Chin. Ann. of Math.
13B; 4(1992), 471—484.

DISTORTION THEOREM FOR BIHOLOMORPHIC MAPPINGS IN TRANSITIVE DOMAINS (II)

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Abstract

Based on [1], this paper gives concrete estimates for distortion theorem for biholomorphic convex mappings in classical domains of types I, II, and III including transitive domains which are holomorphically homeomorphic to these domains.

§1. Introduction

Let $M\subset C^n$ be a transitive domain, bounded or unbounded, m be a point in M. Let G be a Lie group consisting of some holomorphic automorphisms of M and acting transitively on M, K be an isotropy group of G which leaves m fixed. We denote by ψ_z a holomorphic automorphism of M which maps $z\in M$ to $m\in M$, i. e., $\psi_z(z)=m$, and denote the Jacobian of ψ_z by J_{ψ_z}. If M is unbounded, we must assume $\det J_{\psi_k}(m)=1$ for all k in K. Set $K(z, \bar z)=c\det J_{\psi_z}(z)\ \overline{\det J_{\psi_z}(z)}$ with c being constant and denote $K(m, \bar m)^{-1}\left(\frac{\partial}{\partial z_p}K(z, \bar z)\right)_{z=m}$ by C_p. Then $K(z, \bar z)$ is well defined and if M is bounded, $K(z, \bar z)$ is the Bergman kernel function for certain constant c.

Suppose f is a holomorphic mapping of M into C^n. Then we can use $K(z, \bar z)$, C_p and coefficients of the expansion of f to express the det $J_f(z)$. This is the result of Gong and Zheng[1].

Let $M\subset C^n$ be a bounded symmetric domain containing the origin, and M be the canonical Harish-Chandra realization of Hermitian symmetric space G/K. Then G is a semisimple connected noncompact Lie group with finite center, K is a maximal compact subgroup of G which fixes 0.

Let g be the Lie algebra of G, k be the maximal compact subalgebra of g which corresponds to K. Then g has the Cartan decomposition $g=k+p$. Suppose a is the maximal Abelian subspace in p, A is the analytic subgroup in G which corresponds

Manuscript received June 22, 1989.

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to a. Then G has Iwasawa decomposition $G=KAN$.

We can choose a basis of a, $X_1, X_2, \cdots, X_q$, where $q=\dim a=\operatorname{rank} G/K$, and for any $X\in a$, there exists a unique decomposition $X=x_1X_1+x_2X_2+\cdots+x_qX_q$. For every $z\in M$, there exists $X\in a$ and $k\in K$, such that $z=\xi$ (exp $Ad(k)X\cdot o$), where o is the identity coset in G/K, ξ is the canonical holomorphic diffeomorphism of G/K onto M. Because K acts on M just as a subgroup of the unitary group U_n acts on C^n, we have

$$z=\xi(\exp Ad(k)X\cdot o)=k\xi(\exp X\cdot o)=A\tilde{k},$$

where $A=\xi(\exp X\cdot o)=(\tanh x_1, \tanh x_2, \cdots, \tanh x_q, 0, \cdots, 0)\in M$, $\tilde{k}=(k_{ij})\in U_n$.

We say that the holomorphic mapping $f(z)=(f_1(z), f_2(z), \cdots f_n(z))$ which maps M into C^n is normalized if $f(z)=z+\Sigma_{ij}d_{ij}z_iz_j+\cdots$, where

$$d_{ij}=(d_{ij}^{(1)}, \cdots, d_{ij}^{(n)}),\ z=(z_1, \cdots, z_n)\in C^n.$$

A family S of normalizd holomorphic mappings of M into C^n is called an A-invariant family if the following condition is satisfied: the composition of any $f\in S$ with any holomorphic automorphism of M, after normalization, remains a holomorphic mapping in S.

Gong and Zheng[1] proved the following result.

Theorem 1.1. *Under the same assumptions given above, if f is a normalized biholomorphic mapping of M into C^n, $f\in S$ where S is a family if A-invariant biholomorphic mapping, $z=\xi(\exp Ad(k)X\cdot o)\in M$, $X=x_1X_1+\cdots+x_qX_q$, $q=\dim a=$ rank G/K, then the following inequality*

$$|\log(\det J_f(z)/(K(z,\bar{z})/K(0,0))^{1/2}|\leqslant c(S)\sum_{p=1}^{q}\log((1+|\tan hx_p|)/(1-|\tan hx_p|))$$

holds, where $c(S)=\sup\left\{\sum_{i,j=1}^{n}k_{pi}d_{ij}^{(j)},\ f\in S,\ p=1, 2, \cdots, n\right\}$, $(k_{pi})\in U_n$ *and $K(.,.)$ is the Bergman kernel function of M.*

Theorem 1.1 implies the following distortion theorem:

$$\sqrt{\frac{K(z,\bar{z})}{K(0,0)}}\left(\prod_{p=1}^{q}\frac{1-|\tanh x_p|}{1+|\tanh x_p|}\right)^{c(S)}\leqslant|\det J_f(z)|\leqslant\sqrt{\frac{K(z,\bar{z})}{K0,0)}}\left(\prod_{p=1}^{q}\frac{1+|\tanh x_p|}{1-|\tanh x_p|}\right)^{c(S)}.$$

In this paper, we will obtain the estimations of $c(S)$ when M is one of the classical domains of type I, II, III, and S is the family of biholomorphic convex mappings.

