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LARGE DEVIATIONS FOR SYMMETRIC DIFFUSION PROCESSES**

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Abstract

Let $a(x)=(a_{ij}(x))$ be a uniformly continuous, symmetric and matrix-valued function satisfying uniformly elliptic condition, $p(t, x, y)$ be the transition density function of the diffusion process associated with the Dirichlet space $(\mathscr{E}, H_0^1(R^d))$, where

$$\mathscr{E}(u, v)=\frac{1}{2}\int_{R^d}\sum_{i,j}^{d}\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}a_{ij}(x)dx.$$

Then by using the sharpened Aronson's estimates established by D. W. Stroock, it is shown that

$$\lim_{t\to 0} 2t\ln p(t, x, y)=-d^2(x, y).$$

Moreover, it is proved that P_y^ε has large deviation property with rate function

$$I(\omega)=\frac{1}{2}\int_0^1 <\dot\omega(t), a^{-1}(\omega(t)), \dot\omega(t)>dt$$

as $\varepsilon\to 0$ and $y\to x$, where P_y^ε denotes the diffusion measure family associated with the Dirichlet form $(\epsilon\mathscr{E}, H_0^1(R^d))$.

§ 1. Introduction

Suppose that $X(t)$ is a d-dimensional diffusion process on R^d associated with the infinitesimal generator

$$L=\frac{1}{2}\sum_{i,j}^{d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}. \tag{1.1}$$

For any $\epsilon>0$, definea process $X_\epsilon(t)$ to be $X(\epsilon t)$, Then X_ϵ is a diffusion process on R^d associated with the infinitesimal generator

$$L_\epsilon=\frac{1}{2}\epsilon\sum_{i,j}^{d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}. \tag{1.2}$$

We denote by P_x and P_x^ϵ the laws of processes $X(t)$ and $X_\epsilon(t)$ satisfying the conditions that $X(0)=x$, and $X_\epsilon(0)=x$, respectively. S. R. S. Varadhan[9] proved that if $a(\cdot)$ is uniformly elliptic and uniformly Hölder continuous, then P_x^ϵ has large deviation property when $\epsilon\to 0$. D. W. Stroock[7] gave a simpler proof of above results under the additional condition that $a_{ij}(x)$ are C^2. In this direction, M. I.

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Fredilin and A. D. Wentzell[3] developed the theory of random perturbations using the theory of stochastic differential equations.

In this paper, we deal with the large deviations of the symmetric diffusion process associated with the infinitesimal generator

$$L=\frac{1}{2}\sum_{i,j}^{d}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j} \tag{1.3}$$

under less regular conditions made on $a(\cdot)$. Our main results show that if $a_{ij}(x)$ are uniformly continuous, then P_y^ϵ has large deviation property as $\epsilon\to 0$ and $y\to x$, where P_y^ϵ denotes the law of X_ϵ under the condition that $X_\epsilon(0)=y$, X_ϵ denotes the process associated with the infinitesimal generator

$$L_\epsilon=\frac{1}{2}\epsilon\sum_{i,j}^{d}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}. \tag{1.4}$$

It is obvious that the methods given by D. W. Stroock[7] do not work in this case. Using the Aronson's estimate technique which do not depend on the smooth conditions made on $a(\cdot)$, we show that the arguments given by S. R. S. Varadhan[10] also work well in this case. In § 2, we recall Aronson's estimates for heat kernels. In particular using the sharpened Aronson's estimates established by D. W. Stroock, we show that

$$\lim_{t\to 0} 2t\ln p(t, x, y)=-d^2(x, y) \tag{1.5}$$

only under the assumption that $a(\cdot)$ is uniformly continuous and uniformly elliptic, which can be regarded as an extension of the famous S. R. S. Varadhan's theorem in [9].

Under the same conditions, we give a large deviation principle of Markov processes with state space R^d in § 3, and then apply this result to symmetric diffusion processes.

