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ON THE BOREL SUBGROUPS OF LARGE TYPE**

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Abstract

The author defines the large type Borel subgroups of a reductive algebraic group, which are used to discuss Langlands' L-groups and the Langlands classification of the admissible representations of reductive algebraic groups over $\mathbf{R}$(see [1, 2, 5]) and determine all of the Borel subgroups of large type for the classical semisimple Lie groups.

§ 1. Main Theorems

Suppose G is a connected reductive algebraic group over $\mathbf{C}$. To give a definition of G over $\mathbf{R}$ means to give an action of $\Gamma(\mathbf{C}/\mathbf{R})$ (the Galois group) on G; that is, a homomorphism

$$\alpha: \Gamma \to \mathrm{Aut}_{\mathrm{cont}}(G)$$

with the property that $\alpha(\sigma)$ is anti-holomorphic where σ is the non-trivial element of the Galois group $\Gamma(\mathbf{C}/\mathbf{R})$. The corresponding real form of G is

$$G(\mathbf{R}) = \{g \in G;\ \alpha(\gamma)g = g \text{ for all of } \gamma \in \Gamma\}.$$

It determines the Galois action uniquely. Two real forms are said to be equivalent if the Galois actions are conjugate under G.

Every complex reductive group has a compact real form ([3], Chapter 3, Theorem 6.3), it is unique up to equivalence. Fix a real form of G, there is a compact real form of G such that the corresponding Galois action $\alpha_c(\sigma)$ commutes with $\alpha(\sigma)$ (see [3], Chapter 3, Theorem 7.1). It is unique up to conjugation by $G(\mathbf{R})$ (see [3], Chaper 3, Theorem 7.2). The Cartan involution attached to the real form is

$$\theta = \alpha(\sigma)\alpha_c(\sigma), \tag{1.1}$$

θ is a holomorphic automorphism of G, of order 2.

Proposition 1.1. *Suppose G is a connected reductive algebraic group. Every holomorphic involution of G is the Cartan involution for some real form of G. Two involutions are conjugate by G iff the corresponding real forms are equivalent.*

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Proof See [3], Chapter 10, Section 1.

Proposition 1.2. *Two real forms of G are said to be inner (to each other) if the corresponding Cartan involutions θ and θ' differ by an inner automorphism of G:*

$$\theta' = \theta \operatorname{Ad}(X), \tag{1.2}$$

where $\operatorname{Ad}(X)$: $G \to G$ *is defined by* $\operatorname{Ad}(X)g = XgX^{-1}$ *for all of* $g \in G$. *Conjugate automorphisms are always inner: if*

$$\theta' = \operatorname{Ad}(g)\cdot\theta\cdot\operatorname{Ad}(g^{-1}), \tag{1.3}$$

then (1.2) *holds with X equal to $(\theta g)g^{-1}$. In fact, for any $h \in G$*

$$[\theta\cdot\operatorname{Ad}((\theta g)g^{-1})](h) = \theta\cdot(\theta(g)g^{-1}hg\,\theta(g^{-1})) = g(\theta(g^{-1})\theta h\theta(g)g^{-1}$$
$$= g\theta(g^{-1}hg)g^{-1} = [\operatorname{Ad}(g)\theta\operatorname{Ad}(g^{-1})](h).$$

It is easy to verify that the condition for the formula (1.2) to define an involution (assuming that θ is one) is

$$X\theta(X) \in Z(G), \tag{1.4}$$

where $Z(G)$ denotes the center of G.

The relation "inner" is an equivalence relation on real forms. An equivalence class is called an inner class of real forms.

Proposition 1.3. *Suppose G is a connected reductive algebraic group. The inner classes of real forms of G are parameterized by the elements of order 2 in* $\operatorname{Out}(G)$, *where* $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Ad}(G)$ *is called the group of outer automorphisms of G.*

Proof See [1], Proposition 6.12.

By the definition, a Borel subgroup of G is a maximal solvable subgroup of G.

Definition 1.1. *Suppose G is a connected reductive algebraic group. An involution θ of G is called principal if there is a regular semisimple element X in $\mathfrak{g}$ (the Lie algebra of G) such that*

$$\theta X = -X.$$

A real form of G is called quasisplit if there is a Borel subgroup of G defined over $\mathbf{R}$.

Theorem 1.1. *Suppose G is a connected reductive algebraic group. Then each inner class of involution of G contains a unique G-conjugacy class of principal involutions. The corresponding real forms are exactly the quasisplit ones in the inner class.*

Proof See [1], Theorem 6.14.

In order to give some other characterizations of principal involutions, we should give a little general notation.

Definition 1.2. *Suppose G is a connected reductive algebraic group, B is a Borel subgroup of G, and T is a maximal torus in B. The Lie algebra $\mathfrak{g}$ of G has the root space decomposition* *as*

$$\mathfrak{g}=\mathfrak{t}+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha,$$

where Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The Borel subgroup B determines a set Δ^+ of positive roots, and therefore a set $\pi=\pi(B, T)$ of simple roots.

Definition 1.3. *Suppose G is a reductive algebraic gaoup, θ is an involution of G and T is a θ-stable maximal torus in G. For any $\alpha\in\Delta$, we define*

$$\theta\alpha(h)=\alpha(\theta(h)) \text{ for any } h\in T.$$

A root α of T in G is called real if $\theta\alpha=-\alpha$; imaginary if $\theta\alpha=\alpha$; and complex otherwise.

If α is imaginary, then θ preserves the root space $\mathfrak{g}_\alpha$. We say that α is compact if θ acts by $+1$ on $\mathfrak{g}_\alpha$, and non-compact if θ acts by -1 on $\mathfrak{g}_\alpha$.

Proposition 1.4. *Suppose G is a connected reductive algebraic group, and θ is an involution of G. The following conditions are equivalent:*

(a) *The involution θ is principal.*

(b) *The corresponding real form $G(\mathbf{R})$ is quasisplit.*

(c) *There exists a θ-stable maximal torus with no imaginary roots.*

(d) *There exists a θ-stable pair $B\supset T$ (a Borel subgroup and maximal torus), such that every simple root is either complex or non-compact imaginary.*

(e) *If T is any θ-stable maximal torus, then there is a set of positive imaginary roots for which every simple root is non-compact.*

Proof See [1], Proposition 6.24.