§ 2. Some Lemmas

Lemma 2.1. *$c(S)$ in Theorem 1.1 can be expressed as*

$$c(S)=\sup\left\{\sum_{j=1}^{n}d_{ij}^{(j)},\ f\in S,\ i=1, 2, \cdots, q\right\}.$$

Proof If $f(z)\in S$, then

$$f(z\tilde{k})\tilde{k}^{-1}=z+\sum_{i,j}\tilde{d}_{ij}z_iz_j+\cdots\in S,$$

where

$$\tilde{d}_{ij}=(\tilde{d}_{ij}^{(1)},\ \cdots,\ \tilde{d}_{ij}^{(n)}).$$

The components of $\tilde{d}_{ij}$ are

$$\tilde{d}_{sp}(r)=\sum_{i,j,m=1}^{n} d_{ij}^{(m)}k_{si}k_{pj}u_{mr},\quad r=1,\ 2,\ \cdots,\ n,$$

where $\tilde{k}^{-1}=(v_{ij})\in U_n$. It is easy to verify

$$\sum_{p=1}^{n}\tilde{d}_{sp}^{(p)}=\sum_{p,i,j,m=1}^{n} d_{ij}^{(m)}k_{si}k_{pj}v_{mp}$$
$$=\sum_{i,j,m=1}^{n} d_{ij}^{(m)}k_{si}\delta_{jm}=\sum_{i,j=1}^{n}k_{si}d_{ij}^{(j)}.$$

If S is the family of convex biholomorphic mappings, $f\in S$, then

$$F(z)=(f(z\tilde{k})+f(z))/2\in f(M)$$

holds for all $z\in M$. Let $\varphi(z)=f^{-1}(F(z))$. Then φ is a holomorphic mapping which maps M into M with $\varphi(0)=0$. Expand φ as power series at $z=0$

$$\varphi(z)=zJ_\varphi(0)+\sum_{i,j=1}^{n}\varphi_{ij}(0)z_iz_j+\cdots.$$

By the definition of $F(z)$, on the one hand,

$$F(z)=(z\tilde{k}+z)/2+\frac{1}{2}\sum_{i,j=1}^{n}d_{ij}((z\tilde{k})_i(z\tilde{k})_j+z_iz_j)+\cdots;$$

and on the other hand,

$$F(z)=f(\varphi(z))=\varphi(z)+\sum_{i,j=1}^{n}d_{ij}\varphi_i(z)\varphi_j(z)+\cdots$$
$$=zJ\varphi(0)+\sum_{i,j=1}^{n}\varphi_{ij}(0)z_iz_j+\sum_{i,j=1}^{n}d_{ij}(zJ_\varphi(0))_i(zJ_\varphi(0))_j+\cdots,$$

where $\varphi(z)=(\varphi_1(z),\ \cdots,\ \varphi_n(z))$.

These two power series express the same mapping $F(z)$. Comparing the coefficients of the first order terms, we have

$$J_\varphi(0)=(\tilde{k}+I)/2.$$

Comparing the coefficients of the second terms, we have

$$\frac{1}{2}\sum_{i,j=1}^{n}d_{ij}((zk)_i(zk)_j+z_iz_j)$$
$$=\sum_{i,j=1}^{n}\varphi_{ij}(0)\,z_iz_j+\sum_{i,j=1}^{n}d_{ij}\,\frac{1}{2}(z\tilde{k}+z)_i\,\frac{1}{2}(z\tilde{k}+z)_j.$$

Thus

$$\sum_{i,j=1}^{n}\varphi_{ij}(0)z_iz_j=\frac{1}{4}\sum_{i,j=1}^{n}d_{ij}((z\tilde{k})_i(z\tilde{k})_j+z_iz_j)$$
$$-\frac{1}{4}\sum_{i,j=1}^{n}d_{ij}(z\tilde{k})_iz_j-\frac{1}{4}\sum_{i,j=1}^{n}d_{ij}(z\tilde{k})_jz_i$$
$$=\frac{1}{4}\sum_{i,j=1}^{n}\Big(\sum_{p,s=1}^{n}d_{ps}k_{is}k_{jp}+d_{ij}\Big)z_iz_j$$
$$-\frac{1}{4}\sum_{i,j=1}^{u}\Big(\sum_{s=1}^{n}d_{sj}k_{is}\Big)z_iz_j-\frac{1}{4}\sum_{i,j=1}^{n}\Big(\sum_{s=1}^{n}d_{si}k_{js}\Big)z_iz_j.$$

We get the expression of $\varphi_{ij}(0)$ as

$$\varphi_{ij}(0)=\frac{1}{4}\Big(\sum_{p,s=1}^{n} d_{ps}\bar{k}_{is}\bar{k}_{jp}+d_{ij}\Big)-\frac{1}{4}\sum_{s=1}^{n} d_{sj}\bar{k}_{is}-\frac{1}{4}\sum_{s=1}^{n} d_{si}\bar{k}_{js}$$
$$=\sum_{p,q=1}^{n} w_{ip}\, d_{pq}\, w_{jq},$$

where $(w_{ij})=(\bar{k}-I)/2$; the exppression of $\varphi(z)$ as

$$\varphi(z)=z(\bar{k}+I)/2+\sum_{i,j,p,q} w_{ip}d_{pq}w_{jq}z_iz_j+\cdots. \tag{1}$$

Let $\xi=z(\bar{k}+I)/2$, $\eta=z(\bar{k}-I)/2$. Then $\operatorname{Re}(\xi,\eta)=0$. In fact, since $\bar{k}\in U_n$, we have

$$(\xi,\eta)=\xi\bar{\eta}'=z(\bar{k}+I)(\bar{k}-I)\bar{z}'/4$$
$$=[z\bar{k}k'\bar{z}'+zk'\bar{z}'-z\bar{k}\bar{z}'-z\bar{z}']/4=[zk'\bar{z}'-z\bar{k}\bar{z}']/4=-\operatorname{Im} z\bar{k}\bar{z}'/2.$$

Let $e_j=(0,\cdots,0,1,0,\cdots,0)$, all the components are equal to zero except the j-th component which equals 1.