§ 2. Aronson's Estimates

S. R. S. Varadhan [9] proved that if $p(t, x, y)$ is the fundamental solution which solves the equation

$$\frac{1}{2}\sum_{i,j}^{d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}-\frac{\partial}{\partial t}=0,$$

where $a(\cdot)$ is uniformly Hölder continuous and satisfies the condition

$$\lambda I_{R^d}\leqslant a(\cdot)\leqslant\lambda^{-1}I_{R^d} \tag{2.1}$$

for some $\lambda\in(0, 1]$, then

$$\lim_{t\to 0} 2t\ln p(t, x, y)=-d^2(x, y), \tag{2.2}$$

where $d(x, y)$ denotes the geodesic metric determined by matrix a^{-1}. In this section, we show that if $a(x)=(a_{ij}(x))$ is a symmetric, uniformly continuous, matrix-valued function satisfying (2.1), $d \geqslant 3$, $p(t, x, y)$ is the transition density function of diffusion process associated with the generator (1.3), then (2.2) also holds. Our sole contribution to this result is the observation that the sharpened Aronson's estimates established by D. W. Stroock[7] ensure (2.2).

Throughout this paper, we assume that $a(x) = (a_{ij}(x))$ is a uniformly continuous, symmetric matrix valued function satisfying (2.1) and $d \geqslant 3$. For any function $\psi \in C^1(R^d; R)$, define

$$\Gamma_a(\psi) = |\langle \nabla\psi, a.\nabla\psi\rangle|_\infty^{1/2} = \left|\sum_{i,j}^d \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} a_{ij}\right|_\infty^{1/2} \leqslant +\infty, \tag{2.3}$$

where $|\cdot|_\infty$ denotes the essential upper bound with respect to the Lebesgue measure on R^d. By (2.1) one knows that $\Gamma_a(\psi) < \infty$ if and only if $\|\nabla\psi\|_\infty \equiv |\langle\nabla\psi, \nabla\psi\rangle|_\infty^{1/2} < \infty$. Denote by $a^{-1}(x) = (a^{ij}(x))$ the inverse matrix of $a(x)$. It is easy to see that a^{-1} also satisfies (2.1). Let $d(x, y)$ be the geodesic metric from x to y determined by the matrix valued function $a^{-1}(x)$ (hence by $a(x)$), defined by

$$\begin{aligned} d_{a^{-1}}(x, y) &\equiv \sup\left\{\frac{|\psi(x)-\psi(y)|}{\Gamma_a(\psi)}: \psi \in C^1(R^d; R)\right\} \\ &= \sup\left\{\frac{|\psi(x)-\psi(y)|}{\Gamma_a(\psi)}: \psi \in C^1(R^d; R) \text{ and } \|\nabla\psi\|_\infty = 1\right\} \\ &= \sup\{|\psi(x)-\psi(y)|: \Gamma_a(\psi) \leqslant 1\}. \end{aligned} \tag{2.4}$$

For a_{ij} are uniformly continuous, by variational principle, it is easy to check that (see [6])

$$\begin{aligned} d^2_{a^{-1}}(x, y) = \inf\Big\{&\int_0^1 \langle\dot\omega(t), a^{-1}(\omega(t))\cdot\dot\omega(t)\rangle dt: \\ &\omega \in C^1([0, 1]; R^d) \text{ with } \omega(0) = x \text{ and } \omega(1) = y\Big\}, \end{aligned} \tag{2.5}$$

where $\dot\omega$ denotes the generalized derivative of the function ω. Using (2.1), we know that there exists a constant $\beta \in (0, 1]$ depending only on λ and d, such that

$$\beta|x-y| \leqslant d_{a^{-1}}(x, y) \leqslant \beta^{-1}|x-y|.$$

Let $\rho(x)$ equal $\alpha\exp(|x|^2-1)^{-1}$ if $|x|<1$, and 0 if $|x| \geqslant 1$, where α is a constant such that $\int_{R^d}\rho(x)dx = 1$. Then ρ belongs to the Schwarz space. For any $t>0$, define $\rho_t(x) = t^{-\frac{d}{2}}\rho\left(\frac{x}{\sqrt{t}}\right)$. Denote by $\rho_t * a^{-1}$ the matrix $(\rho_t * a^{ij})$, where $\rho_t * a^{ij}$ denotes the convolution of functions ρ_t and a^{ij}, defined by

$$\rho_t * a^{ij}(x) = \int_{R^d}\rho_t(x-y)a^{ij}(y)dy. \tag{2.6}$$

In particular, for any $\xi = (\xi_i) \in R^d$, we have

$$\sum_{i,j}^d \rho_t * a^{ij}(x)\xi_i\xi_j = \int_{R^d}\rho_t(x-y)\sum_{i,j}^d \xi_i\xi_j a^{ij}(y)dy,$$

that is, $\rho_t * a^{-1}$ (hence $(\rho_t * a^{-1})^{-1}$) satisfies (2.1). We note that $\rho_t * a^{ij}$ are bounded smooth functions on R^d.

