_1x_1^i)]=0,\ \text{for all } i,$$

since $(x_1^i-q_1^{(j)}x_1^i)^*(x_1^i-q_1^{(j)}x_1^i)=(x_1^*)^i(x_1)^i-(x_1^*)^iq_1^{(j)}(x_1)^i$ converges strongly to $(x_1^*)^i(x_1)^i-(x_1^*)^iq_1(x_1)^i=(x_1^i-q_1x_1^i)^*(x_1^i-q_1x_1^i)$.

We can construct two sequences of decreasing projections $\{p_n^{(k)}\}$ and $\{q_n^{(k)}\}$ satisfying

(1) $p_n^{(k)}$ is a minimal projection in $M_{r_{n+k-1}}$ and $p_n^{(k)}\downarrow p_n$, where p_n is a minimal projection in A^{**}.

(2) $p_n^{(k)}\perp p_m^{(n-m+k)}$ and $p_n^{(k)}\perp q_m^{(n-m+k)}$ if $n>m$, hence $p_n\perp p_m$ and $p_n\perp q_m$.

(3) $q_n^{(k)}$ is a projection in $M_{r_{n+k-1}}$ and $q_n^{(k)}\downarrow q_n$, where q_n is a non-zero projection in A^{**}.

(4) $q_n^{(k)}\perp q_m^{(n-m+k)}$ and $q_n^{(k)}\perp p_m^{(n-m+k)}$ if $n>m$, hence $q_n\perp q_m$ and $q_n\perp p_m$.

Furthermore (let $\eta_n^{(k)}$ be the pure state of $M_{r_{n+k-1}}$ defined by $p_n^{(k)}ap_n^{(k)}$ $\eta_n^{(k)}(a)$ $p_n^{(k)}$),

(5) $\eta_n^{(k)}(a)=\lambda_n(a)$ if $a\in M_{r_n}$,

(6) $\eta_n^{(k)}(q_n^{(j)})=0$ and $\eta_n^{(k)}[(x_n^i-q_n^{(j)}x_n^i)^*(x_n^i-q_n^{(j)}x_n^i)]=0$ if $k\geqslant j$ and $i=1, 2, \cdots$.

To construct $\{p_n^{(k)}\}$ and $\{q_n^{(k)}\}$, we assume that $p_n^{(1)}$ and $q_n^{(1)}$ have the forms

$$p_n^{(1)}\oplus p_n^{(1)}\oplus\cdots\oplus p_n^{(1)}\oplus 0_{l(n+1)} \text{ and}$$

$$q_n^{(1)}\oplus q_n^{(1)}\oplus\cdots\oplus q_n^{(1)}\oplus 0_{l(n+1)} \text{ in } M_{r_{n+1}}.$$

We take $p_n^{(2)}=0\oplus p_n^{(1)}\oplus 0\oplus\cdots\oplus 0\oplus 0_{l(n+1)}$ and

$$q_n^{(2)}=0\oplus q_n^{(1)}\oplus 0\oplus\cdots\oplus 0\oplus 0_{l(n+1)}.$$

If $p_n^{(k)}$ and $q_n^{(k)}$ are taken in M_{n+k}, we take

$$p_n^{(k+1)}=0\oplus p_n^{(k)}\oplus 0\oplus\cdots\oplus 0\oplus 0_{l(n+k)} \text{ and}$$

$$q_n^{(k+1)}=0\oplus q_n^{(k)}\oplus 0\oplus\cdots\oplus 0\oplus 0_{l(n+k)}.$$

Once $\{p_n^{(k)}\}$ and $\{q_n^{(k)}\}$ are constructed, (1)—(6) are easily checked.

Let f_n be the state of A denfined by $p_n\ a\ p_n=f_n(a)p_n$. Denote the normal extention of f_n by $\tilde{f}_n$, we have, as in the case $n=1$, $\tilde{f}_n[(x_n^i-q_nx_n^i)^*(x_n^i-q_nx_n^i]=0$. Moreover $\tilde{f}_n(q_m)=0$ for all n. Let $q=\sum\limits_{n=1}^{\infty}q_n$. Then q is a projection in A^{**}. Let π_n be the representation associated with f_n. Then there is a unit vector $\xi\in H_{\pi_n}$ such that