Lemma 2.2. *If the set $F=\{z_ke_k\in M\}$ in M is the unit disk, then*

$$|d_{kk}^{(j)}|<1,\ j=1,2,\cdots,n.$$

Proof Taking $\bar{k}=-I$ in (1), we have

$$\varphi(z)=\sum_{i,j,p,q} w_{ip}d_{pq}w_{jq}z_iz_j+\cdots=\sum_{p,q} d_{pq}z_pz_q+\cdots$$

and $\varphi(z)\in M$ since $\varphi(z)$ maps M into M. In particular, we take $z_ke_k\in M$; then

$$\varphi(z_ke_k)=d_{kk}z_k^2+\cdots\in M$$

holds for any $z_k=re^{iy}$, $0\leqslant r<1$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(re^{iy}e_k)e^{-2iy}\,dy=r^2d_{kk}.$$

The j-th component is

$$\frac{1}{2\pi}\int_0^{2\omega}\varphi_j(re^{iy}e_k)e^{-2iy}\,dy=r^2d_{kk}^{(j)}.$$

Then

$$|r^2d_{kk}^{(j)}|\leqslant\frac{1}{2\pi}\int_0^{2\pi}|\varphi_j(re^{iy}e_k)|\,dy\leqslant 1.$$

Letting $r\to 1$, we obtain Lemma 2.2.

Lemma 2.3. *If the set $\{z_je_j+z_ke_k\,|\,z_j=r_je^{iy_j},\ z_k=r_ke^{iy_k},\ 0\leqslant r_j,\ r_k<1,\ 0\leqslant y_j,\ y_k<2\pi\}\subset M$, then*

$$|d_{jk}^{(m)}|\leqslant 1/2,\ j\neq k,\ m=1,2,\cdots,n.$$

Proof Taking $\bar{k}=-I$ in (1), we get

$$\varphi(z_je_j+z_ke_k)=2d_{jk}r_jr_ke^{i(y_j+y_k)}+\cdots$$

by (1), where $z_j=r_je^{iy_j}$, $z_k=r_ke^{iy_k}$. We have

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\varphi(z_je_j+z_ke_k)e^{-i(y_j+y_k)}\,dy_j\,dy_k=2d_{jk}r_jr_k.$$

Taking the m-th component on both sides, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\varphi_m(z_je_j+z_ke_k)e^{-i(y_j+y_k)}\,dy_j\,dy_k=2d_{jk}(m)r_jr_k.$$

Hence $|2d_{jk}(m)|r_jr_k\leqslant 1$.

Let ting $r_j \to 1$, $r_k \to 1$, we prove the lemma.

Lemma 2.4. *If the set $B_2 = \{z_j e_j + z_k e_k \in M\}$ forms the complex ball $|z_j|^2 + |z_k|^2 \leqslant 1$, and for any $\xi, \eta \in B_2$, $|\xi|^2 + |\eta|^2 < 1$,* $\mathrm{Re}(\xi, \eta) = 0$, *there exist $z \in B_2$ and $k \in K$ such that $\xi - \eta = z$, $\xi + \eta = z\bar{k}$, then*

$$|d_{jk}^{(k)}| \leqslant \sqrt{2}/2.$$

Proof Take $\xi = \xi_k e^{iy} e_k = ai|\eta|^{-1}\eta_k e^{iy} e_k$, $\eta = \eta_j e_j + \eta_k e^{iy} e_k$, $a \in R$, and $|\xi_k|^2 + |\eta_j|^2 + |\eta_k|^2 < 1$. Then

$$\varphi(\xi - \eta) = \xi + \sum d_{pq} \eta_p \eta_q + \cdots \in M$$

by (1). The k-th component is

$$\varphi_k(\xi - \eta) = \xi_k e^{iy} + 2 d_{jk}^{(k)} \eta_j \eta_k e^{iy} + T \xi_k (\eta_j)^2 e^{iy} + O(|\eta|^3),$$

where T is a constant. Hence

$$\xi_k + 2 d_{jk}^{(k)} \eta_j \eta_k + T \xi_k \eta_j^2 + O(|\eta|^3) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_k(\xi - \eta) e^{-iy} \, dy,$$

and

$$|\xi_k + 2 d_{jk}^{(k)} \eta_j \eta_k + T \xi_k (\eta_j)^2 + O(|\eta|^3)| \leqslant 1.$$

This implies

$$\frac{|2 d_{jk}^{(k)} \eta_j \eta_k + T \xi_k \eta_j^2 + O(|\eta|^3)|}{|1 - |\xi_k|^2 - 2 d_{jk}^{(k)} \xi_k \eta_j \eta_k - T |\xi_k|^2 \eta_j^2 + O(|\eta|^3)|} \leqslant 1.$$