Lemma 2. 1. *$\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2$ converges to $\Gamma_a(\psi)^2$, as $t\to 0$, uniformly over $\psi\in C^1(R^d; R)$ such that $\|\psi\|_\infty\leqslant 1$.*

Proof Since

$$\begin{aligned}&|\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2-\Gamma_a(\psi)^2|\\&=\|\langle\nabla\psi,\ (\rho_t*a^{-1})^{-1}\nabla\psi\rangle|_\infty-|\langle\nabla\psi,\ a.\ \nabla\psi\rangle|_\infty|\\&\leqslant|\langle\nabla\psi,\ ((\rho_t*a^{-1})^{-1}-a)\nabla\psi\rangle|_\infty\\&\leqslant\|(\rho_t*a^{-1})^{-1}\|_\infty\|a\|_\infty\|\nabla\psi\|_\infty\|\rho_t*a^{-1}-a^{-1}\|_\infty\end{aligned}$$

and the fact $|\rho_t*a^{ij}-a^{ij}|_\infty\to 0$ as $t\to 0$ due to the fact a is uniformly continuous, we get the lemma.

Here we denote by $\|a\|_\infty$ the L_∞-norm of the matrix a.

Corollary 2. 1. *For any $x,\ y\in R^d$, we have*

$$d^2_{a^{-1}}(x,\ y)=\lim_{t\to 0}d^2_{\rho_t*a^{-1}}(x,\ y) \tag{2.7}$$

uniformly over all x, y such that $|x-y|$ is bounded,

Proof Note that $\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2\geqslant\lambda\|\nabla\psi|^2_\infty$. Hence for any $\psi\in C^1(R^d;\ R)$ such that $\|\nabla\psi\|_\infty=1$, we have

$$\left|\frac{|\psi(x)-\psi(y)|^2}{\Gamma_a(\psi)^2}-\frac{|\psi(x)-\psi(y)|^2}{\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2}\right|\leqslant\frac{|x-y|^2}{\lambda^2}|\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2-\Gamma_a(\psi)^2|.$$

Now (2.7) follows from Lemma 2.1 immediately.

Remark 2. 1. Because ρ_t*a^{-1} is continuous, $d^2_{\rho_t*a^{-1}}(x,\ y)$ can be rewritten as

$$d^2_{\rho_t*a^{-1}}(x,\ y)=\inf\left\{\int_0^1\langle\dot\omega(s).\ \rho_t*a^{-1}(\omega(t))\cdot\dot\omega(s)\rangle ds:\ \omega\in C^1([0,\ 1];\ R^d)\text{ with }w(0)=x\text{ and }\omega(1)=y\right\}. \tag{2.8}$$

Now we recall Aronson's estimates for heat kernels. Let $(\mathscr{E},\ H^1_0(R^d))$ be the regular Dirichlet space on $L^2(R^d;\ dx)$ dertermined by the matrix-valued function a, where

$$\mathscr{E}(u,\ v)=\frac{1}{2}\int_{R^d}\langle\nabla u,\ a.\nabla v\rangle(x)dx. \tag{2.9}$$

Then there exists a positive real-valued function $p(t,\ x,\ y)\in C((0,\ \infty)\times R^d\times R^d)$ satisfying

$$p(s+t,\ x,\ y)=\int_{R^d}p(t,\ x,\ z)p(s,\ y,\ z)dz,$$

$$p(t,\ x,\ y)=p(t,\ y,\ x) \tag{2.10}$$

and there exists a constant M depending only on λ and d such that

$$\frac{1}{Mt^{\frac{d}{2}}}\exp\left(-\frac{M|x-y|^2}{t}\right)\leqslant p(t,\ x,\ y)\leqslant\frac{M}{t^{\frac{d}{2}}}\exp\left(-\frac{|x-y|^2}{Mt}\right). \tag{2.11}$$

