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# IDEAL STRUCTURE IN CERTAIN $C^*$-TENSOR PRODUCTS***

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**Abstract**

This paper discusses the ideal structure in tensor product $A \underset{\min}{\otimes} B$ of $C^*$-algebra $A$ and $B$, and introduces the concept of property (I) and property ($K$) with respect to the problem. When $A$ is an $AF$ (or scattered) $C^*$-algebra, it is shown that for any $C^*$-algebra $B$, the ideals in $A \underset{\min}{\otimes} B$ can be expressed by those of $A$ and $B$.

## § 1. Introduction

It is well known that ideal structure in a $C^*$-algebra is very important in the theory of $C^*$-algebras. The ideal structure of type I $C^*$-algebras has been studied intentively[1]. The problem of determining ideal structure in $C^*$-tensor products is very difficult. Takesaki proved that $A \underset{\min}{\otimes} B$ is simple whenever $A$ and $B$ are simple[7]. In general, the ideal in $A \underset{\min}{\otimes} B$ can not de expressed by that of $C^*$-algebras $A$ and $B$. Wassermann proved that there exists a closed two-sided ideal $I$ in $B(H) \underset{\min}{\otimes} B(H)$ such that $I \supsetneqq B(H) \underset{\min}{\otimes} K + K \underset{\min}{\otimes} B(H)$[9], where, as usual, $B(H)$ denotes all bounded linear operators on Hilbert space $H$ and $K$ all compact operators on separable Hilbert space.

In this paper, we consider the condition on $C^*$-algebras $A$ and $B$ under which ideals in $A \underset{\min}{\otimes} B$ can be determined by those of $A$ and $B$. A necessary condition on such $C^*$-algebras will be given. When $A$ is an $AF$ (or scattered) $C^*$-algebra, it is shown that any ideal in $A \underset{\min}{\otimes} B$ can be determined by those of $A$ and $B$.

We note that, in our paper, ideals of a $C^*$-algebra always mean closed two-sided ideals. Let $A$ be a $C^*$-algebra. We use $A^{**}$ to denote the enveloping Von Neumann algebra of $A$. $A^{**}$ is a $W^*$-algebra in $\sigma(A^{**}, A^*)$ topology. We often identify $A$ with its canonical image in $A^{**}$, and hence $A \subset A^{**}$.

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## § 2. Slice Map Property

In this section, we discuss the slice map property. First of all, we have

**Lemma 2.1** *Let $A$ be a $C^*$-algebra, and $e$ be a central projection in $A^{**}$. Then, $A^{**}e\cap A$ is a closed two-sided ideal in $A$. On the other hand, every ideal of $A$ has this form.*

*Proof* Since $e$ is a central projection in $A^{**}$, it is easy to verify that $A^{**}e\cap A$ is a closed two-sided ideal in $A$. On the other hand, by [3, Theorem 2.5], every ideal of $A$ has this form.

Let $B$ be a $C^*$-subalgebra of a $C^*$-algebra $A$. For every $X$ in $B^{**}$, we define $\Phi(X)\in\overline{B}^{\sigma}$ (the $\sigma(A^{**}, A^*)$ closure of $B$ in $A^{**}$) as follows. For any $F\in A^*$, let $\Phi(X)(F)=X(F|_B)$. By Propositions 1.12.3 and 2.11.4 in [4], we have

**Lemma 2.2** *Let $A$ and $B$ as above. Then $\Phi$ is a $W^*$-isomorphism from $B^{**}$ onto $\overline{B}^{\sigma}$.*

**Propostion 2.3.** *Let $B$ be a $C^*$-subalgebra of $C^*$-algebra $A$. If $x\in A$ and $x\overline{\in}B$, then $x\overline{\in}\overline{B}^{\sigma}$. Therefore $A\cap\overline{B}^{\sigma}=B$.*

*Proof* As $x\in A$ and $x\overline{\in}B$, there is an element $f$ in $A^*$ such that $f(x)\neq0$ and $f(B)=0$. If we had that $x\in\overline{B}^{\sigma}$, there would exist a $y$ in $B^{**}$ such that $\Phi(y)=x$, where $\Phi$ is the isomorphism in Lemma 2.2. Hence, $0\neq f(x)=x(f)=\Phi(y)(f)=y(f|_B)=0$, a contradiction. This completes the proof.