Definition 1.5. *Suppose G is a conneted reductive algebraic group, and θ is a principal involution of G with fixed point set K. Let T be a θ-stable maximal torus in G. A set Δ' of positive imaginary roots of T in G is said to be of large type if each simple root is non-compact. A θ-stable Borel subgroup B of G is said to be of large type if it satisfies the property in Proposition 1.9(d). The corresponding Lie algebra $\mathfrak{b}$ is also said to be of large type.*

From the definition and Proposition 1.4, it is easy to see that only for the principal involution θ, the θ-stable Borel subgroup of large type exists.

We will give a new characterization of principal involution.

Proposition 1.6. *Suppose G is a connected reductive algebraic group, and θ is an involution of G. Then the involution is principal if and only if the Satake diagram of the correspoding real form $G(\mathbf{R})$ has no black vertex.*

Proof Let $\mathfrak{g}(\mathbf{R})=\mathfrak{k}(\mathbf{R})+\mathfrak{s}(\mathbf{R})$ be the corresponding Cartan decomposition of $\mathfrak{g}(\mathbf{R})$. Take a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{s}(\mathbf{R})$ and extend $\mathfrak{a}$ to $\mathfrak{t}(\mathbf{R})=\mathfrak{t}_k+\mathfrak{a}$ such that $\mathfrak{t}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Therefore T, the corresponding maximal torus, is also θ-stable.

If the Satake diagram of $G(\mathbf{R})$ has no black vertex, then, by the definition of Satake diagram, the corresponding simple root system ϖ has no imaginary root, i.e.,

$$\theta\alpha \neq \alpha \text{ for all of } \alpha \in \varpi.$$

It is easy to verify that

$$\theta\alpha \neq \alpha \text{ for all of } \alpha \in \Delta,$$

where Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$. Therefore the θ-stable maximal torus T has no imaginary root. By Proposition 1.4., θ is principal.

Suppose θ is principal. It follows from Proposition 1.4 that there exists a θ-stable maximal torus T with no imaginary roots. Clearly in the corresponding Cartan subalgebra $\mathfrak{t}=\mathfrak{t}_k+\mathfrak{t}_s$, $\mathfrak{t}_s$ must be a maximal Abelian subspace of $\mathfrak{s}$. In fact, if $\mathfrak{t}_s$ were not a maximal Abelian subspace of $\mathfrak{s}$, then there would be a θ-stable maximal torus T' with $\mathfrak{t}'_s$, a maximal Abelian subspace of $\mathfrak{s}$, and a Cayley transform $T \to T'$. But this means T has a strongly orthogonal set of non-compact imaginary roots. This is a contradiction. Therefore one can select orderings on $\mathfrak{t}$ and $\mathfrak{t}_s$ which are compatible and the corresponding Satake diagram has no black vertex.

Proposition 1.7. *Suppose G is a connected reductive algebraic group, and θ is an involution of G. Then there is a θ-stable maximal torus T such that there are no real roots of T in G. It has the following additional properties:*

(a) *T is unique up to conjugation by K(the subgroup of elements of G fixed by θ).*

(b) *$T \cap K^0$ is a maximal torus in K^0 (the identity component of K).*

(c) *$T=\mathrm{Cent}_G(T \cap K^0)$, i.e., T is the centralizer of $T \cap K^0$ in G.*

Remark. It is obvious that for the real form $G(\mathbf{R})$ corresponding to θ, $K(\mathbf{R})$ is a maximal compact subgroup of $G(\mathbf{R})$.

Proof Suppose first that there are no real roots for T. Let T_k be the identity component of $T \cap K$. Clearly no roots vanish on T_k, so its centralizer in G is just T. This proves (c). In particular, T_k is its own centralizer in K, so it is a maximal torus in K. This is (b). It follows that T_k is unique up to conjugation by K; so by (c), T is as well.

On the other hand, existence of a T with no real roots follows from the Cayley transform construction (see [1], Definition 6.23). In fact, the centralizer in G of a maximal torus in K^0 is a maximal torus in G with no real roots.

Theorem 1.2. *Suppose G is a connected reductive algebraic group, and θ is a principal involution of G. Take a θ-stable maximal torus T with no real roots (according to Proposition 1.12, it is unique up to conjugation by K). Then for the root system Δ of (G, T), there is an ordering of Δ such that*

$$\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$$

is a θ-stable Borel subalgebra and every simple root is either complex or non-compact *imaginary. Therefore the connected subgroup B of G whose Lie algebra is $\mathfrak{b}$ is the θ-stable Borel subgroup of large type.*

Proof According to Proposition 1.7, for the θ-stable maximal torus T with no real roots, $T\cap K^0$ is a maximal torus of K^0 and $T=\mathrm{Cent}_G(T\cap K^0)$. One can take a regular semisimple element X in $\mathfrak{t}\cap\mathfrak{k}=\mathrm{Lie}(T\cap K^0)$ such that the eigenvalues of $\mathrm{ad}X$ in $\mathfrak{g}$ are either pure imaginary or 0. Clearly

$\mathfrak{b}'=$ the sum of the eigensubspaces with respect to 0 and "positive" pure imaginary eigenvalues

is a Borel subalgebra and $\mathfrak{t}=\mathrm{Lie}(T)$ is just the 0-eigensubspace. It is not hard to see that $\mathfrak{b}'$ is θ-stable by using the fact that $T=\mathrm{Cent}_G(T\cap K^0)$. This means that the set Δ^+ of positive roots is θ-stable. In general, $\mathfrak{b}'$ might not be of large type. But according to the part (e) of Proposition 1.4, one can find an element w in the Weyl group of the imaginary roots with the property that every simple imaginary root in $w\mathfrak{b}'$ is non-compact. Since the action of w commutes with θ, $w\Delta^+$ is also θ-stable. Therefore $\mathfrak{b}=w\mathfrak{b}'$ is of large type.

Theorem 1.2 and its proof give a way to construct the θ-stable Borel subgroups of large type for principal involution. In the next section we will do this.

§ 2. Determination of the Borel Subgroups of Large Type

In this section we will determine all of the Borel subgroups of large type for the classical semisimple groups.

