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A GLOBAL REPRESENTATION OF ALL SOLUTIONS TO A NONLINEAR EQUATION AND ITS APPLICATIONS**

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Abstract

For a nonlinear equation, a global representation for all solutions is obtained. Via this represention, a nonlinear generalized inverse theorem is derived and an application to control systems with mixture constraints is given as well.

§ 1. Introduction

Let U be a Hilbert space and Z be a Banach space. Let h: $U \to Z$ be a continuous map. In this paper, we are interested in the following equation:

$$h(u)=0. \tag{1.1}$$

In the case that $h(\cdot)$ is affine, i. e., for some $G \in \mathscr{L}(U, Z)$ and some $z \in Z$,

$$h(u)=Gu-z, \ \forall u \in U, \tag{1.2}$$

we know that by assuming $GG^* \in \mathscr{L}(Z)$ to be invertible, one can solve (1.1) explicitly. In fact, in this case, all possible solutions of (1.1) can be represented as

$$u=G^*(GG^*)^{-1}z+(I-G^*(GG^*)^{-1}G)v, \tag{1.3}$$

with $v \in U$ as a parameter([1]). Our goal of this paper is to obtain a representation of all solutions to the nonlinear equation (1.1) under some suitable conditions which includes (1.2) as a special case. This representation is derived in § 2 and the basic idea is taken from [4] (see [2, 3] also). In § 3, as an application of the obtained representation, we establish a generalized inverse theorem for nonlinear maps. Some continuity of the inverse map is also studied. Finally, in § 4, we prove the equivalence of two control systems, one of which is subject to a mixture constraint and the other has no constraints.

To close the introduction, we should point out that our result is also applicable to singular perturbation problems in control systems ([2, 3]).

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§ 2. Solution Representation

In this section, we will obtain a global representation for all solutions of (1.1). To this end, let us first make the following assumption.

(H1) The map $h: U \to Z$ is continuously differentiable. Moreover, there is a constant $\beta_0>0$ and a nondecreasing function L_0: $[0, \infty)\to[0, \infty)$, such that

$$|h_u^*(u)z^*|_U \geqslant \frac{\beta_0|z^*|_{Z^*}}{1+|u|_U}, \quad \forall u\in U,\ z^*\in Z^*, \tag{2.1}$$

$$\|(h_u(u)-h_u(\hat{u})\|_{\mathscr{L}(U,Z)} \leqslant L_0(|u|\vee|\hat{u}|)|u-\hat{u}|_U, \quad \forall u,\ \hat{u}\in U, \tag{2.2}$$

where $|u|\vee|\hat{u}|\equiv\max\{|u|, |\hat{u}|\}$.

Lemma 2.1. *Let $G\in\mathscr{L}(U, Z)$ satisfy*

$$|G^*z^*|_U\geqslant\sigma|z^*|_z, \quad \forall z^*\in Z^*, \tag{2.3}$$

for some $\sigma>0$. Then, $(GG^*)^{-1}\in\mathscr{L}(Z, Z^*)$ and

$$\|(GG)^{-1}\|_{\mathscr{L}(Z,Z^*)}\leqslant\sigma^{-2}, \tag{2.4}$$

$$\|G^*(GG^*)^{-1}\|_{\mathscr{L}(Z,U)}\leqslant\sigma^{-1}, \tag{2.5}$$

$$\|G^*(GG^*)^{-1}G\|_{\mathscr{L}(Z)}\leqslant 1. \tag{2.6}$$

Proof From (2.3), we know $(GG^*)^{-1}\in\mathscr{L}(Z, Z^*)$ (see [5], p.205). Next, for any $z\in Z$ with $|z|_Z=1$, we have

$$\sigma^2|(GG^*)^{-1}z|^2_{Z^*}\leqslant(G^*(GG^*)^{-1}z,\ G^*(GG^*)^{-1}z)_U=\langle(GG^*)^{-1}z,\ z\rangle_{Z^*,Z}\leqslant|(GG^*)^{-1}z|_{Z^*}.$$

Tnen (2.4) follows. Again, for $z\in Z$ with $|z|_Z=1$,

$$|G^*(GG^*)^{-1}z|^2_U=(G^*(GG^*)^{-1}z,\ G^*(GG^*)|^1z)_U=\langle(GG^*)^{-1}z,\ z\rangle_{Z^*,Z}\leqslant\sigma^{-2}.$$

Hence, (2.5) follows. Finally, for any $u\in U$ with $|u|_U=1$,

$$|G^*(GG^*)^{-1}Gu|^2_U=(G^*(GG^*)^{-1}Gu,\ G^*(GG^*)^{-1}Gu)_U$$
$$=\langle(GG^*)^{-1}Gu,\ Gu\rangle_{Z^*,Z}=(G^*(GG^*)^{-1}Gu,\ u)\leqslant|G^*(GG^*)^{-1}Gu|_U.$$

Therefore, (2.6) follows.