$p(t,\ x,\ y)$ generates the Dirichlet space $(\mathscr{E},\ H^1_0(R^d))$ in the sense of M. Fukushima[4]. The inequality (2.11) is called Moser's inequality or Aronson's estimate. As a result of the Aronson's estimate, we have the following Nash's inequality:

There exist constants $C>0$ and $\alpha\in(0,1)$ which only depend on λ and d, such that

$$|p(t',x',y')-p(t,x,y)|\leqslant\frac{C}{\delta^d}\left(\frac{|t'-t|^{\frac{1}{2}}\vee|x'-x|\vee|y'-y|}{\delta}\right)^{\alpha}\tag{2.12}$$

for all $(t,x,y),(t',x',y')\in[\delta^2,\infty)\times R^d\times R^d$ with $|x-x'|\vee|y-y'|\leqslant\delta$.

Using the E. B. Davies' method for obtaining off-diagonal estimates, one has the following stronger estimates for the upper bound, whose proof can be found in D. W. Stroock [7].

Theorem 2.1. *There is a constant $K>0$, which only depends on λ and d, such that for any $\delta\in(0,1]$ we have*

$$p(t,x,y)\leqslant\frac{K}{(\delta t)^{d/2}}\exp\left(-\frac{d^2_{a^{-1}}(x,y)}{2(1+\delta)t}\right)\tag{2.13}$$

for any $(t,x,y)\in(0,\infty)\times R^d\times R^d$.

We note that there is not any smooth assumption made on a. For the lower bound, we have the following Theorem 2.5 under the additional condition that a is smooth, which was proved by D. W. Stroock[7].

Theorem 2.2. *Assume $a(x)$ is uniformly continuous. There is a constant $M\geqslant1$, depending only on λ and d, such that for any $\delta\in(0,1]$ we have*

$$p(t,x,y)\geqslant\frac{e^{-M/\delta}}{Mt^{d/2}}\exp\left(-\frac{(1+\delta)d^2_{\rho_t*a^{-1}}(x,y)}{2t}\right)\tag{2.14}$$

for any $(t,x,y)\in(0,\infty)\times R^d\times R^d$.

Proof We can choose a sequence of bounded smooth, symmetric, matrix-valued functions $a^n(x)=(a^n_{ij}(x))$ satisfying (2.1), such that $|a^{ij}_n-a^{ij}|_\infty\to0$ when $n\to\infty$, where $(a^{ij}_n)\equiv(a^n)^{-1}$. Then $a^n_{ij}(\cdot)\to a_{ij}(\cdot)$ almost everywhere. Denote by $p_n(t,x,y)$ the transition density function correponding to $a^n(\cdot)$. By Nash's inequality (2.12), we know that $p_n(t,x,y)\to p(t,x,y)$, when $n\to\infty$, uniformly on any compact subset of $(0,\infty)\times R^d\times R^d$. Since

$$|\Gamma_{(\rho_t*a_n^{-1})^{-1}}(\psi)^2-\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi)^2|\leqslant|\nabla\psi,(\rho_t*a_n^{-1})^{-1}\rho_t*(a_n^{-1}-a^{-1})(\rho_t*a^{-1})^{-1}\nabla\psi|_\infty,$$

we have

$$\Gamma_{(\rho_t*a_n^{-1})^{-1}}(\psi)^2\to\Gamma_{(\rho_t*a^{-1})^{-1}}(\psi^2),$$

when $n\to\infty$, uniformly over $\psi\in C^1(R^d;R)$ such that $\|\nabla\psi\|_\infty\leqslant1$. So we get

$$\lim_{n\to\infty}d^2_{\rho_t*a_n^{-1}}(x,y)=d^2_{\rho_t*a^{-1}}(x,y).$$

For any n, we have (see [7], Theorem 3.9)

$$p_n(t,x,y)\geqslant\frac{e^{-M/\delta}}{Mt^{d/2}}\exp\left(-\frac{(1+\delta)d^2_{\rho_t*a_n^{-1}}(x,y)}{2t}\right).$$

Let $n\to\infty$, one deduces (2.14).