First according to Proposition 1.6 and Satake diagrams we have the following

Theorem 2.1. *For the classical semisimple groups, we have the following results:*

Group G	Number of inner classes	Principal involution θ	Quasisplit real form $G(\mathbf{R})$
$\mathrm{SL}(n,\mathbf{C})$	2	$\theta(g)={}^tg^{-1}\quad g\in G$	$\mathrm{SL}(n,\mathbf{R})$
		$\theta(g)=\begin{pmatrix}1&&&&\\&-1&&&\\&&\ddots&&\\&&&1&\\&&&&-1\end{pmatrix}g\begin{pmatrix}1&&&&\\&-1&&&\\&&\ddots&&\\&&&1&\\&&&&-1\end{pmatrix}$ $(n=2k)$	$\mathrm{SU}(k,k)$
		$\theta(g)=\begin{pmatrix}1&&&&&\\&-1&&&&\\&&\ddots&&&\\&&&1&&\\&&&&-1&\\&&&&&1\end{pmatrix}g\begin{pmatrix}1&&&&&\\&-1&&&&\\&&\ddots&&&\\&&&1&&\\&&&&-1&\\&&&&&1\end{pmatrix}$ $(n=2k+1)$	$\mathrm{SU}(k,k+1)$
$\mathrm{SO}(2n+1,\mathbf{C})$ $(n\geqslant 2)$	1	$\theta(g)=I_{n+1,n}gI_{n+1,n}$	$\mathrm{SO}_0(n+1,n)$
$\mathrm{Sp}(n,\mathbf{C})$ $(n\geqslant 3)$	1	$\theta(g)={}^tg^{-1}$	$\mathrm{Sp}(n,\mathbf{R})$
$\mathrm{SO}(2n,\mathbf{C})$ $(n\geqslant 4)$	2	$\theta(g)=I_{n,n}gI_{n,n}$	$\mathrm{SO}_0(n,n)$
		$\theta(g)=I_{2k+1,2k-1}gI_{2k+1,2k-1}$ $(n=2k)$	$\mathrm{SO}_0(2k+1,2k-1)$
		$\theta(g)=I_{2k+2,2k}gI_{2k+2,2k}$ $(n=2k+1)$	$\mathrm{SO}_0(2k+2,2k)$

Here

$$I_{n+1,n}=\begin{bmatrix} & I_n & \\ I_n & & \\ & & -1 \end{bmatrix},$$

$$I_{2k+2,2k}=\begin{bmatrix} I_2 & & & & & \\ & -I_2 & & & & \\ & & \ddots & & & \\ & & & I_2 & & \\ & & & & -I_2 & \\ & & & & & I_2 \end{bmatrix}, \quad I_{2k+1,2k-1}=\begin{bmatrix} I_2 & & & & & \\ & -I_2 & & & & \\ & & \ddots & & & \\ & & & I_2 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix},$$

$$I_{n,n}=\begin{cases} \begin{pmatrix} I_2 & & & & \\ & -I_2 & & & \\ & & \ddots & & \\ & & & I_2 & \\ & & & & -I_2 \end{pmatrix} & (n=2k), \\ \begin{pmatrix} I_2 & & & & & & \\ & -I_2 & & & & & \\ & & \ddots & & & & \\ & & & I_2 & & & \\ & & & & -I_2 & & \\ & & & & & 1 & \\ & & & & & & -1 \end{pmatrix} & (n=2k+1). \end{cases}$$

Proof It is just simple calculation.

In the next part, we will give the θ-stable Borel subgroup of large type for each class of principal involutions in Theorem 2.1.

$$G=\mathrm{SL}(2k, \mathbf{C}) \quad (n=2k),$$
$$\theta(g)=\mathrm{diag}(1-1\cdots1-1)g\,\mathrm{diag}(1-1\cdots1-1),\ g\in G,$$
$$\theta(X)=\mathrm{diag}(1-1\cdots1-1)X\,\mathrm{diag}(1-1\cdots1-1),\ X\in\mathfrak{g},$$
$$G(\mathbf{R})=\mathrm{SU}(k, k).$$

Take $T=\{$diagonal matrices in $G\}$. This is a θ-stable maximal torus in G. $\mathfrak{t}=\mathrm{Lie}(T)=\{$diagonal matrices with trace 0 in $G\}$. E_{ij} denotes the matrix whose (i, j)-component is 1 and the others are 0. Let $H=\mathrm{diag}(h_1h_2\cdots h_n)\in\mathfrak{t}$. We define $e_i\in\mathfrak{t}^*$ by

$$e_i(H)=h_i(i=1, \cdots, n).$$

Then the root system Δ of (G, T) consists of

$$e_i-e_j,\ 1\leqslant i\neq j\leqslant n.$$

Put $B=\{$upper triangular matrices in $G\}$. Then B is a θ-stable Borel subgroup of large type containing T. In fact, the root subspace $\mathfrak{g}_{e_i-e_j}$ of $\mathfrak{g}$ with root (e_i-e_j) is $\mathbf{C}E_{ij}$. If we choose the ordering in $\mathfrak{t}^*$ as follows

$$e_1>e_2>\cdots>e_n,$$

then the corresponding set Δ^+ of positive roots consists of

$$e_i-e_j,\ 1\leqslant i<j\leqslant n,$$

and the simple root system is $\{\alpha=e_1-e_2,\ \cdots,\ \alpha_{n-1}=e_{n-1}-e_n\}$. Because $\theta(e_i)=e_i(i=1,\ \cdots,\ n)$, every root is an imaginary root, and $\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ consists of all of the upper triangular matrices in $\mathfrak{g}$. It is obvious that $\theta(E_{ii+1})=-E_{ii+1}$. Hence $\mathfrak{b}$ is θ-stable and all of the simple root are non-compact imaginary roots. This means $\mathfrak{b}$ is a θ-stable Borel subalgebra of large type. Therefore B is the θ-stable Borel subgroup of large type.

$$G=\mathrm{SL}(2k+1,\ \mathbf{C})\quad (n=2k+1),$$
$$\theta(g)=\mathrm{diag}(1-1\cdots1-1\ 1)\,g\,\mathrm{diag}(1-1\cdots1-1\ 1),\ g\in G,$$
$$\theta(X)=\mathrm{diag}(1-1\cdots1-1\ 1)\,X\,\mathrm{diag}(1-1\cdots1-1\ 1),\ X\in\mathfrak{g},$$
$$G(\mathbf{R})=\mathrm{SU}(k,\ k+1)\cong\mathrm{SU}(k+1,\ k).$$

In this case, the situation is similar to the above case. Take $T=\{$diagonal matrices in $G\}$ and $B=\{$upper triangular matrices in $G\}$. Then the same argument shows B is a θ-stable Borel subgroup of large type containing T.

The two cases are the simplest cases.

$G=\mathrm{SL}(n,\ \mathbf{C})$, $\theta(g)={}^tg^{-1}$, $g\in G$; $\theta(X)=-{}^tX$, $X\in\mathfrak{g}$. $G(\mathbf{R})=\mathrm{SL}(n,\ \mathbf{R})$.