Choose ξ, η such that $1 - |\xi|^2 - |\eta|^2 = o(|\eta|^2)$, and $\lim\limits_{|\eta| \to 0} \eta_j |\eta|^{-1} = b_j$,

$$\lim_{|\eta| \to 0} \eta_k |\eta|^{-1} = b_k, \quad \lim_{|\eta| \to 0} \xi_k = a_k. \text{ Then}$$

$$\frac{|2 d_{jk}^{(k)} b_j b_k + T a_k b_j^2|}{|1 - 2 d_{jk}^{(k)} \bar{a}_k b_j \eta_k - T |a_k|^2 (b_j)^2|} \leqslant 1$$

when $|\eta| \to 0$. By the choise of ξ, η, we have $|a_k| = 1$. choosing $b_k = 0$, $|b_j| = 1$, $T b_j^2 > 0$, we get $|T| \leqslant 1/2$. Taking b_j, b_k such that $2 d_{jk}^{(k)} a_k b_j b_k \geqslant 0$, we have

$$\frac{|2 d_{jk}^{(k)} b_j b_k| - |T b_j^2|}{1 - |2 d_{jk}^{(k)} b_j b_k| + |T b_j^2|} \leqslant 1.$$

Hence

$$4 |d_{jk}^{(k)} b_j b_k| \leqslant 1 + 2 |T b_j^2| \leqslant 1 + |b_j|^2,$$

i. e.,

$$|d_{jk}^{(k)}| \leqslant (1 + |b_j|^2) / (4 |b_j b_k|).$$

We get $|d_{jk}^{(k)}| \leqslant \sqrt{2}/2$ when we take $|b_j| = 1/\sqrt{3}$, $|b_k| = \sqrt{2/3}$ since

$$|b_j|^2 + |b_k|^2 = 1.$$

§ 3. Distortion Theorem of Convex Mappings on Classical Domains of Type I

The classical domain of type I: $R_I \subset C^{mn}$ $(m \leqslant n)$, is defined as $z = (z_{11}, \cdots, z_{1n}, \cdots, z_{m1}, \cdots, z_{mn})$ if and only if

$$z = z^{(m,n)} = (z_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$$

satisfies $I-Z\overline{Z}'>0$, where I is the identity matrix. In this case,

$$G=SU(m,n),\ K=S(U_m\times U_n),$$

and Z can be expressed as $Z=U\Lambda V$, where $U\in U_n$, $V\in U_n$,

$$\Lambda=\begin{pmatrix}\lambda_1 & & 0\cdots 0\\ & \ddots & \\ 0 & \lambda_m, & \cdots 0\end{pmatrix},\ 1>\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m\geqslant 0,$$

(cf. [2, 3]) $\lambda_j=\tanh x_j$, $j=1, 2, \cdots, m$.

If all components of z are zero except $z_{rs}=1$, we denote this vector be e_{rs}. Consider $z_0=z_{ij}e_{ij}+z_{pq}e_{pq}$. There are four possibilities: (1) $i\neq p$, $j\neq q$; (2) $i=p$, $j\neq q$; (3) $i\neq p$, $j=q$; (4) $i=p$, $j=q$.

(1) If $i\neq p$, $j\neq q$, then $z_0\in R_I$ if and only if $|z_{ij}|<1$, $|z_{pq}|<1$. Set $\{z_0\in R_I\}$ is a polydisc. By Lemma 3, $|d^{(pj)(iq)}_{(ij)(pq)}|\leqslant 1/2$.

(2) If $i=p$, $j\neq q$, then $z_0\in R_I$ if and only if $1-|z_{ij}|^2-|z_{pq}|^2>0$. Set $\{z_0\in R_I\}$ is a ball. By Lemma 4, $|d^{(iq)(ij)}_{(ij)(pq)}|\leqslant\sqrt{2}/2$.

(3) If $i\neq p$, $j=q$. Since $I-\overline{Z}Z'>0$ and $I-Z'Z>0$ are equivalent to each other, we have $|d^{(pj)(ij)}_{(ij)(pq)}|\leqslant\sqrt{2}/2$ by the same reason as 2).

(4) If $i=p$, $j=q$, then take $z_0=z_{ij}e_{ij}$ and $z_0\in R_I$ if and only if $|z_{ij}|<1$. Set $\{z_0\in R_I\}$ is unit disc. By Lemma 2.2, $|d^{(ij)}_{(ij)(ij)}|\leqslant 1$.

Substituting all these estimations to Lemma 2.1, we get the upper bound of $c_1(S)$ of R_I:

$$\begin{aligned}c_1(S)&\leqslant(mn-m-n+1)/2+(\sqrt{2}/2)(m+n-2)+1\\&=(mn+1)/2+(\sqrt{2}-1)(m+n-2)/2.\end{aligned}$$

Now we consider the lower bound of $c_1(S)$.

We know that the holomorphic automorphism of R_I are

$$W=Q(Z-P)(I-\overline{P}'Z)^{-1}R^{-1},$$

where P, Q, R satisfy

$$\begin{aligned}&P=P^{(m,n)}=(p_{ij})\in R_I,\ \text{i.e. } I-P\overline{P}'>0,\\&Q=Q^{(m,m)},\ \overline{Q}'Q=(I-P\overline{P}')^{-1},\\&R=R^{(n,n)},\ \overline{R}'R=(I-\overline{P}'P)^{-1}.\end{aligned}$$

The holomorphic automorphisms of R_I are convex mappings since R_I is convex. obviously,

$$\begin{aligned}W+QPR^{-1}&=Q(Z-P+P-P\overline{P}'Z)(I-\overline{P}'Z)^{-1}R^{-1}\\&=Q(I-P\overline{P}')Z(I-\overline{P}'Z)^{-1}R^{-1}\\&=(\overline{Q}')^{-1}Z(I-\overline{P}'Z)^{-1}R^{-1}.\end{aligned}$$

Let $F(Z)=Z(I-\overline{P}'Z)^{-1}$. Then $F(Z)$ is a normalized biholomorphic convex mapping which maps R_I into C^{mn}.