**Remark.** By Lemma 2.2, we often identify $B^{**}$ with $\overline{B}^{\sigma}$. Then $A\cap B^{**}=B$ whenever $B\subset A$.

A $C^*$-algebra $B$ is called scattered if $B^{**}$ is atomic, i. e., $B^{**}$ is a direct sum of some type $I$ factor[5]. Quigg proved that $(A\underset{\min}{\otimes}B)^{**}=A^{**}\hat{\otimes}B^{**}$ if and only if $A$ or $B$ is a scattered $C^*$-algebra, where $A^{**}\hat{\otimes}B^{**}$ is the $W^*$-tensor product of $A^{**}$ and $B^{**}$ (See [5]). We also know from [5] that any $C^*$-subalgebra of a scattered $C^*$-algebra is scattered.

Let $A$ and $B$ be $C^*$-algebras. For any $\varphi\in A^*$, we define a linear map $\sum a_i\otimes b_i\mapsto\sum\varphi(a_i)b_i$ from $A\otimes B$ to $B$. This map can be extended to a bounded linear map from $A\underset{\min}{\otimes}B$ to $B$. We denote this map by $R_\varphi$. $R_\varphi$ is called the right $C^*$-slice map determined by $\varphi$.

Let $A$ and $B$ be $C^*$-algebras, and $C$ a $C^*$-subalgebra of $B$. The triple $(A, B, C)$ is said to verify the slice map conjecture if $x\in A\underset{\min}{\otimes}B$ and, for every $\varphi\in A^*$, $R_\varphi(x)\in C$ implies that $x$ is in $A\underset{\min}{\otimes}C$ (See [8]).

Archbold and Batty proposed a problem whether the triple $(A, B, C)$ verifies the slice map conjecture for any nuclear $C^*$-algebra $B$, where $A$ is any $C^*$-algebra and $C$ is any $C^*$-subalgebra of $B$. The following theorem partially answers the problem affirmatively.

**Theorem 2.4.** *If $B$ is a scattered $C^*$-aloebra and $C$ is any $C^*$-subalgebra of $B$, then for any $C^*$-algebra $A$ the triple $(A, B, C)$ verifies the slice map conjecture.*

*Proof* As $B$ is scattered, we have $(A \underset{\min}{\otimes} B)^{**} = A^{**} \hat{\otimes} B^{**}$. Now, suppose that $x$ is in $A \underset{\min}{\otimes} B$ and $R_\varphi(x)$ is in $C$ for every $\varphi$ in $A^*$. An element $x'$ in $(A \underset{\min}{\otimes} C)^{**}$ can be defined by $x'(\varphi \otimes \psi) = \psi(R_\varphi(x))$ for any $\varphi \in A^*$ and $\psi \in C^*$ since $A^* \underset{\min^*}{\otimes} C^* = (A \underset{\min}{\otimes} C)^*$. It is clear that $\Phi(x')(\varphi \otimes \psi) = x'(\varphi \otimes \psi|_{A \underset{\min}{\otimes} C})$, where $\Phi$ is the isomorphism in Lemma 2.2. Therefore, $x = \Phi(x')$, and $x \in \overline{(A \underset{\min}{\otimes} C)}^\sigma$. By Proposition 2.3, we know that $A \underset{\min}{\otimes} C = \overline{(A \underset{\min}{\otimes} C)}^\sigma \cap (A \underset{\min}{\otimes} B)$, hence $x$ is in $A \underset{\min}{\otimes} C$. The proof is completed.

## § 3. Ideal Structure

In this section, we will obtain our main results on ideal structure in certain $C^*$-tensor products. We first have

**Theorem 3.1.** *Let $A$ be a scattered $C^*$-algebra, and $B$ any $C^*$-algebra. Then any ideal of $A \underset{\min}{\otimes} B$ can be expressed as the form $\sum_{\lambda \in \Lambda}^{\oplus} I_\lambda \underset{\min}{\otimes} J_\lambda$, where $I_\lambda$ and $J_\lambda$ are ideals of $A$ and $B$ respectively, and $\Lambda$ is an index set.*