Theorem 2.3. *Assume $a(x)$ is uniformly continuous. Then*

$$\lim_{t\to0}2t\ln p(t,x,y)=-d^2_{a^{-1}}(x,y)\tag{2.15}$$

uniformly over x, y such that $|x-y|$ is bounded

Proof By Theorem 2.4, for any $\delta\in(0, 1]$ we have

$$2t\ln p(t, x, y)\leqslant 2t\ln K-dt\ln\delta t-\frac{d_{a^{-1}}^2(x, y)}{1+\delta}.$$

Hence by (2.6) we get

$$\overline{\lim_{t\to 0}}\, 2t\ln p(t, x, y)\leqslant -d_{a^{-1}}^2(x, y)$$

uniformly over x, y such that $|x-y|$ is bounded. By the same argument, using Theorem 2.2 and Corollary 2.2, we get

$$\liminf_{t\to 0} 2t\ln p(t, x, y)\geqslant -d_{a^{-1}}^2(x, y).$$

The proof is complete.

Remark 2.2. Using Moser's inequality, (2.15) and conditional diffusion processes, we have proved that Theorem 2.3 also holds if $p(t, x, y)$ is the transition density function associated with the generator

$$L=\frac{1}{2}\sum_{i,j}^{d}\frac{\partial}{\partial x_i}a_{ij}(\cdot)\frac{\partial}{\partial x_j}+\sum_{i}^{d}b_i(\cdot)\frac{\partial}{\partial x_i},$$

where we only assume that $b_i(x)$ are bounded measurable (see [2]).

§ 3. Large Deviation Property

In this section, we will show that the Moser's inequality and the asymptotic relation (2.15) essentially ensure a diffusion process in a small interval having large deviation property. In particular, diffusion processes with continuous diffusion coefficients in small intervals have large deviation properties. We basically follow the argument of S. R. S. Varadhan[10], but there are a few difficulties to overcome. Moreover we use the Moser's inequality, without assuming any smooth conditions on diffusion coefficients as in [10].

Throughout this section, we assume that $p(t, x, y)\in C((0, \infty)\times R^d\times R^d)$ is a positive transition density function (without symmetric assumption) satisfying (2.11) for any $(t, x, y)\in(0, 1]\times R^d\times R^d$ for some constant $M>0$. Let $\Omega=C([0, 1]; R^d)$ with the topology of uniform convergence on $[0, 1]$. Assume there exists a continuous, symmetric and matrix-valued function $a(x)=(a_{ij}(x)): R^d\to R^d\otimes R^d$ satisfying (2.1) for some constans λ. Let I be the rate function on Ω associated with the function $a(x)$ defined by

$$I(\omega)=\frac{1}{2}\int_0^1\langle\dot{\omega}(s), a^{-1}(\omega(s))\cdot\dot{\omega}(s)\rangle ds. \tag{3.1}$$

Then $I: \Omega\to[0, \infty]$ is lower semicontinuous, and has compact level sets. Moreover, let $d(x, y)$ be the geodesic metric function defined by (2.4) satisfying the following conditions

$$\lim_{t\to 0} 2t \ln p(t, x, y) = -d^2(x, y) \tag{3.2}$$

uniformly over x, y such that $|x-y|$ is bounded.

Example Let $a(x)=(a_{ij}(x))$: $R^d \to R^d \otimes R^d$ $(d \geqslant 3)$ be a symmetric, bounded, uniformly continuous and matrix-valued function satisfying (2.1), $p(t, x, y)$ be the transition density function associated with the regular Dirichlet space (2.9). Then by (2.5) and Theorem 2.3, it is easy to check that $p(t, x, y)$, $d_{a^{-1}}(x, y)$ and I satisfy above conditions.