Lemma 2.2. *Let G, $\hat{G}\in\mathscr{L}(U, Z)$ satisfy*

$$|G^*z^*|_V\geqslant\sigma|z^*|_{Z^*},\quad |\hat{G}^*z^*|_U\geqslant\hat{\sigma}|z^*|_{Z^*},\quad \forall z^*\in Z^*, \tag{2.7}$$

for some σ, $\hat{\sigma}>0$. Then, for any z, $\hat{z}\in Z$,

$$\|G^*(GG^*)^{-1}-\hat{G}^*(\hat{G}\hat{G}^*)^{-1}\|\leqslant\frac{3}{\sigma\hat{\sigma}}\|G-\hat{G}\|, \tag{2.8}$$

$$|G^*(GG^*)^{-1}z-\hat{G}^*(\hat{G}\hat{G}^*)^{-1}\hat{z}|\leqslant 3\left(\frac{|z|}{\sigma}\vee\frac{|\hat{z}|}{\hat{\sigma}}\right)\frac{\|G-\hat{G}\|}{\sigma\vee\hat{\sigma}}+\frac{|z-\hat{z}|}{\sigma\vee\hat{\sigma}}. \tag{2.9}$$

Proof Without loss of generality, let $\hat{\sigma}\leqslant\sigma$. Then, by Lemma 2.1,

$$\|G^*(GG^*)^{-1}-\hat{G}^*(\hat{G}\hat{G}^*)^{-1}\|$$
$$\leqslant\|(G^*-\hat{G}^*)(GG^*)^{-1}\|+\|\hat{G}^*[(GG^*)^{-1}-(\hat{G}\hat{G}^*)^{-1}\|$$
$$\leqslant\frac{1}{\sigma^2}\|G-\hat{G}\|+\|\hat{G}^*(\hat{G}\hat{G}^*)^{-1}[\hat{G}\hat{G}^*-GG^*](GG^*)^{-1}\|$$

$$+\|\hat{G}^*(\hat{G}\hat{G}^*)^{-1}(\hat{G}-G)G^*(GG^*)^{-1}\|$$
$$\leqslant\left(\frac{2}{\sigma^2}+\frac{1}{\sigma\hat{\sigma}}\right)\|G-\hat{G}\|\leqslant\frac{3}{\sigma\hat{\sigma}}\|G-\hat{G}\|.$$

Thus, (2.8) holds. Now, we prove (2.9) (Still let $\hat{\sigma}\leqslant\sigma$).

$$|G^*(GG^*)^{-1}z-\hat{G}^*(\hat{G}\hat{G}^*)^{-1}\hat{z}|$$
$$\leqslant\|G^*(GG^*)^{-1}-\hat{G}^*(\hat{G}\hat{G}^*)^{-1}\|\,|\hat{z}|+\|G^*(GG^*)^{-1}\|\,|z-\hat{z}|$$
$$\leqslant\frac{1}{\sigma}\,|z-\hat{z}|+\frac{3}{\sigma\hat{\sigma}}|\hat{z}|\,\|G-\hat{G}\|=\frac{1}{\sigma\vee\hat{\sigma}}\,|z-\hat{z}|+\frac{3}{\hat{\sigma}}|\hat{z}|\frac{\|G-\hat{G}\|}{\sigma\vee\hat{\sigma}}.$$

Hence, (2.9) follows.