*Proof* As $A$ is a scattered $C^*$-algebra, $(A \underset{\min}{\otimes} B)^{**} = A^{**} \hat{\otimes} B^{**}$, and $A^{**}$ is a direct sum of some type $I$ factor, i. e., $A^{**} = \sum_{i \in \Gamma}^{\oplus} B(H_i)$. Putting $H = \sum_{i \in \Gamma}^{\oplus} H_i$ and denoting the projection from $H$ onto $H_i$ by $p_i$ for every $i$ in $\Gamma$, we can easily see that $p_i$'s are in $A^{**}$. We also have $(A \underset{\min}{\otimes} B)^{**} = A^{**} \hat{\otimes} B^{**} = \sum_{i \in \Gamma}^{\oplus} B(H_i) \hat{\otimes} B^{**}$.

Regarding $B(H_i)$ as a subalgebra of $A^{**}$, we can easily prove that the center of $(A \underset{\min}{\otimes} B)^{**}$ is $\sum_{i \in \Gamma}^{\oplus} \mathbf{C} p_i \otimes Z$, where $Z$ is the center of $B^{**}$. Thus any central projection $p$ in $(A \underset{\min}{\otimes} B)^{**}$ can be written as $p = \sum_{\lambda \in \Lambda}^{\oplus} p_\lambda \otimes q_\lambda$, where $q_\lambda$'s are projections in $Z$ and $\Lambda$ is some subset of $\Gamma$.

Any weak closed ideal in $(A \underset{\min}{\otimes} B)^{**}$ is associated with a central projection $p$ in $(A \underset{\min}{\otimes} B)^{**}$. By above, there is an index set $\Lambda \subset \Gamma$ such that $p = \sum_{\lambda \in \Gamma}^{\oplus} p_\lambda \otimes q_\lambda$. Therefore, $(A \underset{\min}{\otimes} B)^{**} p = (A^{**} \hat{\otimes} B^{**}) p = \sum_{\lambda \in \Lambda}^{\oplus} A^{**} p_\lambda \hat{\otimes} B^{**} q_\lambda$. Now, putting $A^{**} p_\lambda \cap A = I_\lambda$ and $B^{**} q_\lambda \cap B = J_\lambda$, one can easily show that $I_\lambda^{**} = A^{**} p_\lambda$ and $J_\lambda^{**} = B^{**} q_\lambda$. Since any ideal of a scattered $C^*$-algebra is also scattered, it follows from [5] that $(I_\lambda \underset{\min}{\otimes} J_\lambda)^{**} = I_\lambda^{**} \hat{\otimes} J_\lambda^{**}$ for every $\lambda \in \Lambda$. Thus we get $\sum_{\lambda \in \Lambda}^{\oplus} I_\lambda \underset{\min}{\otimes} J_\lambda = (\sum_{\lambda \in \Lambda}^{\oplus} I_\lambda \underset{\min}{\otimes} J_\lambda)^{**} \cap (A \underset{\min}{\otimes} B) =$

$(\sum_{\lambda\in\Lambda}^{\oplus}(I_\lambda \underset{\min}{\otimes} J_\lambda)^{**} \cap A \underset{\min}{\otimes} B = \sum_{\lambda\in\Lambda}^{\oplus} I_\lambda^{**}\hat{\otimes}J_\lambda^{**}) \cap (A \underset{\min}{\otimes} B = \sum_{\lambda\in\Lambda}^{\oplus} A^{**}p_\lambda\hat{\otimes}B^{**}q_\lambda) \cap (A \underset{\min}{\otimes} B) =$ $((A^{**}\hat{\otimes}B^{**})p) \cap (A \underset{\min}{\otimes} B) = (A \underset{\min}{\otimes} B)^{**}p \cap (A \underset{\min}{\otimes} B)$, i. e., we have the following equation $(A \underset{\min}{\otimes} B)^{**}p \cap (A \underset{\min}{\otimes} B) = \sum_{\lambda\in\Lambda}^{\oplus} I_\lambda \underset{\min}{\otimes} J_\lambda$. The theorem follows from Lemma 2.1.