Let $x(t)$ be the coordinate process on Ω, and $\mathscr{F}$, $\mathscr{F}_t$ be its natural filtration. By Moser's inequality (2.11) and Kolmogorov's criterion, it is easy to see that for any $x \in R^d$, there is a unique probability P_x such that $(\Omega, \mathscr{F}, \mathscr{F}_t, x(t), P_x)$ is a strongly Feller continuous, Markov process with transition probability $P(t, x, dy) = p(t, x, y)dy$. For each $\epsilon \in (0, 1]$, set $p_\epsilon(t, x, y) = p(\epsilon t, x, y)$; then $p_\epsilon(t, x, y)$ also is a transition density function on R^d and satisfies

$$\frac{1}{M(\epsilon t)^{d/2}} \exp\left(-\frac{M|x-y|^2}{\epsilon t}\right) \leqslant p_\epsilon(t, x, y) \leqslant \frac{M}{(\epsilon t)^{d/2}} \exp\left(-\frac{|x-y|^2}{M\epsilon t}\right). \tag{3.3}$$

In particular, there is a unique probability P_x^ϵ for each $x \in R^d$, such that $(\Omega, \mathscr{F}, \mathscr{F}_t, x(t), P_x^\epsilon)$ is a strongly Feller continuous, Markov process with transition density function $p_\epsilon(t, x, y)$. For any subset $B \subset \Omega$, define $B_x \equiv \{\omega: \omega \in B$ with $\omega(0) = x\}$. The main result in this section is the following

Theorem 3.1. *For any closed subset $C \subset \Omega$, and open subset $G \subset \Omega$, we heve*

$$\overline{\lim_{\substack{\epsilon\to 0\\ y\to x}}} \epsilon \ln P_y^\epsilon(C) \leqslant -\inf_{\omega \in C_x} I(\omega),$$

$$\liminf_{\substack{\epsilon\to 0\\ y\to x}} \epsilon \ln P_y^\epsilon(G) \geqslant -\inf_{\omega \in G_x} I(\omega). \tag{3.4}$$

As a consequence of this result applied to the diffusion process associated with the infinitesimal generator (1.3) (see Example), we have the following

Theorem 3.2. *Assume that $a(x) = (a_{ij})$ is a bounded, uniformly continuous, symmetric and matrix-valued function on $R^d (d \geqslant 3)$, which satisfies (2.1) for some constant $\lambda \in (0, 1]$. For each $\epsilon > 0$, let $(x(t), P_x^\epsilon)$ be the symmetric strongly Feller continuous, Markov process associated with the Dirichlet space $(\epsilon \mathscr{E}, H_0^1(R^d))$, where $\mathscr{E}$ is defined by (2.9) and $x(t)$ is the coordinate process on Ω. Then P_y^ϵ has large deviation property with rate function I defined by (3.1), when $\epsilon \to 0$ and $y \to x$, in the sense of Theorem 3.1, that is, (3.4) holds.*

The remains of this paper are devoted to proving Theorem 3.1.

First we prove the upper bound.

For each partition π: $0 = t_0 < t_1 < \cdots < t_n = 1$, denote by T_π the map from Ω to $R^{d\times(n+1)}$, defined by

$$T_\pi \omega = \{\omega(t_0), \cdots, \omega(t_n)\}. \tag{3.5}$$

If A is Borel set, and $C=T_x^{-1}A$, then

$$P_y^\epsilon(C)=\int_{A_y}\prod_{j=1}^{n}p_\epsilon(t_j-t_{j-1},\ y_{j-1},\ y_j)dy_j, \tag{3.6}$$

where $A_y=\{(y_0,\ \cdots,\ )\in A:\ y_0=y\}$. Using (3.6), (3.2), Moser's inequality (2.11) and the followihg

$$\inf_{\substack{\omega:\omega(t_j)=x_j\\ j=1,\cdots,n}} I(\omega)=\frac{1}{2}\sum_{j=1}^{n-1}\frac{d^2(x_{j+1},\ x_j)}{t_{j+1}-t_j} \tag{3.7}$$

one can easily check (for the details, see [10], Lemma 3.1)

Lemma 3. 1. *If $A\subset R^{d\times(n+1)}$ is a closed subset, $C=T_x^{-1}A$, then*

$$\varlimsup_{\substack{\epsilon\to0\\ y\to x}}\epsilon\ln P_y^\epsilon(C)\leqslant-\inf_{\omega\in C_x}I(\omega). \tag{3.8}$$

Lemma 3. 2. *There is a constant $C\geqslant1$, which depends on M and d, such that for any $\epsilon\in(0,\ 1]$ we have*

$$P_x^\epsilon(\sup_{s\in[0,t]}|x(s)-x(0)|\geqslant r)\leqslant C\exp\left(-\frac{r^2}{C\epsilon t}\right) \tag{3.9}$$

for any $(t,\ x)\in(0,\ 1]\times R^d$, $r>0$.