(1) $n=2k$.

It is easy to verify that

$$\mathfrak{t}=\left\{H=\begin{bmatrix} h_1 & -h_{k+1} & & & \\ h_{k+1} & h_k & & & \\ & & \ddots & & \\ & & & h_k & -h_{2k} \\ & & & h_{2k} & h_k \end{bmatrix};\ h_i\in\mathbf{C},\ \sum_{i=1}^{k}h_i=0\right\}$$

is a θ-stable Cartan subalgebra of $\mathrm{sl}(2k,\ \mathbf{C})$ with no real roots. In fact, let e_i be the linear function on $\mathfrak{t}$ defined by

$$e_i(H)=\begin{cases} h_i+\sqrt{-1}\,h_{k+i}, & 1\leqslant i\leqslant k,\\ h_{n-i+1}-\sqrt{-1}\,h_{k+n+1-i}, & k+1\leqslant i\leqslant n.\end{cases}$$

Then the root system of $(\mathrm{sl}(2k,\ \mathbf{C}),\ \mathfrak{t})$ is

$$\Delta=\{\pm(e_i-e_j);\ 1\leqslant i<j\leqslant n\}.$$

Clearly there is no real root in Δ.

We can take

$$J_{a_1,\cdots,a_n}=\begin{bmatrix} \sqrt{-1} & a_1-a_{k+1} & & & \\ a_{k+1} & \sqrt{-1}a_1 & & & \\ & & \ddots & & \\ & & & \sqrt{-1}a_k & -a_{2k} \\ & & & a_{2k} & \sqrt{-1}a_k \end{bmatrix},$$

where $a_1>a_2>\cdots>a_k$, $a_1+a_2+\cdots+a_k=0$; $a_{k+1}>a_{k+2}>\cdots>a_{2k}>0$ and $a_{k+i}>a_i(i=1,\cdots,k)$, $a_{k+i}-a_{k+j}>a_i-a_j(1\leqslant i<j\leqslant k)$. (This is possible. For example, we can take $a_1>\cdots>a_k$ such that $\sum_i a_i=0$ and $a_i-a_{i+1}=1(i=1,\cdots,k)$ and take $a_{k+1}>\cdots>a_{2k}>a_1$, $a_{k+i}-a_{k+i+1}=2(i=1,\cdots,k)$.) Clearly $J_{a_1,\cdots,a_n}$ is a regular semisimple element in $\mathfrak{t}\cap\mathfrak{k}$.

Let us discuss the eigenvalues and eigenvectors of ad $J_{a_1\cdots a_n}$ on sl $(2k, \mathbf{C})$. We have the following results:

The eigenvalues of ad $J_{a_1,\cdots,a_n}$ are 0 (multiplicity $n-1$), $\pm2\sqrt{-1}a_{k+i}$ (multiplicity 1) $(i=1,\cdots,k)$ and $\sqrt{-1}(a_s-a_t+a_{k+s}+a_{k+t})$ $(1\leqslant s\neq t\leqslant k)$ (multiplicity 1).

In order to express the eigenvectors we introduce some notations. Put

$$A=\begin{pmatrix}-1 & \sqrt{-1}\\ \sqrt{-1} & 1\end{pmatrix},\ B=\begin{pmatrix}1 & \sqrt{-1}\\ -\sqrt{-1} & 1\end{pmatrix},$$

$$C=\begin{pmatrix}1 & -\sqrt{-1}\\ \sqrt{-1} & 1\end{pmatrix},\ D=\begin{pmatrix}1 & \sqrt{-1}\\ \sqrt{-1} & -1\end{pmatrix},$$

$$A_{ij}=\begin{bmatrix} & 0 & \\ & \vdots & \\ 0\cdots & A & \cdots 0\\ & \vdots & \\ & 0 & \end{bmatrix}\ i^{\text{th}}\text{row (}A\text{ in the }j^{\text{th}}\text{ column)}$$

and B_{ij}, C_{ij}, D_{ij} are similar.

Therefore we have that

$\mathfrak{t}$ is just the 0-eigenspace of ad $J_{a_1,\cdots,a_n}$;

the $2\sqrt{-1}a_{k+i}$-eigenspace is $\mathbf{C}A_{ii}$;

the $-2\sqrt{-1}a_{k+i}$-eigenspace is $\mathbf{C}D_{ii}$;

the $\sqrt{-1}(a_s-a_t+a_{k+s}+a_{k+t})$-eigenspace is $\mathbf{C}A_{st}$ $(1\leqslant s\neq t\leqslant k)$;

the $\sqrt{-1}(a_s-a_t-a_{k+s}-a_{k+t})$-eigenspace is $\mathbf{C}D_{st}$ $(1\leqslant s\neq t\leqslant k)$;

the $\sqrt{-1}(a_s-a_t+a_{k+s}-a_{k+t})$-eigenspace is $\mathbf{C}B_{st}$ $(1\leqslant s\neq t\leqslant k)$;

the $\sqrt{-1}(a_s-a_t-a_{k+s}+a_{k+t})$-eigenspace is $\mathbf{C}C_{st}$ $(1\leqslant s\neq t\leqslant k)$.

According to the choice of a_i's, the "positive" pure imaginary eigenvalues of ad$J_{a_1,\cdots,a_n}$ are $\sqrt{-1}a_{k+i}(1\leqslant i\leqslant k)$, $\sqrt{-1}(a_s-a_t+a_{k+s}+a_{k+t})(1\leqslant s\neq t\leqslant k)$, $\sqrt{-1}(a_s-a_t+a_{k+s}-a_{k+t})(1\leqslant s<t\leqslant k)$ and $\sqrt{-1}(a_s-a_t-a_{k+s}+a_{k+t})(1\leqslant t<s\leqslant k)$. Therefore

$$\mathfrak{b}=\mathfrak{t}+\sum_{1\leqslant i,j\leqslant k}\mathbf{C}A_{ij}+\sum_{1\leqslant i<j\leqslant k}\mathbf{C}B_{ij}+\sum_{1\leqslant j<i\leqslant k}\mathbf{C}C_{ij}$$

is a Borel subalgebra of $\mathrm{sl}(n,\mathbf{C})$. Because ${}^tA_{ij}=A_{ji}$, ${}^tB_{ij}=C_{ji}$, we can conclude that $\theta(\mathfrak{b})=\mathfrak{b}$, i.e., $\mathfrak{b}$ is a θ-stable Borel subalgebra.