The (ks) entry of

$$F(Z)=(F^{(ks)}(Z))=Z(I-\overline{P}'Z)=Z+Z\overline{P}'Z+\cdots$$

$$\text{is } F^{(ks)}(Z)=z_{ls}+\sum_{i,j} z_{ki}\bar{p}_{ji}z_{js}+\cdots$$

and

$$\frac{\partial F^{(ks)}(Z)}{\partial z_{ks}}=1+\sum_j \bar{p}_{js}z_{js}+\sum_i \bar{p}_{ki}z_{ki}+\cdots,$$

$$\frac{\partial^2 F(ks)(Z)}{\partial z_{ii}\,\partial z_{ks}}=\bar{p}_{ki}\delta_{ki}+\bar{p}_{is}\delta_{is}+\cdots.$$

Taking $P=\begin{pmatrix}\lambda & & 0\cdots 0\\ & \ddots & \\ 0 & \lambda & \cdots 0\end{pmatrix}$, $0\leqslant\lambda<1$, we have

$$\frac{1}{2}\sum_{(ks)}\frac{\partial F^{(ks)}}{\partial z_{ii}\,\partial z_{ks}}=(m-1+n-1+2)\lambda/2=(m+n)\lambda/2.$$

Hence $c(S)>(m+n)/2$.

Let $\xi=(I_m, 0)$. Then

$$F_1(Z)=Z(I-\bar{\xi}'Z)^{-1}=\lim_{\lambda\to 1} Z(I-\bar{P}'Z)^{-1},\ P=(\lambda I_m,\ 0)$$

is a normlized biholomorphic convex mapping which maps R_I into C^{mn}. The differentiation of F_1 is

$$dF_1=dZ(I-\bar{\xi}'Z)^{-1}+Z(I-\bar{\xi}'Z)^{-1}\bar{\xi}'dZ(I-\bar{\xi}'Z)^{-1}$$
$$=(I+Z(I-\bar{\xi}'Z)^{-1}\bar{\xi}')dZ(I-\bar{\xi}'Z)^{-1}.$$

The determinant of Jacobian of F_1 a t Z is

$$\det J_{F_1}(Z)=\det(I+Z(I-\bar{\xi}'Z)^{-1}\bar{\xi}')^m\det(I-\bar{\xi}'Z)^{-n}.$$

Taking $Z=\begin{pmatrix}\lambda_1 & & 0\cdots 0\\ & \ddots & \\ 0 & \lambda_m & \cdots 0\end{pmatrix}$, we have

$$\det J_{F_1}(Z)=\Big(\prod_{j=1}^m(1-\lambda_j)\Big)^{-m-n}.$$

Finally, we get the estimation of $|\det J_{F_1}(Z)|$

$$\prod_{j=1}^m(1+|\lambda_j|)^{-m-n}\leqslant|\det J_{F_1}(Z)|\leqslant\prod_{j=1}^m(1-|\lambda_j|)^{-m-n}.$$

This estimation is precise, i. e., the equalities holds at some points.

We conclude it as

Theorem 3.1. *If $f(Z)$: $R_I\to C^{mn}$ is a normalized biholomorphic convex mapping, then*

$$\frac{\Big(\prod_{j=1}^m(1-\lambda_j)\Big)^{c_1(s)-(m+n)/2}}{\Big(\prod_{j=1}^m(1+\lambda_j)\Big)^{c_1(s)+(m+n)/2}}\leqslant|\det J_f(Z)|\leqslant\frac{\Big(\sum_{j=1}^m(1+\lambda_j)\Big)^{c_1(s)-(m+n)/2}}{\Big(\prod_{j=1}^m(1-\lambda_j)\Big)^{c_1(s)+(m+n)/2}}$$

where

$$Z=U\begin{pmatrix}\lambda_1 & 0 & \\ & \ddots & \\ 0 & \lambda_m, & 0\end{pmatrix}V,\ U\in U_m,\ V\in U_n,\ 1>\lambda_1\geqslant\cdots\geqslant\lambda_m\geqslant 0,$$

and $c_1(S)$ is a constant which satisfies

$$(m+n)/2 \leqslant c_1(S) \leqslant (mn+1)/2 + (\sqrt{2}-1)(m+n-2)/2.$$

Conjecture 1. $c_1(S)=(m+n)/2$, and $F_1(Z)=Z(I-(I_m, 0)'Z)^{-1}$ is one of the extremal mappings.

If the conjecture is true, then

$$\left(\prod_{j=1}^{m}(1+\lambda_j)\right)^{-m-n} \leqslant |\det J_f(Z)| \leqslant \left(\prod_{j=1}^{m}(1-\lambda_j)\right)^{-m-n}.$$

§ 4. Distortion Theorem of Convex Mappings in Classical Domain of Type II

The classical domain of type II, $R_{II}\subset C^{n(n+1)/2}$, is defined as

$z=(z_{11}, \cdots, z_{1n}, z_{22}, \cdots, z_{2n}, \cdots, z_{nn})$ if and only if $Z=Z^{(n)}$

$=(z_{ij})_{1\leqslant i,j\leqslant n}$ satisfies $I-Z\bar{Z}>0$, $Z=Z'$. In this case $G=S_p(n, R)$

$=S_p(n, C)\cap SU(n, n)$, $K=U_n$, and Z can be expressed as

$$Z=U\Lambda U', \quad U\in U_n,$$

$$\Lambda=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}, \quad 1>\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0,\ \lambda_j=\tanh x_j,\ j=1, 2, \cdots, n.$$

If all components of z are zero except $z_{rs}=1$, we denote this vector by e_{rs}. Consider $z_0=z_{jk}e_{jk}+z_{pq}e_{pq}$. At first we consider the case $j\neq p$, $k\neq q$. There are four possibilities: (1) j, k, p, q, are mutually unequal to each other; (2) $j=k$ and j, p, are mutually unequal to each other; (3) $p=q$ and j, k, p are mutualy unequal to each other; (4) $j=k$, $p=q$ and $j\neq p$.