Proof Let $\zeta_r=\inf\{t\geqslant0;\ |x(t)-x|\geqslant r\}$. By strongly Markov property, we get

$$P^\epsilon(t,\ x,\ \overline{B(x,\ r)^c})=E^{P_x^\epsilon}\ [P^\epsilon(t-\zeta_r,\ x(\zeta_r),\ \overline{B(x,\ r)^c}),\ \zeta_r<t],$$

where $B(x,\ r)=\{y:\ |y-x|<r\}$, $P^\epsilon(t,\ x,\ dy)=p_\epsilon(t,\ x,\ y)\ dy$. By the lower bound of Moser's inequality, one knows that there is a constant $\delta>0$, depending only on M and d, such that for any $\xi\in\partial(B(x,\ r))$, $s>0$, and $\epsilon\in(0,\ 1]$, we have

$$P^\epsilon(s,\ \xi,\ \overline{B(x,\ r)^c})=P(\epsilon s,\ \xi,\ \overline{B(x,\ r)})^c\geqslant\delta.$$

On the other hand, using the upper bound of Aronson's estimate, one easily check that

$$P^\epsilon(t,\ x,\ \overline{B(x,\ r)^c})=P(\epsilon t,\ x,\ \overline{B(x,\ r)^c})\leqslant A\exp\left(-\frac{r^2}{B\epsilon t}\right),$$

where A and B are constants depending on M and d. Hence

$$P_x^*(\sup_{s\in]0,t[}|x(s)-x(0)|\geqslant r)=P_x^\epsilon(\zeta_r<t)\leqslant\delta_-^{1}P^\epsilon(t,\ x,\ \overline{B(x,\ r)^c})$$

$$\leqslant\delta^{-1}A\exp\left(-\frac{r^2}{B\epsilon t}\right),$$

the proof is complete.

For any n, denote by π_n the partition: $0=t_0<t_1<\cdots<t_n=1$, $t_i=\frac{1}{n}$. Let $x^n(t)$ be the process such that $x^n(t_i)=x(t_i)$, $(i=0,\ \cdots,\ n)$ and joins the successive ones by one of geodesics connecting $x(t_i)$ and $x(t_{i+1})$. For simplicity, we denote by $d(x,\ y)$ the $d_{a^{-1}}(x,\ y)$.

Lemma 3. 3. *For any $\delta>0$*

$$\varlimsup_{n\to\infty}\varlimsup_{\epsilon^*\to0}\sup_x\epsilon\ln P_x^\epsilon(\sup_{t\in[0,1]}d(x(t),\ x^n(t))\geqslant\delta)=-\infty. \tag{3.10}$$

Proof By defintiion, for any $t\in(t_j, t_{j+1})$ we have

$$d(x^n(t), x^n(t_j))\leqslant d(x^n(t_{j+1}), x^n(t_j))=d(x(t_{j+1}), x(t_j))\leqslant \sup_{t\in[t_j,t_{j+1}]} d(x(t), x(t_j)).$$

Hence we have

$$\sup_{t\in[0,1]} d(x(t), x^n(t))\leqslant 2\sup_{0\leqslant j\leqslant n-1}\sup_{t\in[t_j,t_{j+1}]} d(x(t), x(t_j)).$$