It is easy to verify that the root subspaces $\mathfrak{g}_{e_i-e_j}$ are as follows:

$\mathfrak{g}_{e_i-e_{n-i+1}}=\mathbf{C}A_{ii}$, $\mathfrak{g}_{e_{n-i+1}-e_i}=\mathbf{C}D_{ii}$ $(1\leqslant i\leqslant k)$;

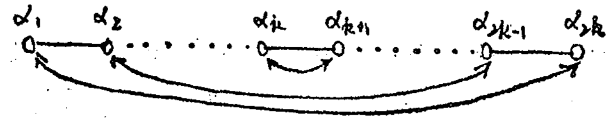
$\mathfrak{g}_{e_i-e_j}=\mathbf{C}B_{ij}(1\leqslant i<j\leqslant k)$; $\mathfrak{g}_{e_j-e_i}=\mathbf{C}C_{ij}(1\leqslant i<j\leqslant k)$;

$$\mathfrak{g}_{e_i-e_{n-j+1}}=\mathbf{C}A_{ij}(1\leqslant i\neq j\leqslant k);\ \mathfrak{g}_{e_{n-j+1}-e_i}=\mathbf{C}D_{ij}(1\leqslant i\neq j\leqslant k);$$
$$\mathfrak{g}_{e_i-e_j}=\mathbf{C}C_{n-i+1,n-j+1}(k<i<j\leqslant n);\ \mathfrak{g}_{e_j-e_i}=\mathbf{C}B_{n-i+1,n-j+1}(k<i<j\leqslant n).$$

If we take the ordering $e_1>e_2>\cdots>e_n$ on $\mathfrak{t}^*$, then

$$\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha,$$

here Δ^+ is the set of positive roots and the simple root system consists of $\alpha_i=e_i-e_{i+1}(i=1,\cdots,n-1)$. It is clear that $\theta B_{i,i+1}=-C_{i+1,i}$, so $\theta\alpha_i=\alpha_{2k-i}(i=1,\cdots,k-1)$. This means that $\alpha_i(i\neq k)$ are complex roots and from $\theta\alpha_k=\alpha_k$ and $\theta A_{kk}=-A_{kk}$ we conclude that α_k is a non-compact imaginary root. Therefore $\mathfrak{b}$ is of large type and the corresponding Borel subgroup is also of large type.



(2) $n=2k+1$.

Similar to (1), we have

$$\mathfrak{t}=\left\{H=\begin{bmatrix} h_1 & -h_{k+2} & & & & \\ h_{k+2} & h_1 & & & & \\ & & \ddots & & & \\ & & & h_k & -h_{2k+1} & \\ & & & h_{2k+1} & h_k & \\ & & & & & h_{k+1}\end{bmatrix};\ h_i\in\mathbf{C},\ \sum_{i=1}^{k+1}h_i=0\right\}$$

is a θ-stable Cartan subalgebra of sl $(2k+1,\mathbf{C})$. Let e_i be the linear function on $\mathfrak{t}$ defind by

$$e_i(H)=\begin{cases} h_i+\sqrt{-1}\,h_{k+1+i}, & 1\leqslant i\leqslant k,\\ h_{k+1}, & i=k+1,\\ h_{n-i+1}-\sqrt{-1}\,h_{n-i+k+2}, & k+2\leqslant i\leqslant 2k+1.\end{cases}$$

Then the root system of (sl $(2k+1,\mathbf{C})$, $\mathfrak{t}$) is

$$\Delta=\{\pm(e_i-e_j);\ 1\leqslant i<j\leqslant 2k+1\}.$$

Put

$A'_{ij}=\begin{pmatrix}A_{ij} & 0\\ 0 & 0\end{pmatrix}$; B'_{ij}, C'_{ij} and D'_{ij} are similar;

$$L_i=\begin{bmatrix} & & 0 & & 0\\ 0\cdots 1 & & -\sqrt{-1} & \cdots 0 & 0\end{bmatrix};\quad M_i=\begin{bmatrix} & 0\\ & \vdots\\ & 1\\ 0 & -\sqrt{-1}\\ & \vdots\\ & 0\\ 0 & 0\end{bmatrix}\ 2i^{\text{th}}\text{ row}$$

$2i^{\text{th}}$ column $\qquad (i=1,\cdots,k).$

Then we have

$\mathfrak{g}_{e_i-e_j}=\mathbf{C}B'_{ij}(1\leqslant i<j\leqslant k)$; $\mathfrak{g}_{e_i-e_j}=\mathbf{C}C'_{ij}(1\leqslant i<j\leqslant k)$;

$\mathfrak{g}_{e_i-e_{k+1}}=\mathbf{C}M_i(1\leqslant i\leqslant k)$; $\mathfrak{g}_{e_{k+1}-e_{n-i+1}}=\mathbf{C}L_i(1\leqslant i\leqslant k)$;

$\mathfrak{g}_{e_i-e_{n-j+1}}=\mathbf{C}A'_{ij}(1i,\ jk)$; $\mathfrak{g}_{e_{n-j+1}-e_i}=\mathbf{C}D'_{ij}(1\leqslant i,\ j\leqslant k)$;

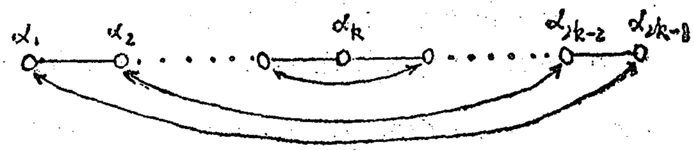
$\mathfrak{g}_{e_i-e_j}=\mathbf{C}C'_{n-i+1,n-j+1}(k+2\leqslant i<j\leqslant n)$; $\mathfrak{g}_{j-e_i}=\mathbf{C}B'_{n-i+1,n-j+1}(k+2\leqslant i<j\leqslant n)$.

If we take the ordering $e_1>e_2>\cdots>e_n$, then

$$\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha=\mathfrak{t}+\sum_{1\leqslant i,j\leqslant k}\mathbf{C}A'_{ij}+\sum_{1\leqslant i<j\leqslant k}\mathbf{C}B'_{ij}+\sum_{1\leqslant i<j\leqslant k}\mathbf{C}C'_{ji}+\sum_{i=1}^{k}\mathbf{C}M_i+\sum_{i=1}^{k}\mathbf{C}L_i$$

is a Borel subalgebra. Because $\theta A'_{ij}=-A'_{ij}$, $\theta B'_{ij}=-C'_{ij}$, $\theta M_i=-L_i$ and $\theta(\mathfrak{t})=\mathfrak{t}$, $\mathfrak{b}$ is θ-stable. The simple root system π is $\{\alpha_i=e_i-e_{i+1}(i=1,\cdots,n-1)\}$. It is easy to see that $\theta\alpha_i=\alpha_{2k-i+1}(i=1,\cdots,k)$, so all of the simple roots are complex roots.