(1) If j, k, p, q are mutually unequal to each other, then $z_0\in R_{II}$ if and only if $|z_{jk}|<1$, $|z_{pq}|<1$. Set $\{z_0\in R_{II}\}$ is a polydisc. By Lemma 3, $|d^{(pq)}_{(jk)(pq)}|\leqslant 1/2$.

(2) If $j=k$, j, p, q are mutually unequal to each other, then $z_0\in R_{II}$ if and only if $|z_{jj}|<1$, $|z_{pq}|<1$. Set $\{z_0\in R_{II}\}$ is a polydisc. By Lemma 2.3, $|d^{(pq)}_{(jj)(pq)}|\leqslant 1/2$.

(3) If $p=q$, j, k, p are mutually unequal to each other. By the same reason as 2) $|d^{(pp)}_{(jk)(pp)}|\leqslant 1/2$.

(4) If $j=k$, $p=q$, $j\neq p$, then $z_0\in R_{II}$ if and only if $|z_{jj}|<1$, $|z_{pp}|<1$. Set $\{z_0\in R_{II}\}$ is a polydisc. By Lemma 2.3, $|d^{(pp)}_{(jj)(pp)}|\leqslant 1/2$.

Thus $|d^{(pq)}_{(jk)(pq)}|\leqslant 1/2$ holds when $j\neq p$, $k\neq q$.

Next, we consider $j=p$ $k\neq q$ or $k=q$, $j\neq p$. We only need to consider one case, for example, $j=p$, $k\neq q$. There are two possibilities: (1) j, k, q are mutually unequal to eacd other, (2) $j=k$, $j\neq q$.

(1) If $j=p$, $k\neq q$, j, k, q are mutually unequal to each other, then $z_0\in R_{II}$ if and only if $1-|z_{jk}|^2-|z_{jq}|^2>0$. $\{z_0\in R_{II}\}$ is a ball.

By Lemma 2.4, $|d^{(jq)}_{(jk)(jq)}|\leqslant\sqrt{2}/2$.

(2) If $j=p=k$, $j\neq q$, then $z_0\in R_{II}$ if and only if

$$(1-|z_{jj}|^2-|z_{jq}|^2)(1-|z_{jq}|^2)-|z_{jj}|^2|z_{jq}|^2>0.$$

It does not satisfy the conditions of Lemma 2.4. Consider $z(t)=z_{jj}e_{jj}+z_{jq}e_{jq}$. Let

$$\begin{pmatrix} u_{jj} & u_{jq} \\ u_{qj} & u_{qq}\end{pmatrix}\in U_2,$$

E_{jk} is an $n\times n$ matrix, all the entries are equal to zero except the jk-th entry which is 1. Let $U=I-E_{jj}-E_{qq}+u_{jj}E_{jj}+u_{jq}E_{jq}+u_{qj}E_{qj}+u_{qq}E_{qq}\in U_n$. The corresponding $Z(t)=z_{jj}E_{jj}+z_{jq}e^{it}(E_{jq}+E_{qj})$. Let $A(t)=UZ(t)U'=(a_{ij}(t))$. $A(t)$ is a symmetric matrix since $Z(t)$ is. $A(t)\in R_{II}$ if $Z(t)\in R_{II}$. Let $a(t)=(a_{11}(t),\ \cdots,\ a_{1n}(t),\ a_{22}(t),\ \cdots,\ a_{2n}(t),\ \cdots,\ a_{nn}(t))=z(t)V$. V is not a unitary matrix. Let $\xi=(z(t)V+z(t))/2$, $\eta=(z(t)V-z(t))/2$. Then $\xi+\eta=a(t)$, $\xi-\eta=z(t)$. By (1),

$$\varphi(z(t))'=\varphi(\xi-\eta)=\xi+\sum d_{(pq)(rs)}\eta_{pq}\eta_{rs}+\cdots$$

and

$$a(t)=(z_{jj}u_{jj}^2+2z_{jq}e^{it}u_{jj}u_{jq})e_{jj}+(z_{jj}u_{jj}u_{qj}+z_{jq}e^{it}(u_{jq}u_{qj}+u_{jj}u_{qq}))e_{jq}$$
$$+(z_{jj}u_{qj}^2+2z_{jq}e^{it}u_{qj}u_{qq})e_{qq}.$$

We can evaluate $\xi(t)$ and $\eta(t)$,

$$\xi(t)=(z_{jj}(1+(u_{jj})^2)/2+z_{jq}e^{it}u_{jj}u_{jq})e_{jj}+\frac{1}{2}(z_{jj}u_{jj}u_{qj}+z_{jq}e^{it}(1+u_{jq}u_{qj}+u_{jj}u_{qq}))e_{jq}$$
$$+((z_{jj}(u_{qj})^2/2)+z_{jq}e^{it}u_{qj}u_{jq})e_{qq},$$

$$\eta(t)=(z_{jj}(-1+(u_{jj})^2)/2+z_{jq}e^{it}u_{jj}u_{jq})\,e_{jj}+\frac{1}{2}(z_{jj}u_{jj}u_{qj}+z_{jq}e^{it}(-1+u_{jq}u_{qj}+u_{jj}u_{qq}))e_{jq}$$
$$+(z_{jj}(u_{qj})^2/2+z_{jq}e^{it}u_{qj}u_{qq})e_{qq}.$$