**Corollary 3. 2.** *For any $C^*$-algebra $A$, every ideal in $A \underset{\min}{\otimes} K$ is of the form $I \underset{\min}{\otimes} K$ for some ideal $I$ of $A$, where $K$ is the $C^*$-algebra consisting of compact operators on some separable Hilbert space.*

**Proposition 3. 3.** *Let $A = \overset{n}{\underset{i=1}{\oplus}} A_i$. If, for any $C^*$-algebra $B$, every ideal in $A_i \underset{\min}{\otimes} B$ is of the form $A_i \underset{\min}{\otimes} I_i$, where $I_i$'s are some ideals of $B$, then every ideal in $A \underset{\min}{\otimes} B$ is of the form $\overset{n}{\underset{i=1}{\oplus}} A_i \underset{\min}{\otimes} J_i$ for some ideals $J_i$ of $B$.*

*Proof* Let $D = D_1 \oplus D_2$ be the direct sum of $C^*$ algebras $D_1$ and $D_2$. It is clear that $D^{**} = D_1^{**} \oplus D_2^{**}$. Any central projection $e$ in $D^{**}$ can be written as $e = e_1 \oplus e_2$, where $e_1$ and $e_2$ are central projections in $D_1$ and $D_2$ respectively. We note that $D^{**}e \cap D = (D_1^{**}e_1 \cap D_1) \oplus (D_2^{**}e_2 \cap D_2)$, i. e., every ideal of $D$ can be expressed as a direct sum of ideals in $D_1$ and $D_2$. Thus the proposition follows from the assumption and induction.

**Corollary 3. 4.** *If $A$ is a finite dimensional $C^*$-algebra, for any $C^*$-algebra $B$, every ideal in $A \underset{\min}{\otimes} B$ can be expressed as the form $\sum_i I_i \underset{\min}{\otimes} J_i$, where $I_i'$s and $J_i'$s are ideals of $A$ and $B$ respectively.*

To prove Theorem 3.6, we need the following

**Lemma 3. 5.** *If $A$ is the direct limit of $\{A_\alpha, \alpha \in I_1\}$ and $B$ is the direct limit of $\{B_\beta, \beta \in I_2\}$, then $A \underset{\min}{\otimes} B$ is the direct limit of $\{A_\alpha \underset{\min}{\otimes} B_\beta, (\alpha, \beta) \in I_1 \times I_2\}$.*

*Proof* By [4] or [6], we may assume that $A = \overline{\bigcup_{\alpha\in I_1} A_\alpha}$ and $B = \overline{\bigcup_{\beta\in I_2} B_\beta}$, where $A_\alpha$ and $B_\beta$ are $C^*$-subalgebras of $A$ and $B$ respectively, and $A_\alpha \subset A_{\alpha'}$ $(B_\beta \subset B_{\beta'})$ whenever $\alpha < \alpha'$ $(\beta < \beta')$. So, it suffices to prove that $A \underset{\min}{\otimes} B = \overline{\bigcup_{(\alpha,\beta)\in I_1\times I_2} A_\alpha \underset{\min}{\otimes} B_\beta}$.

It is clear that $A \underset{\min}{\otimes} B \supset \overline{\bigcup A_\alpha \underset{\min}{\otimes} B_\beta}$. On the other hand, for any $x = \sum_{i=1}^{n} a_i \otimes b_i$ in $A \otimes B$ (algebraic tensor product of $A$ and $B$) and any positive number $\varepsilon$, letting $M = \max\{\|a_i\|, \|b_i\|, i = 1, 2, \cdots, n\}$, one can find $\alpha_0$, $\beta_0$ and $a_i' \in A_{\alpha_0}$, $b_i' \in B_{\beta_0}$ such that $\|a_i - a_i'\| < \varepsilon/2nM$ and $\|b_i - b_i'\| < \varepsilon/2nM$ $(i = 1, 2, \cdots, n)$. Thus we have

$$\left| x - \sum_{i=1}^{n} a_i' \otimes b_i' \right| < \varepsilon + \varepsilon^2/4n^2M^2,$$

and $x$ is in $\overline{\bigcup A_\alpha \underset{\min}{\otimes} B_\beta}$. Therefore, equation $A \underset{\min}{\otimes} B = \overline{\bigcup A_\alpha \underset{\min}{\otimes} B_\beta}$ follows from the density of $A \otimes B$ in $A \underset{\min}{\otimes} B$.