Therefore $\mathfrak{b}$ is of large type.



$G=SO(2n+1,\ \mathbf{C})$,

$\theta(g)=I_{n+1,n}gI_{n+1,n}$, $g\in G$; $\theta(X)=I_{n+1,n}XI_{n+1,n}$, $X\in\mathfrak{g}$, where

$$I_{n+1,n}=\begin{bmatrix} & I_n & \\ I_n & & \\ & & -1\end{bmatrix},$$

$G(\mathbf{R})=SO_0(n+1,\ n)$.

(1) $n=2k$.

$$\mathfrak{t}=\left\{H=\begin{bmatrix}H_1 & H_2 & \\ H_2 & H_1 & \\ & & 0\end{bmatrix}\right\},$$

where

$$H_1=\begin{bmatrix} & h_1 & & \\ & & \ddots & \\ & & & h_k\\ -h_1 & & & \\ & \ddots & & \\ & & -h_k & \end{bmatrix},\ H_2=\begin{bmatrix} & -h_{k+1} & & \\ & & \ddots & \\ & & & -h_n\\ h_{k+1} & & & \\ & \ddots & & \\ & & h_n & \end{bmatrix}\quad (h_i\in\mathbf{C})$$

is a θ-stable Cartan subalgebra of $SO(2n+1,\ \mathbf{C})$. Let e_i be the linear function defined by

$$e_{2i-1}(H)=(h_i+h_{k+i})\sqrt{-1}\quad (i=1,\ \cdots,\ k),$$
$$e_{2i}(H)=(h_i-h_{k+i})\sqrt{-1}\quad (i=1,\ \cdots,\ k),$$

then the root system of $(SO(2n+1,\ \mathbf{C}),\ \mathfrak{t})$ is

$$\Delta=\{\pm(e_i\pm e_j),\ \pm e_i;\ i=1,\ \cdots,\ n\}$$

and one can verify that there is no real roots in Δ.

If

$$X=\begin{bmatrix} X_{11} & X_{12} & X_{14} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}\in SO(2n+1, \mathbf{C}),$$

then

$$\theta(X)=\begin{bmatrix} X_{22} & X_{21} & -X_{23} \\ X_{12} & X_{11} & -X_{13} \\ -X_{32} & -X_{31} & X_{33} \end{bmatrix},$$

where $X_{ij}(1\leqslant i,\ j\leqslant 2)$ are n by n, X_{13}, X_{23} are n by 1, X_{31} and X_{32} are 1 by n and X_{33} is 1 by 1.

One can check $\theta(\alpha)=\alpha\ \forall\alpha\in(SO(2n+1, \mathbf{C}), \mathfrak{t})$. Taking the ordering $e_1>e_2>\cdots>e_n$ on $\mathfrak{t}^*$, the corresponding set of the positive roots is

$$\Delta^+=\{e_i\pm e_j,\ e_i;\ 1\leqslant i<j\leqslant n\},$$

then $\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ is a θ-stable Borel subalgebra. To verify this, we only need to compute the root subspaces corresponding to Δ^+. But it is too long to write the results here, so we omit them. Also we can verify that for every root α_i in the simple root system $\{\alpha_1=e_1-e_2,\ \cdots,\ \alpha_{n-1}=e_{n-1}-e_n,\ \alpha_n=e_n\}$, $\theta(X)=-X$ for any $X\in\mathfrak{g}_{\alpha_i}$, so $\alpha_i(1\leqslant i\leqslant n)$ is non-compact; therefore $\mathfrak{b}$ is of large type.

(2) $n=2k+1$.

Let H_1 and H_2 be the same as in the case of $n=2k$. Then

$$\mathfrak{t}=\left\{H=\begin{bmatrix} H_1 & H_2 & & \\ & & & -h_n \\ -H_2 & H_1 & & \\ & & & h_n \\ & h_n & -h_n & \end{bmatrix};\ h_i\in\mathbf{C}\right\}$$

is a θ-stable Cartan subalgebra of $\mathfrak{so}(2n+1, \mathbf{C})$. Let e_i be the linear function defined by

$$e_{2i-1}(H)=(h_i+h_{k+i})\sqrt{-1}\quad(i=1,\cdots,k),$$
$$e_{2i}(H)=(h_i-h_{k+i})\sqrt{-1}\quad(i=1,\cdots,k),$$
$$e_n(H)=\sqrt{-2}\,h_n.$$

Then the root system of $(\mathfrak{so}(2n+1, \mathbf{C}), \mathfrak{t})$ is

$$\Delta=\{\pm e_i\pm e_j,\ \pm e_i;\ 1\leqslant i<j\leqslant n\}.$$

One can verify $\theta(\alpha)=\alpha\forall\alpha\in\Delta$. Therefore there is no real root in Δ. Taking the ordering $e_1>e_2>\cdots>e_n$ on $\mathfrak{t}^*$, the corresponding set of positive roots is $\Delta^+=\{e_i\pm e_j,\ e_i\ (1\leqslant i<j\leqslant n)\}$ and the simple root system is $\pi=\{\alpha_i=e_i-e_{i+1}\ (i=1,\cdots,n-1),\ \alpha_n=e_n\}$.