Using Markov property and Lemma 3.4, one deduces that

$$\begin{aligned}
&P_x^\epsilon[\sup_{t\in]0,1]} d(x(t), x^n(t))\geqslant\delta]\\
&\leqslant P_x^\epsilon\Big[\sup_{0\leqslant j\leqslant n-1}\sup_{t\in[t_j,t_{j+1}]} d(x(t), x(t_j))\geqslant\frac{\delta}{2}\Big]\\
&\leqslant\sum_{j=0}^{n-1} P_x^\epsilon\Big[\sup_{t\in[t_j,t_{j+1}]} d(x(t), x(t_j))\geqslant\frac{\delta}{2}\Big]\\
&\leqslant\sum_{j=0}^{n-1}\sup_{x\in R^d} P_x^\epsilon\Big[\sup_{t\in[0,\frac{1}{n}]} d(x(t), x(0))\geqslant\frac{\delta}{2}\Big]\\
&\leqslant nM\exp\Big(-\frac{n\delta^2\beta^2}{4M\epsilon}\Big).
\end{aligned}$$

Now (3.10) follows immediately.

Using above lemma and the same argument as [10], we can get the upper bound. For completion, we outline the proof as follows.

Let π_n be the partition of [0, 1] as above. For any $\omega\in\Omega$, define $\omega_n\in\Omega$ such that $\omega_n(t_i)=\omega(t_i)$ $(i=0, \cdots, n)$ and joins the successive ones by one of geodesics connecting $\omega(t_i)$ and $\omega(t_{i+1})$, such that $x_n(t, \omega)=x(t, \omega_n)$. Let $C\subset\Omega$ be a closed subset. For any $\delta>0$, define

$$C_\delta=\{\omega\in C: d(\omega(0), x)<\delta\},$$

$$C_\delta^\delta=\{\gamma: \sup_{t\in[0,1]} d(\gamma(t), \omega(t))<\delta \text{ for some } \omega\in C_\delta\},$$

$$I^\delta(\omega)=\inf\{I(\gamma): \sup_{t\in[0,1]} d(\gamma(t), \omega(t))<\delta\},$$

$$\alpha_\delta=\inf_{\omega\in C_\delta^\delta} I(\omega).$$

Because $y\to x$, we can assume $d(y, x)<\delta$. Note the fact $\omega\in C_\delta$ implies $I^\delta(\omega)\geqslant\alpha_\delta$, hence

$$P_y^\epsilon(C)=P_y^\epsilon(C_\delta)\leqslant P_y^\epsilon(I^\delta(\omega)\geqslant\alpha_\delta).$$

On the other hand

$$P_y^\epsilon(\omega: I^\delta(\omega)\geqslant\alpha_\delta)\leqslant P_y^\epsilon\{\omega: \sup_{t\in]0,1]} d(\omega(t), \omega_n(t))\geqslant\delta\}+P_y^\epsilon\{\omega: I(\omega_n)\geqslant\alpha_\delta\}.$$

But

$$I(\omega_n)=\frac{1}{2}\sum_{j=0}^{n-1}\frac{d^2(\omega(t_j), \omega(t_{j+1}))}{t_{j+1}-t_j}.$$

Using Lemma 3.2 for set $\{\omega: I(\omega_n)\geqslant\alpha_\delta\}$ we get

$$\overline{\lim_{\substack{\epsilon\to0\\ y\to x}}}\,\epsilon\ln P_y^\epsilon(\omega: I(\omega_n)\geqslant\alpha_\delta)\leqslant-\alpha_\delta.$$

By Lemma 3.3, one can check

$$\overline{\lim_{\substack{\epsilon\to 0\\ y\to x}}}\ \epsilon \ln P_y^\epsilon(C) \leqslant -\alpha_\delta$$

for any $\delta>0$. Since I is lower semicontinuous, we have

$$\lim_{\delta\to 0} \alpha_\delta = \inf_{\omega\in C_x} I(\omega).$$

Thus we have proved the upper bound.

By the proof of Lemma 3.4 in [10], we know that lower bound of Theorem 3.1 is a consequence of Lemma 3.3, Theorem 2.3 and upper bound. For the details, we refer to [10].

Remark 3.1. Theorem 3.2 also holds for the diffusion associated with the operator

$$L=\frac{1}{2}\sum_{i,j}^{d}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}+\sum_{i}^{d}b_i(x)\frac{\partial}{\partial x_i},$$

where (a_{ij}) is continuous and satisfies (2.1), (b_i) is bounded measurable (compare with Theorem 3.1 in [3]).

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