**Theorem 3.6.** *Let $A$ be the direct limit of $\{A_\alpha, \alpha\in\Lambda\}$ and $B$ a simple $C^*$-algebra. If any ideal in $A_\alpha \underset{\min}{\otimes} B(\alpha\in\Lambda)$ is of the form $I_\alpha \underset{\min}{\otimes} B$ for some ideal $I_\alpha$ of $A_\alpha$, then every ideal in $A\underset{\min}{\otimes} B$ can be written as $I\underset{\min}{\otimes} B$ for some ideal $I$ of $A$.*

*Proof* As in the proof of Lemma 3.5, we may assume that $A=\overline{\bigcup_{\alpha\in\Lambda} A_\alpha}$. By Lemma 3.5, we have $A\underset{\min}{\otimes} B=\overline{\bigcup A_\alpha\underset{\min}{\otimes} B}$. Just as in the proof of Lemma 12.4.1 in [4], we know that any ideal $J$ of $A\underset{\min}{\otimes} B$ can be expressed as $J=\overline{\bigcup_{\alpha\in\Lambda} J_\alpha}$, where $J_\alpha=(A_\alpha\underset{\min}{\otimes} B)\cap J$ is an ideal in $A_\alpha\underset{\min}{\otimes} B(\alpha\in\Lambda)$.

By assumption, $J_\alpha=I_\alpha\underset{\min}{\otimes} B$ for some ideal $I_\alpha$ of $A$. We claim that $J_\alpha\subset J_{\alpha'}$ whenever $\alpha<\alpha'$. In fact, for any $x$ in $I_\alpha$, taking $b\neq 0$ in $B$ one can choose an $f$ in $B^*$ such that $f(b)=1$ by the Hahn-Banach theorem. Since $b\otimes x\in B\underset{\min}{\otimes} I_\alpha\subset B\underset{\min}{\otimes} I_{\alpha'}$, $R_f(b\otimes x)\in I_{\alpha'}$ and $R(b\otimes x)=x$. So, $x\in I_\alpha$ implies that $x\in I_{\alpha'}$. It is obvious that $I=\overline{\bigcup_\alpha I_\alpha}$ is an ideal of $A$ and $J=I\underset{\min}{\otimes} B$.

**Corollary 3.7.** *Let $A$ be an AF-algebra and $B$ any simple $C^*$-algebra. Then, every ideal in $A\underset{\min}{\otimes} B$ is of the form $I\underset{\min}{\otimes} B$ for some ideal $I$ of $A$.*

Using the same method as in the proof of Theorem 3.6, one can prove

**Proposition 3.8.** *Let $A$ be the direct limit of $\{A_\alpha, \alpha\in\Lambda\}$ and $B$ any $C^*$-algebra. If, for every $\alpha\in\Lambda$, any idealin $A_\alpha\underset{\min}{\otimes} B$ is of the form $A_\alpha\underset{\min}{\otimes} J_\alpha$ for some ideal $J_\alpha$ of $B$, then every ideal in $A\underset{\min}{\otimes} B$ can be expressed as the form $A\underset{\min}{\otimes} J$ for some ideal $J$ of $B$.*

In particular, we get

**Corollary 3.9.** *If $A$ is a UHF-algebra, then for any $C^*$-algebra $B$ every ideal in $A\underset{\min}{\otimes} B$ is of the form $A\underset{\min}{\otimes} I$ for some ideal $I$ of $B$.*

## § 4. Propertys (I) and (K)

To study ideal structure in $C^*$-tensor products more generally, we introduce the definition of property (I), which will help us to understand ideals in $C^*$-tensor products. By Theorem 3.1, we see that any scattered $C^*$-algebra has property (I). Moreover, we introduce the concept of property (K) and get the relationship between property (I) and property (K). The $C^*$-algebra with property (K) is closely related with the $C^*$-algebraic extension theory (See [2], Theorems $A$ and $B$). Therefore, there exists close relationship between the extension theory and the ideal structure in $C^*$-tensor products.

Suppose that $\{I_\lambda, \lambda\in\Lambda\}$ is a family of ideals in $C^*$-algebra $A$. We use the notation $I=\sum_{\lambda\in\Lambda} I_\lambda$ to mean that $x\in I$ if and only if there exists a sequence $\{x_n\}$ converging to $x$ in norm, where $x_n\in \sum_{\lambda\in\Lambda_n} I_\lambda$, and $\Lambda_n$ is a finite subset of $\Lambda$. It is easy to prove that $I$ is an ideal in $A$.