$$\langle\pi_n[(x_n^i-q_nx_n^i)^*(x_n^i-q_nx_n^i)]\xi,\ \xi\rangle=0,\ \text{for all i, i. e.,}$$

$$\pi_n(q_n)\pi_n(x_n^i)\xi=\pi_n(x_n^i)\xi \text{ for all i.}$$

Since $q\geqslant q_n$, $\pi_n(q)\pi_n(x_n^i)\xi=\pi_n(x_n^i)\tau$. Hence

$$\tilde{f}_n[(x_n^i-qx_n^i)^*(x_n^i-q_n^i)]=0,\ \text{for all } i.$$

Suppose that f is a weak * limit of $\{f_n\}$. Since $f_n(e_1)=\lambda_n(e_1)\geqslant\prod\limits_{i=1}^{\infty}r_i/q_{i+1}$, $f\neq 0$.

Because $x_n^i - qx_n^i$ converges to $x^i - qx^i$ in norm, for all i

$$\tilde{f}[(x^i - qx^i)^* - qx^i)] = 0.$$

Since $\tilde{f}_n(q_m) = 0$ for all n and m, $\tilde{f}_n(q) = 0$. Hence $\tilde{f}(q) = 0$ and $\tilde{f}((qx^i)(qx^i)^*) = 0$ for all i. We may assume that f is a state of A. Clearly $(x^i - qx^i)(x^i - qx^i)^*$ and $(qx^i)(qx^i)^*$ are Borel affine functions on Q for all i. By Lemma 1.2, there is a pure state ρ of A such that $\rho(x^i) = 0$ for all i. Thus for every polynomial $p(t)$ with $p(0) = 0$, $\rho(p(x)) = 0$. Let π be the (faithful, separable) irreducible representation of A associated with ρ and H_π the corresponding (separable) Hilbert space. There is a unit vector $\xi \in H$ such that $\rho(y) = \langle \pi(y)\xi, \xi \rangle$ for all $y \in A$. If $\pi(x)\xi = 0$, then the $\ker\pi(x)$ is a non-trivial invariant subspace for $\pi(x)$. So we may assume that $\pi(x)\xi \neq 0$. But then the closure of $H_0 = \{\pi(p(x))\xi \mid p(t)$ polynomials with $p(0) = 0\}$ is a non-trivial invariant subspace for $\pi(x)$, since $\xi \perp H_0$.

Remark. The method used in Theorem 1.1 may apply to some other finite separable simple AF C^*-algebra.

§ 2.

Let A be a C^*-algebra and x an element in A. Suppose that $\lambda \in \mathrm{Sp}(x)$. We will consider the following eigenvalue problem: Is there an irreducible representation π of A such that λ is an eigenvalue of $\pi(x)$? Corollary 2.1 and Lemma 2.2 are versions of Fredholm Alternative, while Corollary 2.2 is a generalised Weyl's theorem.

Lemma 2. 1. *Let A be a C^*-algebra and $x \in A$. Suppose λ is a non-zero number in $Sp(x)$. Then there is an irreducible representation π of A such that*

$$\lambda \in \mathrm{Sp}(\pi(x)).$$

Proof Suppose that for every irreducible representation π of A, $\lambda \notin \mathrm{Sp}(\pi(x))$. Let $y(\pi)$ be the inverse of $\lambda - \pi(x)$ in $\pi(\tilde{A})$. Let π_a be the atomic representation of A. If

$$\sup\{\|y(\pi)\| \mid \pi \text{ irreducible}\} < \infty,$$

then $y = \bigoplus_\pi y(\pi)$ would be the inverse of $\lambda - \pi_a(x)$. Thus there is a sequence $\{\pi_k\}$ of irreducible repesentations such that $\|y(\pi_k)\| \to \infty$ as $k \to \infty$. Consequently, there are $f_k \in H_{\pi_a}$ such that $\|f_k\| = 1$ and

$$\|\pi_a(\lambda - x)f_k\| \to 0, \text{ as } k \to \infty,$$

Let g be a weak * limit of the vectors states $\langle \cdot\, f_k, f_k \rangle$, then $g[(\lambda - x)^*(\lambda - x)] = 0$. Since $\lambda \neq 0$, $g \neq 0$. Thus we have a pure state ρ of $\tilde{A}$ such that $\rho[(\lambda - x)^*(\lambda - x)] = 0$. Moreover, $\rho | A \neq 0$, since $|\lambda|^2 \neq 0$. Thus $\rho|_A$ is a multiple of a pure state of A. Let π_ρ be the irreducible representation of A associated with $\rho | A$. Then there is $\xi \in H_{\pi_\rho}$,

$\xi \neq 0$, such that

$$[\lambda-\pi_\rho(x)]\xi=0. \text{ Hence } \lambda\in \mathrm{Sp}(\pi_\rho(x)),$$

a contradiction.