Next, we consider the following problem:

$$\begin{cases}\dot{y}(s)=-h_u^*(y(s))\,[h_u(y(s))h_u^*(y(s))]^{-1}h(u),\ s\in(0,1],\\ y(0)=u.\end{cases}\tag{3.10}$$

Lemma 2.3. *Let* (H1) *hold. Then, for any* $u\in U$, *there exists a unique solution* $y(\cdot)$ *of* (2.10) *satisfying*

$$|y(s)|\leqslant\left(|u|+\frac{|h(u)|}{\beta_0}\right)e^{|h(u)|/\beta_0},\ s\in[0,1].\tag{3.11}$$

Proof For any y, $\hat{y}\in U$, by Lemma 2.2, one has

$$\|h_u^*(y)\,[h_u(y)h_u^*(y)]^{-1}-h_u^*(\hat{y})\,[h_u(\hat{y})h_u^*(\hat{y})]^{-1}\|$$
$$\leqslant\frac{3(1+|y|)(1+|\hat{y}|)}{\beta_0^2}\|h_u(y)-h_u(\hat{y})\|$$
$$\leqslant\frac{3(1+|y|)(1+|\hat{y}|)}{\beta_0^2}L_0(|y|\vee|\hat{y}|)\,|y-\hat{y}|.\tag{3.12}$$

Thus, we have the local existence and uniqueness of the solution $y(\cdot)$ of (2.10). Let $t_0\leqslant 1$, such that the solution $y(\cdot)$ is defined on $[0, t_0)$. Then, by Lemma 2.1, we have

$$|y(s)|\leqslant|u|+\int_0^s|h_u^*(y(r))\,[h_u(y(r))]^{-1}h(u)|\,dr$$
$$\leqslant|u|+\int_0^s\frac{1+|y(r)|}{\beta_0}|h(u)|\,dr.\tag{3.13}$$

By Gronwall's inequality,

$$|y(s)|\leqslant\left(|u|+\frac{|h(u)|}{\beta_0}\right)e^{h(u)/\beta_0},\ s\in[0,t_0).$$

Then, combining (2.12), we see the global solution uniquely exists and (2.11) is satisfied.

Now, let us define

$$P_0(u)\equiv y(1;u),\tag{2.14}$$

where $y(\cdot)$ is the solution of (2.10). Our main result of this paper is the following

Theorem 2.4. *Let* (H1) *hold. Then,*

$$\{u\in U\mid h(u)=0\}=\{P_0(u)\mid u\in U\}.\tag{2.15}$$

Moreover, the mapping P_0: $U\to U$ *satisfies the following*

$$P_0(P_0(u))=P_0(u),\ \forall u\in U, \tag{2.16}$$
$$h(u)=0,\ \textit{iff}\ P_0(u)=u. \tag{2.17}$$

Proof By (2.10), we see

$$h(P_0(u))=h(y(1;\,u))=h(u)+\int_0^1 h(y(s))\dot{y}(s)ds=h(u)-\int_0^1 h(u)ds=0.$$

Hence, $P_0(u)\in\{u\in U\,|\,h(u)=0\}$ or $h(P_0(u))=0$, $\forall u\in U$. Conversely, if $h(u)=0$, then $y(s)\equiv u$ satisfies (2.10). Thus,

$$u=y(1;\,u)=P_0(u).$$

Thus we obtain (2.15) and (2.17). Finally, (2.16) is obvious.

The above theorem gives a representation of all solutions of the nonlinear equation (1.1). For the case that $h(\cdot)$ is affine (i. e., (1.2) holds), the problem (2.10) reads

$$\begin{cases}\dot{y}(s)=-G^*(GG^*)^{-1}(Gu-z),\\ y(0)=u.\end{cases} \tag{2.18}$$

Thus,

$$P_0(u)=y(1)=u-\int_0^1 G^*(GG^*)^{-1}(Gu-z)ds=[I-G^*(GG^*)^{-1}G]u+G^*(GG^*)^{-1}z.$$

Hence our result recovers the known result for affine mapping case.

Remark 2.5. Our result can be extended to the case that U is also a Banach space. In that case the duality mapping $J: U\to U^*$ will be involved in (2.10) and Lemmas 2.1 and 2.2 should also be modified. Here, we take U to be a Hilbert space just for notational simplicity. This remark also applies to the following sections.

§ 3. A Generalized Inverse Theorem

In this section, we will use the result of the previous section to derive a generalized inverse theorem for nonlinear mappings. To this end, let X and Z be Banach spaces and U be a Hilbert space. Let g: $X\times U\to Z$ be a given map. We consider the following equation in $u\in U$:

$$g(x,\,u)=0, \tag{3.1}$$

where $x\in X$ is regarded as a parameter. By a generalized inverse theorem, we mean that from (3.1) we can obtain a function P: $X\times U\to U$, such that the following holds: $\forall x\in X$,

$$\{u\in U\,|\,g(x,u)=0\}=\{P(x,\,u)\,|\,u\in U\}. \tag{3.2}$$

From now on unless there might be some ambiguity, we use $\|\cdot\|$ and $|\cdot|$ for the norms of operators and vectors, respectively, in possibly different spaces which can be identified from the context. Now, let us make some hypotheses on the map g.