Let $A$ be a $C^*$-algebra. We say that $A$ has property (I) if, for any $C^*$-algebra $B$, every ideal in $A\underset{\min}{\otimes} B$ can be expressed as the form $\sum_{\lambda\in\Lambda} I_\lambda\underset{\min}{\otimes} J_\lambda$, where $I_\lambda$ and $J_\lambda$ are ideals of $A$ and $B$ respectively. We first have

**Proposition 4.1.** *If a $C^*$-algebra has property* (I), *so does its quotients.*

*Proof* Given a $C^*$-algebra $A$ with property (I), we let $I_0$ be an ideal of $A$ and $q$ the canonical map from $A$ onto $A/I_0$. Then $q\otimes 1$ is a homomorphism from $A\underset{\min}{\otimes} B$ onto $(A/I_0)\underset{\min}{\otimes} B$. For any ideal $J$ in $(A/I_0)\underset{\min}{\otimes} B$, there is an ideal $J'$ in $A\underset{\min}{\otimes} B$ such that $J=q\otimes 1(J')$. Since $A$ has property (I), $J'$ can be expressed as $J'=\sum I_\lambda\underset{\min}{\otimes} J_\lambda$, where $I_\lambda$ and $J_\lambda$ are ideals of $A$ and $B$ respectively. Then, we have $J=q\otimes 1(J')=q\otimes 1(\sum I_\lambda\underset{\min}{\otimes} J_\lambda)=\sum q(I_\lambda)\underset{\min}{\otimes} J_\lambda=\sum I'_\lambda\underset{\min}{\otimes} J_\lambda$, where $I'_\lambda$ is an ideal of $A/I_0$. So, the proposition follows.

A $C^*$-algebra $B$ is said to have property $(K)$ if, for any $C^*$-algebra $A$ and any ideal $J$ of $B$, Ker $(q\otimes 1)=J\underset{\min}{\otimes} A$, where $q$ is the canonical map from $B$ onto $B/J$ and $q\otimes 1$ the homomorphism from $B\underset{\min}{\otimes} A$ onto $(B/J)\underset{\min}{\otimes} A$ induced by $q$.

Now, we give a necessary condition of property (I).

**Proposition 4.2.** *Property* (I) *implies the property* (K).

*Proof* If there were a $C^*$-algebra $A$ having property (I) butnot having property (K), then there would be anideal $I$ of $A$ and a $C^*$-algebra $B$ such that $J=\text{Ker }(q\otimes 1)\supsetneqq I\underset{\min}{\otimes} B$.

Obviously, $J$ is an ideal in $A\underset{\min}{\otimes} B$. Since $A$ has property (I), one can write $J$ as $J=\sum I_\lambda\underset{\min}{\otimes} J_\lambda$, where $I_\lambda$ and $J_\lambda$ are ideals of $A$ and $B$ respectively. By the assumption, there is an index $\lambda_0$ such that $I_{\lambda_0}\not\subset I$, i. e., there exists an element $x$ in $I_{\lambda_0}$ such that $x$ is not in $I$. For any non-zero element $y$ in $J_{\lambda_0}$, $x\otimes y$ lies in $I_{\lambda_0}\otimes J_{\lambda_0}\subset I_{\lambda_0}\underset{\min}{\otimes} J_{\lambda_0}\subset J$. But we have $q\otimes 1(x\otimes y)=q(x)\otimes y\neq 0$. The contradiction shows that $A$ has property (K).

The following proposition characterizes property (K).

**Proposition 4.3.** *$C^*$-algebra $B$ has property* (K) *if and only if, for any $C^*$-algebra $A$ and any ideal $I$ of $B$, the triple $(A, B, I)$ verifies the slice map conjecture.*

*Proof* Let $q$ be the canonical map from $B$ to $B/I$. Then, for every $x=\sum_{i=1}^n x_i\otimes y_i$

in $A\otimes B$ and any $\varphi\in A^*$, one has

$$q\circ R_\varphi(x)=q\circ R_\varphi\Big(\sum_{i=1}^{n}x_i\otimes y_i\Big)=q\Big(\sum_{i=1}^{n}\varphi(x_i)y_i\Big)=\sum_{i=1}^{n}\varphi(x_i)q(y_i)$$
$$=R_\varphi\Big(\sum_{i=1}^{n}x_i\otimes q(y_i)\Big)=R_\varphi\circ(1\otimes q)(x).$$

Thus, by continuity of $R_\varphi$ and $q$ and the density of $A\otimes B$ in $A\underset{\min}{\otimes}B$, we have the equation $q\circ R_\varphi=R_\varphi\circ(1\otimes q)$.