In the same way as the case of $n=2k$, we can verify that $\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ is a θ-stable Borel subalgebra of $\mathfrak{so}(2n+1, \mathbf{C})$ and $\alpha_i(1\leqslant i\leqslant n)$ is non-compact. Therefore

$\mathfrak{b}$ is of large type.

$$G=\mathrm{sp}(n,\mathbf{C})=\left\{A\in GL(2n,\mathbf{C});\ {}^tAJA=J,\ \text{where } J=\begin{bmatrix} & -I_n\\ I_n & \end{bmatrix}\right\},$$

$$\theta(g)={}^tg^{-1},\ g\in\mathrm{sp}(n,\mathbf{C});\ \theta(X)=-{}^tX,\ X\in\mathrm{sp}(n,\mathbf{C});$$

$G(\mathbf{R})=\mathrm{sp}(n,\mathbf{R})$, which is the normal real form of G.

$$X\in\mathrm{sp}(n,\mathbf{C})\ \text{iff}\ X=\begin{bmatrix} X_1 & X_2\\ X_3 & -{}^tX_1\end{bmatrix}\ X_i\in M_n(C),\ {}^tX_2=X_2,\ {}^tX_3=X_3.$$

$$\mathfrak{t}=\left\{H=\begin{bmatrix} & -H'\\ H' & \end{bmatrix},\ \text{where } H'=\mathrm{diag}(h_1h_2\cdots h_n),\ h_i\in\mathbf{C}\right\}$$

is a θ-stable Borel subalgebra of $\mathrm{sp}(n,\mathbf{C})$. Let e_i be the linear function on $\mathfrak{t}$ defined by $e_i(h)=\sqrt{-1}\,h_i(i=1,\cdots,n)$. Then the root system of $(\mathrm{sp}(n,\mathbf{C}),\mathfrak{t})$ is

$$\Delta=\{\pm e_i\pm e_j, 2e_i;\ (1\leqslant i<j\leqslant n)\},$$

and $\theta(\alpha)=\alpha$ for all of $\alpha\in\Delta$.

The root subspaces of $(\mathrm{sp}(n,\mathbf{C}),\mathfrak{t})$ are as follows.

Put

$$P_i=\begin{bmatrix}\sqrt{-1}E_{ii} & E_{ii}\\ E_{ii} & -\sqrt{-1}E_{ii}\end{bmatrix},\quad P_{-i}=\begin{bmatrix}-\sqrt{-1}E_{ii} & E_{ii}\\ E_{ii} & \sqrt{-1}E_{ii}\end{bmatrix},$$

$$G_{ij}=\begin{bmatrix}\sqrt{-1}(E_{ij}=E_{ji}) & E_{ij}+E_{ji}\\ E_{ij}+E_{ji} & -\sqrt{-1}(E_{ij}+E_{ji})\end{bmatrix},$$

$$G_{-i,-j}=\begin{bmatrix}-\sqrt{-1}(E_{ij}+E_{ji}) & E_{ij}+E_{ji}\\ E_{ij}+E_{ji} & \sqrt{-1}(E_{ij}+E_{ji})\end{bmatrix},$$

$$G_{i,-j}=\begin{bmatrix}\sqrt{-1}(E_{ij}+E_{ji}) & E_{ij}+E_{ji}\\ -(E_{ij}+E_{ji}) & \sqrt{-1}(E_{ij}-E_{ji})\end{bmatrix},$$

$$G_{-i,j}=\begin{bmatrix}-\sqrt{-1}(E_{ij}-E_{ji}) & E_{ij}+E_{ji}\\ -(E_{ij}+E_{ji}) & -\sqrt{-1}(E_{ij}+E_{ji})\end{bmatrix}.$$

We have

$$\mathfrak{g}_{2e_i}=\mathbf{C}P_i;\ \mathfrak{g}_{-2e_i}=\mathbf{C}P_{-i};\ \mathfrak{g}_{e_i+e_j}=\mathbf{C}G_{ij}\ (1\leqslant i\neq j\leqslant n);$$
$$\mathfrak{g}_{-e_i-e_j}=\mathbf{C}G_{-i,-j}(1\leqslant i\neq j\leqslant n);$$
$$\mathfrak{g}_{e_i-e_j}=\mathbf{C}G_{i,-j}(1\leqslant i<j\leqslant n);\ \mathfrak{g}_{-e_i+e_j}=\mathbf{C}G_{-i,j}\ (1\leqslant i<j\leqslant n).$$

It is easy to verify that $\theta P_i=-P_i$; $\theta P_{-i}=-P_{-i}$; $\theta G_{ij}=-G_{ij}$; $\theta G_{-i,-j}=-G_{-i,-j}$; $\theta G_{i,-j}=G_{i,-j}$; $\theta G_{-i,j}=G_{-i,j}$. This means that $\pm 2e_i(i=1,\cdots,n)$ and $\pm(e_i\pm e_j)$ $(1\leqslant i\neq j\leqslant n)$ are non-compact imaginary roots and $\pm(e_i-e_j)\,(1\leqslant i<j\leqslant n)$ are compact imaginary roots.

Take the ordering on $\mathfrak{t}^*$ defined by $e_1>-e_2>e_3>-e_4>\cdots>(-1)^{n-1}e_n$. Then the set of positive roots is

$$\Delta^+=\{(-1)^{i-1}e_i\pm(-1)^{j-1}e_j,\ (-1)^{i-1}2e_i;\ (1\leqslant i<j\leqslant n)\}$$

and the simple root system π is

$$\{\alpha_1 = e_1 - (-e_2) = e_1 + e_2,\ \alpha_2 = -e_2 - e_3,\ \cdots,\ \alpha_{n-1} = (-1)^{n-2}(e_{n-1}+e_n), \alpha_n = (-1)^{n-1}2e_n\}.$$

Because every $\alpha \in \pi$ is a non-compact imaginary root, $\mathfrak{b} = \mathfrak{t} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ is a θ-stable Borel subalgebra of large type.

$G = SO(2n, \mathbf{C})$

$$\theta(g) = I_{n,n} g I_{n,n},\ g \in so(2n, \mathbf{C}),$$
$$\theta(X) = I_{n,n} X I_{n,n},\ X \in so(2n, \mathbf{C}),$$

$G(R) = SO_0(n, n)$.

$X \in SO(2n, \mathbf{C})$ can be written as $X = (X_{ij})$, where $X_{ij} \in M_2(\mathbf{C})$ and ${}^tX_{ii} = -X_{ii}$, ${}^tX_{ij} = -X_{ji}$. If $n = 2k$, $\theta(X) = ((-1)^{i+j} X_{ij})$; If $n = 2k+1$, $\theta(X) = (Y_{ij})$ where $Y_{ij} = (-1)^{i+j} X_{ij}$ $(1 \leqslant i, j \leqslant 2k)$; $Y_{in} = (-1)^{i+1} X_{in} I_{1,1}$; $Y_{ni} = (-1)^{i+1} I_{1,1} X_{ni}$; $(i = 1, \cdots, n-1)$ $Y_{nn} = I_{1,1} X_{nn} I_{1,1}$.

$$\mathfrak{t} = \left\{ H = \begin{bmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_n \end{bmatrix}, \text{ where } H_i = \begin{bmatrix} & h_i \\ -h_i & \end{bmatrix}\ h_i \in \mathbf{C} \right\}.$$