(H2) The map $g(x, u)$ is Frechet differentiable in u and both $g(x, u)$ and $g_u(x, u)$ are continuous. Moreover, there exist continuous functions $\beta, \gamma: [0, \infty) \to (0, \infty)$, $L: [0, \infty) \times [0, \infty) \to (0, \infty)$, with the properties that β and γ are nondecreasing and L is nondecreasing in each of its arguments, such that for all $x, \hat{x} \in X$, $u, \hat{u} \in U$, $z^* \in Z^*$ and $\theta, r \geqslant 0$,

$$|g_u^*(x, u)z^*|_U \geqslant \frac{\beta(|x|_X)}{1+|u|_U}|z^*|_{Z^*}, \tag{3.3}$$

$$\|g_u(x, u) - g_u(\hat{x}, \hat{u})\| \leqslant L(|x| \vee |\hat{x}|, |u| \vee |\hat{u}|)(|x-\hat{x}| + |u-\hat{u}|), \tag{3.4}$$

$$|g(x, u) - g(\hat{x}, \hat{u})| \leqslant (1+|u| \vee |\hat{u}|) L(|x| \vee |\hat{x}|, |u| \vee |\hat{u}|)(|x-\hat{x}| + |u-\hat{u}|), \tag{3.5}$$

$$|g(x, u)| \leqslant \beta(|x|)\gamma(|u|), \tag{3.6}$$

$$L(\theta, r) \leqslant \beta(\theta)\gamma(r). \tag{3.7}$$

Let us look at (3.3)—(3.7). We see that (3.3) gives the invertibility of $g_u(x, u)g_u^*(x, u)$, and (3.4)—(3.5) are common local Lipschitz conditions, while (3.6) and (3.7) give sort of coercivity conditions on the fnnction $g(\cdot)$. All these conditions are satisfied for the case that

$$g(x, u) = G(x)u + g_0(x),$$

with some general conditions on $G(\cdot)$ and $g_0(\cdot)$.

By Theorem 2.4, we have the following generalized nonlinear inverse theorem.

Proposition 3.1. *Let* (H2) *hold. Then, for all* $x \in X$, (3.2) *holds with the map* $P: X \times U \to U$ *defined by the following:*

$$P(x, u) = y(1; x, u), \quad \forall (x, u) \in X \times U, \tag{3.8}$$

and $y(\cdot; x, u)$ *solves the following problem:*

$$\begin{cases} \dot{y}(s) = -g_u^*(x, y(s))[g_u(x, y(s))g_u^*(x, y(s))]^{-1}g(x, u), & s \in (0, 1], \\ y(0) = u. \end{cases} \tag{3.9}$$

Moreover,

$$P(x, P(x, u)) = P(x, u), \quad \forall (x, u) \in X \times U, \tag{3.10}$$

and

$$g(x, u) = 0 \quad \textit{iff} \quad u = P(x, u). \tag{3.11}$$

It is not hard to see that in order to have Proposition 3.1, we only need (3.3) and the following (a weaker condition than (3.4)):

$$\|g_u(x, u) - g_u(\hat{x}, u)\| \leqslant L(|x| \vee |\hat{x}|, |u|)(|x-\hat{x}|), \quad \forall x, \hat{x} \in X, u \in U. \tag{3.4'}$$

Next, let us give some further properties of the map P.