Thus

$$\varphi(z)=\xi+d_{(jj)(jj)}(\eta_{jj})^2+2d_{(jj)(jq)}\eta_{jj}\eta_{jq}+2d_{(jj)(qq)}\eta_{jj}\eta_{qq}+2d_{(jq)(qq)}\eta_{jq}\eta_{qq}$$
$$+d_{(jq)(jq)}(\eta_{jq})^2+d_{(qq)(qq)}(\eta_{qq})^2+\cdots.$$

Taking $u_{jj}=\cos\alpha e^{iy}$, $u_{jq}=-\sin\alpha e^{i(\alpha-t)}$, $u_{qj}=\sin\alpha e^{i(y+t)}$, $u_{qq}=\cos\alpha e^{i\alpha}$, we have

$$\xi_{jj}(t)=\frac{1}{2}(1+(\cos\alpha)^2e^{2iy})z_{jj}-\sin\alpha\cos\alpha\, e^{yi(+\alpha)}z_{jq},$$

$$\xi_{jq}(t)=\frac{1}{2}(\sin\alpha\cos\alpha\, e^{i(2y+t)}z_{jj}+z_{jq}e^{it}((-(\sin\alpha)^2+(\cos\alpha)^2)e^{i(y+\alpha)}+1)),$$

$$\xi_{qq}(t)=\frac{1}{2}(\sin\alpha)^2e^{2i(y+t)}z_{jj}+z_{jq}\sin\alpha\cos\alpha\, e^{i(y+\alpha+2t)},$$

$$\eta_{jj}(t)=\frac{1}{2}((\cos\alpha)^2e^{2iy}-1)z_{jj}-\sin\alpha\cos\alpha\, e^{i(y+\alpha)}z_{jq},$$

$$\eta_{jq}(t)=\frac{1}{2}(\sin\alpha\cos\alpha\, e^{i(2y+t)}z_{jj}+z_{jq}e^{it}((-(\sin\alpha)^2+(\cos\alpha)^2)e^{i(y+\alpha)}-1)),$$

$$\eta_{qq}(t)=\xi_{qq}(t).$$

Substituting all these formulas to the previous equality, we can get

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_{jq}(z(t))e^{-it}\,dt$$
$$=\xi_{jq}(0)+2d^{(jq)}_{(jj)(jq)}\eta_{jj}(0)\eta_{jq}(0)+T(\eta_{jj}(0))^2\xi_{jq}(0)+O(|\eta(0)|^3).$$

This implies

$$|\xi_{jq}(0)+2d^{(jq)}_{(jj)(jq)}\eta_{jj}(0)\eta_{jq}(0)+T(\eta_{jj}(0))^2\xi_{jq}(0)+O(|\eta(0)|^3)|<1.$$

Take $|z_{jj}|=o(1-|z_{jq}|^2)$. Then

$$|\eta_{jj}(0)|^2+|\eta_{jq}(0)|^2$$
$$=(\cos\alpha)^2(\sin\alpha)^2\,|z_{jq}|^2+\frac{1}{4}|1+((\sin\alpha)^2-(\cos\alpha)^2)e^{i(y+x)}|^2|z_{jq}|^2+o(1-|z_{jq}|^2)$$
$$=((\sin\alpha)^2-(\sin\alpha)^4)\,|z_{jq}|^2+\frac{1}{2}[(1-2(\sin\alpha)^2)(1-\cos(y+x))$$
$$+2(\sin\alpha)^4]\,|z_{jq}|^2+o(1-z_{jq}|^2).$$

Take $(\sin\alpha)^2=O(1-|z_{jq}|^2)$. Then

$$|\eta_{jj}(0)|^2+|\eta_{jq}(0)|^2$$
$$=((\sin\alpha)^2+(1-\cos(y+x))(1-2(\sin\alpha)^2))\,|z_{jq}|^2/2+o(1-|z_{jq}|^2).$$

Take
$$1-\cos(y+x)=O(1-|z_{jq}|^2),$$
$$(\sin\alpha)^2+(1-\cos(y+x))/2=1-|z_{jq}|^2+o(1-|z_{jq}|^2).$$

Then
$$|\eta_{jj}(0)|^2+|\eta_{jq}(0)|^2=1-|z_{jq}|^2+o(1-|z_{jq}|^2).$$

Moreover,

$$1-|\xi_{jq}(0)|^2-|\eta_{jj}(0)|^2-|\eta_{jq}(0)|^2=1-|z_{jq}|^2+\sin^2\alpha|z_{jq}|^2+o(1-|z_{jq}|^2).$$

Using the same process as we did in the proof of Lemma 2.4, we have

$$|d^{(jk)}_{(jj)(jk)}|\leqslant\sqrt{3}/2.$$

By Lemma 2.2, $|d^{(jj)}_{(jj)(jj)}|\leqslant 1$. Observing all these estimations and using Lemma 2.1, we obtain the upper bound of $c_2(S)$,

$$c_2(S)\leqslant(n(n+1)/2-n)/2+\sqrt{3}\,(n-1)/2+1$$
$$=(n(n+1)/2+1)/2+(\sqrt{3}-1)(n-1)/2.$$

Now we consider the lower bound of $c_2(S)$.