This is a Cartan subalgebra of so $(2n, \mathbf{C})$ which is θ-stable. Let e_i be the linear function on $\mathfrak{t}$ defined by

$$e_i(H) = -\sqrt{-1} h_i \quad (i = 1, \cdots, n).$$

Then the root system of $(so(2n, \mathbf{C}), \mathfrak{t})$ is

$$\Delta = \{\pm e_i \pm e_j\ 1 \leqslant i \neq j \leqslant n\}.$$

To describe the root subspaces we introduce some notations. Put

$$B = \begin{bmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{bmatrix},\ A = \begin{bmatrix} -1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix} \text{ as above}$$

$$F_{ij} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & B & \cdots \\ & \vdots & & \vdots & \\ \cdots & -{}^tB & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i^{\text{th}} \text{ row} \\ \\ j^{\text{th}} \text{ row} \end{matrix} \quad (1 \leqslant i < j \leqslant n),$$

i^{th} column j^{th} column

$$F_{ji} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -{}^tB & \cdots \\ & \vdots & & \vdots & \\ \cdots & B & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i^{\text{th}} \text{ row} \\ \\ j^{\text{th}} \text{ row} \end{matrix} \quad (1 \leqslant i < j \leqslant n),$$

i^{th} column j^{th} column

$$T_{ij} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & D & \cdots \\ & \vdots & & \vdots & \\ \cdots & -{}^tD & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i^{\text{th}} \text{ row} \\ \\ j^{\text{th}} \text{ row} \end{matrix} \quad (1 \leqslant i < j \leqslant n),$$

i^{th} column j^{th} column

$$T_{ji}=\begin{bmatrix} \cdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & 0 & \cdots & A & \cdots \\ & \vdots & & \vdots & \\ \cdots & -{}^tA & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix}\begin{matrix} i^{\text{th}}\text{ row} \\ \\ j^{\text{th}}\text{ row}\end{matrix} \quad (1\leqslant i<j\leqslant n).$$

i^{th} column j^{th} column

Then $\mathfrak{g}_{e_i-e_j}=\mathbf{C}F_{ij}(1\leqslant i\neq j\leqslant n)$, $\mathfrak{g}_{e_i+e_j}=\mathbf{C}T_{ij}(1\leqslant j<i\leqslant n)$ and

$$\mathfrak{g}_{-e_i-e_j}=\mathbf{C}T_{ij}\quad(1\leqslant i<j\leqslant n).$$

(a) $n=2k$. In this case, $\theta(\alpha)=\alpha$ for all of $\alpha\in\Delta$ and

$$\theta F_{ij}=(-1)^{i+j}F_{ij},\ \theta T_{ij}=(-1)^{i+j}T_{ij}.$$

Take the ordering on $\mathfrak{t}^*$ defined by $e_1>e_2>\cdots>e_n$. Then the set of positive roots is

$$\Delta^+=\{e_i\pm e_j(1\leqslant i<j\leqslant n)\}$$

and the simple root system is

$$\pi=\{\alpha_1=e_1-e_2,\ \alpha_2=e_2-e_3,\ \cdots,\ \alpha_{n-1}=e_{n-1}-e_n,\ \alpha_n=e_{n-1}+e_n\}.$$

Because $\theta F_{ii+1}=-F_{ii+1}$ and $\theta T_{nn-1}=-T_{nn-1}$, $\alpha_i(i=1,\ \cdots,\ n)$ is a non-compact imaginary root. Therefore the θ-stable Borel subalgebra

$$\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha=\mathfrak{t}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}F_{ij}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}T_{ji}$$

is of large type.

(b) $n=2k+1$. It is easy to show that $\theta(e_i)=e_i\quad(i=1,\ \cdots,\ n-1)$, $\theta(e_n)=-e_n$. And we have

$$\begin{matrix}\theta(F_{in})=(-1)^iT_{ni};\ \theta(T_{ni})=(-1)^iF_{in};\\ \theta(F_{ni})=(-1)^iT_{in};\ \theta(T_{in})=(-1)^iF_{ni}.\end{matrix}\quad(i=1,\ \cdots,\ n-1)$$

Take the ordering on $\mathfrak{t}^*$ defined by $e_1>e_2>\cdots>e_n$ again. Then the subalgebra

$$\mathfrak{b}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha=\mathfrak{t}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}F_{ij}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}T_{ji}$$

is θ-stable Borel subalgebra of large type because $\alpha_1,\ \cdots,\ \alpha_{n-2}$ are non-compact imaginary roots and α_{n-1}, α_n are complex roots $(\theta\alpha_{n-1}=\alpha_n,\ \theta\alpha_n=\alpha_{n-1})$.

$G=\mathrm{so}(2n,\ \mathbf{C})\quad(n=2k)$.

$\theta(g)=I_{2k+1,2k-1}gI_{2k+1,2k-1},\quad g\in\mathrm{so}(2n,\ \mathbf{C})$,

$\theta(X)=I_{2k+1,2k-1}XI_{2k+1,2k-1},\quad X\in\mathrm{so}(2n,\ \mathbf{C})$.

$G(\mathbf{R})=\mathrm{SO}_0(2k+1,\ 2k-1)$.

This case is similar to the case of $G=\mathrm{so}(2n,\ \mathbf{C})(n=2k+1)$, $\theta(g)=I_{n,n}gI_{n,n}$. Using the same notations, we have

$$\mathfrak{b}=\mathfrak{t}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}F_{ij}+\sum_{1\leqslant i<j\leqslant n}\mathbf{C}T_{ji}$$

is a θ-stable Borel subalgebra of large type.

$G=\mathrm{so}(2n,\ \mathbf{C})\quad(n=2k+1)$.

$\theta(g)=I_{2k+2,2k}gI_{2k+2,2k},\quad g\in\mathrm{so}(2n,\ \mathbf{C})$

$\theta(X)=I_{2k+2,2k}XI_{2k+2,2k},\quad X\in\mathrm{so}(2n,\ \mathbf{C})$,

$G(\mathbf{R})=\mathrm{SO}_0\quad(2k+2,\ 2k)$.

It is similar to the case of $G=\mathrm{so}(2n,\ \mathbf{C})(n=2k)$ and $\theta(g)=I_{n,n}\ gI_{n,n}$. The θ-stable Borel subalgebra of large type is

$$\mathfrak{b}=\mathfrak{t}+\sum_{1\le i<j\le n}\mathbf{C}F_{ij}+\sum_{1\le i<j\le n}\mathbf{C}T_{ji}.$$

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