Theorem 3.1. *Let* (H2) *hold. Then there exists a nondecreasing function* $\hat{L}$: $[0, \infty) \to [0, \infty)$, *such that*

$$|P(x, u) - P(\hat{x}, \hat{u})| \leqslant \hat{L}(|u| \vee |\hat{u}|)(|x-\hat{x}| + |u-\hat{u}|), \quad \forall x, \hat{x} \in X, u, \hat{u} \in U. \tag{3.12}$$

Proof Let $y(\cdot)=y(\cdot;x,u)$ and $\hat{y}(\cdot;\hat{x},\hat{u})$ be the solutions of (3.9) corresponding to (x,u) and $(\hat{x},\hat{u})$, respectively. Then, as (2.13), we have

$$|y(s)|\leqslant\left(|u|+\frac{|g(x,u)|}{\beta(|x|)}\right)e^{|g(x,u)|/\beta(|x|)}$$
$$\leqslant(|u|+\gamma(|u|))e^{\gamma(|x|)}\equiv L_1(|u|),\ \forall s\in[0,1]. \tag{3.13}$$

Similarly,

$$|\hat{y}(s)|\leqslant L_1(|\hat{u}|),\ \forall s\in[0,1]. \tag{3.14}$$

Thus, by (2.9), we have

$$|g_u^*(x,y(r))[g_u(x,y(r))g_u^*(x,y(r))]^{-1}g(x,u)$$
$$-g_u^*(\hat{x},\hat{y}(r))[g_u(\hat{x},\hat{y}(r))g_u^*(\hat{x},\hat{y}(r))]^{-1}g(\hat{x},\hat{u})|$$
$$\leqslant 3\left(\frac{|g(x,u)|(1+|u|)}{\beta(|x|)}\vee\frac{g(\hat{x},\hat{u})|(1+|\hat{u}|)}{\beta(|\hat{x}|)}\right)\frac{\|g_u(x,y(r))-g_u(\hat{x},\hat{y}(r))\|}{\frac{\beta(|x|)}{1+|y(r)|}\vee\frac{\beta(|\hat{x}|)}{1+|\hat{y}(r)|}}$$
$$+\frac{|g(x,u)-g(\hat{x},\hat{u})|}{\frac{\beta(|x|)}{1+|y(r)|}\vee\frac{\beta(|\hat{x}|)}{1+|\hat{y}(r)|}}$$
$$\leqslant 3([\gamma(|u|)(1+|u|)]\vee[\gamma(|\hat{u}|)(1+|\hat{u}|)])\frac{\|g_u(x,y(r))-g_u(\hat{x},\hat{y}(r))\|}{\frac{\beta(|x|)\vee\beta(|\hat{x}|)}{1+|y(r)|\vee|\hat{y}(r)|}}$$
$$+\frac{|g(x,u)-g(\hat{x},\hat{u})|}{\frac{\beta(|x|)\vee\beta(|\hat{x}|)}{1+|y(r)|\vee|\hat{y}(r)|}}$$
$$\leqslant\frac{3(1+|u|\vee|\hat{u}|)\gamma(|u|\vee|\hat{u}|)}{\beta(|x|\vee|\hat{x}|)}[1+L_1(|u|\vee|\hat{u}|)]\|g_u(x,y(r))-g_u(\hat{x},\hat{y}(r))\|$$
$$+\frac{1+L_1(|u|\vee|\hat{u}|)}{\beta(|x|\vee|\hat{x}|)}|g(x,u)-g(\hat{x},\hat{u})|$$

while, by (3.7), we have

$$\frac{\|g_u(x,y(r))-g_u(\hat{x},\hat{y}(r))\|}{\beta(|x|\vee|\hat{x}|)}$$
$$\leqslant\frac{L(|x|\vee|\hat{x}|,L_1(|u|\vee|\hat{u}|))}{\beta(|x|\vee|\hat{x}|)}(|x-\hat{x}|+|y(r)-\hat{y}(r)|)$$
$$\leqslant\gamma(L_1(|u|\vee|\hat{u}|))(|x-\hat{x}|+|y(r)-\hat{y}(r)|).$$

Similarly,

$$\frac{|g(x,u)-g(\hat{x},\hat{u})|}{\beta(|x|\vee|\hat{x}|)}\leqslant\frac{(1+|u|\vee|\hat{u}|)L(|x|\vee|\hat{x}|,|u|\vee|\hat{u}|)}{\beta(|x|\vee|\hat{x}|)}(|x-\hat{x}|+|u-\hat{u}|)$$
$$\leqslant(1+|u|\vee|\hat{u}|)\gamma(|u|\vee|\hat{t}|)(|x-\hat{x}|+|u-\hat{u}|).$$

Hence, for some nondecreasing function $\tilde{L}:[0,\infty)\to[0,\infty)$, one has

$$|g_u^*(x,y(r))[g_u(x,y(r))g_u^*(x,y(r))]^{-1}g(x,u)$$
$$-g_u^*(\hat{x},\hat{y}(r))[g_u(\hat{x},\hat{y}(r))g_u^*(\hat{x},\hat{y}(r))]^{-1}g(\hat{x},\hat{u})|$$
$$\leqslant\tilde{L}(|u|\vee|\hat{u}|)(|x-\hat{x}|+|u-\hat{u}|+|y(r)-\hat{y}(r)|). \tag{3.15}$$

Then, our conclusion follows from (3.9) and the Gronwall's inequality.