Now, suppose that $(A, B, I)$ verifies the slice omap conjecture. Given $x$ in ker $(1\otimes q)$, we know that $q\circ R_\varphi(x)=R_\varphi(1\otimes q)(x)=0$, and thus $R_\varphi(x)$ is in $I$ for any $\varphi\in A^*$. So, $x$ belongs to $A\underset{\min}{\otimes}I$, and we have ker $(1\otimes q)=A\underset{\min}{\otimes}I$. This shows that $B$ has property (K).

Conversely, suppose that $B$ has property (K). If $x$ is in $A\underset{\min}{\otimes}B$ such that $R_\varphi(x)\in I$ for every $\varphi\in A^*$, then $R_\varphi\circ(1\otimes q)(x)=q\circ R_\varphi(x)=0$. Since the set $\{R_\varphi, \varphi\in A^*\}$ separates elements in $A\underset{\min}{\otimes}(B/I)$, we get $(1\otimes q)(x)=0$, and thus $x$ belongs to ker $(1\otimes q)=A\underset{\min}{\otimes}I$.

**Proposition 4.4.** *Suppose that $A$ is the $C^*$-algebra with property* (I). *Then any ideal $J$ of $A$ has property* (I).

*Proof* It is clear that $J\underset{\min}{\otimes}B$ is an ideal of $A\underset{\min}{\otimes}B$. For any ideal $J'$ in $J\underset{\min}{\otimes}B$, $J'$ is an ideal of $A\underset{\min}{\otimes}B$. By the assumption on $A$, we obtain $J'=\sum_\lambda I_\lambda\underset{\min}{\otimes}J_\lambda$, where $I_\lambda$ and $J_\lambda$ are ideas of $A$ and $B$ respectively. By Propositions 4.2 and 4.3, $(J_\lambda, A, J)$ verifies the slice map conjecture. Then, any $x\in J_\lambda\underset{\min}{\otimes}I_\lambda\subset J'\subset B\underset{\min}{\otimes}J$, $R_\varphi(x)$ is in $J$ for every $\varphi\in J_\lambda^*$, and thus $x\in J_\lambda\underset{\min}{\otimes}J$. Therefore, $J_\lambda\underset{\min}{\otimes}I_\lambda\subset J_\lambda\underset{\min}{\otimes}J$. Using the method similar to the proof in Theorem 3.6, one can derive $I_\lambda\subset J$, i. e., $I_\lambda$ is an ideal of $J$. So, $J$ has property (I).

**Proposition 4.5.** *Let $A$ and $B$ be $C^*$-algebras with property* (I). *Then $A\underset{\min}{\otimes}B$ has property* (I).

*Proof* For any $C^*$-algebra $C$, $(A\underset{\min}{\otimes}B)\underset{\min}{\otimes}C=A\underset{\min}{\otimes}(B\underset{\min}{\otimes}C)$. Since $A$ has property (I), any ideal $I$ in $(A\underset{\min}{\otimes}B)\underset{\min}{\otimes}C$ is of the form $I=\sum_\lambda I_\lambda\underset{\min}{\otimes}J$, where $I_\lambda$ and $J_\lambda$ are ideals of $A$ and $B\underset{\min}{\otimes}C$, respectively. For every $\lambda$, $J_\lambda=\sum_i J'_{\lambda,i}\underset{\min}{\otimes}K_{\lambda,i}$ since $B$ has property (I), where $J'_{\lambda,i}$ and $K_{\lambda,i}$ are ideals of $B$ and $C$, respectively. Therefore, we get $I=\sum_\lambda\sum_i(I_\lambda\underset{\min}{\otimes}J'_{\lambda,i})\underset{\min}{\otimes}K_{\lambda,i}$, i. e., $A\underset{\min}{\otimes}B$ has property (I).

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