Corollary 2.1. *Let A be a liminal C^*-algebra and x an element in A. Suppose that $\lambda\in \mathrm{Sp}(x)$ is a non-zero number. Then there is an irreducible representation π of A such that λ is an eigenvalue of $\pi(x)$.*

Lemma 2.2. *Let A be an AF C^*-algebra and $x\in A$. Suppose that $\lambda\in \mathrm{Sp}(x)$ is a non-zero number. Then there is an irreducible representation π of A such that λ is an eigenvalue of $\pi(x)$.*

Proof There are finite dimensional C^*-subalgebra B_n of A and elements $x_n\in B_n$ such that

$$\|x-x_n\|\to 0, \text{ as } n\to\infty.$$

If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lambda\in \mathrm{Sp}(x_{n_k})$, then for every faithful representation π of A, there are $f_k\in H_\pi$ with $\|f_k\|=1$ such that

$$[\lambda-\pi(x_{n_k})]f_k=0.$$

Thus

$$\|[\lambda-\pi(x)]f_k\|\leqslant\|[\lambda-x_{n_k})]f_k\|+\|\pi(x)-\pi(x_{n_k})\|\to 0.$$

Otherwise, we have

$$(\lambda-x_n)^{-1}(\lambda-x)=1+(\lambda-x_n)^{-1}(x_n-x).$$

If $\{\|(\lambda-x_n)^{-1}\|\}$ is bounded, then

$$\|(\lambda-x_n)^{-1}(\lambda-x)-1\|\leqslant\|(\lambda-x_n)^{-1}\|\cdot\|x_n-x\|\to 0$$

as $n\to\infty$. Let y be a weak limit of $\{(\lambda-x_n)^{-1}\}$ in A^{**}. Then $y(\lambda-x)=1$, a contradiction. Thus we may assume that

$$\|(\lambda-x_n)^{-1}\|\to\infty, \text{ as } n\to\infty.$$

Suppose that $A\subset B(H)$. Then there are $f_n\in H$ with $\|f_n\|=1$ such that

$$\|(\lambda-x_n)f_n\|\to 0, \text{ as } n\to\infty.$$

So in any case there are states $\{\phi_n\}$ of $\widetilde{A}$ such that

$$\phi_n[(\lambda-x)^*(\lambda-x)]\to 0, \text{ as } n\to\infty.$$

As in the proof of Lemma 1.1, there is a pure state ρ of $\widetilde{A}$ such that

$$\rho[(\lambda-x)^*(\lambda-x)]=0.$$

Since $|\lambda|^2\neq 0$, $\rho|_A$ is a non-zero multiple of a pure state of A. Consequently, there is a non-zero vector $\xi\in H_\rho$ such that $[\lambda-\pi_\rho(x)]\xi=0$, where π_ρ is the irreducible representation of A associated with ρ. This completes the proof.

Corollary 2.2. *Let A be a C^*-algebra and I a closed ideal of A. Suppose that I is an AF C^*-algebra or a liminal C^*-algebra, and $x\in A$, $k\in I$. If $\lambda\in \mathrm{Sp}(x+k)$ but $\lambda\notin \mathrm{Sp}(x)$, then there is an irreducible representation π of A such that λ is an eigenvalue of $\pi(x+k)$.*

Proof $(\lambda-(x+k))=(\lambda-x)(1-(\lambda-x)^{-1}k)$. Since $\lambda\in \mathrm{Sp}(x+k)$, $1\in \mathrm{Sp}[(\lambda-x)^{-1}k]$. By Corollary 2.1 or Lemma 2.2, there is an irreducible representation π of I, and a unit vector $\pi\in H_\pi$ such that

$$(1-\pi(\lambda-x)^{-1}k)\xi=0.$$

Extending π to A, we have $\pi(\lambda-(x+k))\xi=0$. This completes the proof.

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