§ 4. An Applicalion to Controol Systems

In this section, we give an application of our results to a control system in some Banach space. Let X, Z and U be the same as in § 2. Let e^{At} be a C_0–semigroup on X with the infinitesimal generator A. Let $T>0$ be given. We consider the following evolutionary system

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-r)}f(x(r),\ u(r))dr,\ \forall t\in[0,\ T], \tag{4.1}$$

with a mixture constraint of the following type:

$$g(x(t),\ u(t))=0,\ \text{a. e. } t\in[0,\ T]. \tag{4.2}$$

Let us make the following hypotheses on the map f.

(H3) There exists a constant $K>0$, such that

$$|f(x,\ u)-f(\hat{x},\ \hat{u})|\leqslant K(|x-\hat{x}|+|u-\hat{u}|),\ \forall x,\ \hat{x}\in X,\ u,\ \hat{u}\in U. \tag{4.3}$$

Proposition 4. 1. *Let* (H2) *and* (H3) *hold. Let P be the operator defined by* (3.8)—(3.9). *Then, for any $x_0\in X$ and $u(\cdot)\in L^\infty(0,\ T;\ U)$, there exists a unique solution of the following*

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-r)}f(x(r),\ P(x(r),\ u(r)))dr,\ \forall t\in[0,\ T]. \tag{4.4}$$

Proof By (H3) and Theorem 3.1, we have, for all x, $\hat{x}\in X$ and $u\in U$,

$$\begin{aligned}&|f(x,\ P(x,\ u))-f(\hat{x},\ P(\hat{x},\ u))|\\&\leqslant K\{|x-\hat{x}|+|P(x,\ u)-P(\hat{x},\ u)|\}\leqslant K[1+\hat{L}(|u|)]\,|x-\hat{x}|.\end{aligned}$$

Hence, our conclusion follows.

For any $x_0\in X$, we let

$$S_g(x_0)=\{x(\cdot)\,|\,x(\cdot)\text{ solves (4.1)—(4.2) for some } u(\cdot)\in L^\infty(0,\ T;\ U)\},$$

and let

$$S_P(x_0)=\{x(\cdot)\,|\,x(\cdot)\text{ solves (4.4) for some } u(\cdot)\in L^\infty(0,\ T;\ U)\}.$$

The main result of this section is the following

Theorem 4. 1. *Let (H2) and (H3) hold. Then, for any $x_0\in X$,*

$$S_g(x_0)=S_P(x_0)\neq\varnothing. \tag{4.5}$$

Proof By Proposition 4.1, we know that for any $x_0\in X$,

$$S_P(x_0)\neq\varnothing. \tag{4.6}$$

Now, let $u(\cdot)\in L^\infty(0,\ T;\ U)$ and $x(\cdot)$ be the solution of (4.4) corresponding to $u(\cdot)$ and x_0. Then, let us set

$$v(t)=P(x(t),\ u(t)),\ t\in[0,\ T]. \tag{4.7}$$

From (3.13), we have

$$|v(t)|\leqslant L_1(|u(t)|),\ \forall t\in[0,\ T]. \tag{4.8}$$

Thus, $v(\cdot)\in L^\infty(0, T; U)$. Hence, $x(\cdot)$ solves (4.1)—(4.2) with $v(\cdot)$ given by (4.7). Conversely, if $x(\cdot)$ solves (4.1)—(4.2) with $u(\cdot)\in L^\infty(0, T; U)$, then, by Theorem 2.1 and (4.2), we know that

$$u(t)=P(x(t), u(t)), \text{ a. e. } t\in[0, T]. \tag{4.9}$$

Thus, $x(\cdot)$ solves (4.4) with the same $u(\cdot)\in L^\infty(0, T; U)$. Hence, (4.5) follows.

The above result gives the equivalence between two control systems in which one has a mixture constraint and the other has no constraints.

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