We know that the biholomorphic automorphisms of R_{II} are

$$W=R(Z-P)(I-\bar{P}Z)^{-1}\bar{R}^{-1},$$

where P, R satisfy

$$P=P'=P^{(n)}=(p_{ij})\in R_{II},\ \text{i. e.}\ I-P\bar{P}>0,$$
$$R=R^{(n)},\ \bar{R}(I-\bar{P}P')R'=I.$$

The biholomorphic automorphisms of R_{II} are convex mappings since R_{II} is convex. Obviously

$$W+RPR^{-1}=R(Z-P+P+P\bar{P}Z)(I-\bar{P}Z)^{-1}\bar{R}^{-1}$$
$$=R(I-P\bar{P})Z(I-\bar{P}Z)^{-1}R^{-1}=\bar{R}'^{-1}Z(I-\bar{P}Z)^{-1}\bar{R}^{-1}.$$

Let $F(Z)=Z(I-\bar{P}Z)^{-1}$. Then $F(Z)$ is a normalized biholomorphic convex mapping which maps R_{II} into $C^{n(n+1)/2}$. The (ks) entry of

$$F(Z)=(F^{(ks)}(Z))=Z(I-\bar{P}Z)^{-1}=Z+Z\bar{P}Z+\cdots$$

is

$$F^{(ks)}(Z)=z_{ks}+\sum_{i,j}z_{ki}\bar{p}_{ij}z_{js}$$

and

$$\frac{\partial^2 F^{(ks)}(Z)}{\partial z_{ii}\,\partial z_{ks}}=\bar{p}_{ki}\delta_{ki}+\bar{p}_{is}\delta_{is}+\cdots.$$

Taking $P=\begin{pmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{pmatrix}$, $0\leqslant\lambda<1$, we have

$$\frac{1}{2}\sum_{(ks)}\frac{\partial^2 F^{(ks)}}{\partial z_{ii}\,\partial z_{ks}}\bigg|_{z=0}=\frac{(n-1+2)\lambda}{2}=\frac{(n+1)\lambda}{2}.$$

Hence $c_2(S)\geqslant(n+1)/2$.

Let $\lambda\to1$. Then

$$F_2(Z)=Z(I-Z)^{-1}=\lim_{\lambda\to1}Z(I-\bar{P}Z)^{-1},\quad P=\begin{pmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{pmatrix}$$

is a normalized biholomorphic convex mapping which maps R_{II} into $C^{n(n+1)/2}$. The differentiation of F_2 is

$$\begin{aligned}dF_2&=dZ(I-Z)^{-1}+Z(I-Z)^{-1}dZ(I-Z)^{-1}\\&=(I+Z(I-Z)^{-1})dZ(I-Z)^{-1}=(I-Z)^{-1}dZ(I-Z)^{-1}.\end{aligned}$$

The determinant of Jacobian of F_2 at Z is

$$\det J_{F_2}(Z)=\det(I-Z)^{-n-1}$$

Taking $Z=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$, we have

$$\det J_{F_2}(Z)=\left(\prod_{j=1}^{n}(1-\lambda_j)\right)^{-n-1}.$$

Finally, we get the estimations of $|\det J_{F_2}(Z)|$

$$\left(\prod_{j=1}^{n}(1+\lambda_j)\right)^{-n-1}\leqslant|\det J_{F_2}(Z)|\leqslant\left(\prod_{j=1}^{n}(1-\lambda_j)\right)^{-n-1}.$$

The estimation is precise, i. e., the equalities hold at some points.

We conclude it as

Theorem 4.1. *If ($f(z)$: $R_{II}\to C^{n(n+1)/2}$ is a normalized biholomrphic convex mapping, then*

$$\frac{\prod_{j=1}^{n}(1-\lambda_j)^{c(S)-(n+1)/2}}{\prod_{j=1}^{n}(1+\lambda_j)^{c(S)-(n+1)/2}}\leqslant|\det J_{F_2}(Z)|\leqslant\frac{\prod_{j=1}^{n}(1+\lambda_j)^{c(S)-n(n+1)/2}}{\prod_{j=1}^{n}(1-\lambda_j)^{c(S)-(n+1)/2}},$$

where $Z=U\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}U'$, $U\in U_n$, $1>\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant0$, *and* $c_2(S)$ *is a constant which* *satisfies*

$$(n+1)/2 \leqslant c_2(S) \leqslant (n(n+1)/2+1)/2+(\sqrt{3}-1)(n-1)/2$$

Conjecture 2. $c_2(S)=(n+1)/2$, and $F_2(Z)=Z(I-UZ)^{-1}$, $U=U'\in U_n$, are extremal mappings.

If the conjecture is true, then

$$\prod_{j=1}^{n}(1+\lambda_j)^{-n-1} \leqslant |\det J_f(Z)| \leqslant \prod_{j=1}^{n}(1-\lambda_j)^{-n-1}.$$

§ 5. Distortion Theorem of Convex Mappings on Classical Domains of Type III

The classical domain of type III, $R_{III}\subset C^{n(n-1)/2}$, is defined as

$z=(z_{12}, \cdots, z_{1n}, z_{23}, \cdots, z_{2n}, \cdots, z_{n-1,n})\in R_{III}$ if and only if

$z=Z^{(n)}=(z_{ij})$ satisfies $I+Z\bar{Z}\geqslant 0$, $Z=-Z'$. In this case

$G=SO^*(2n)$, $K=U_n$, and Z can be expressed as $Z=UMU'$, $U\in U_n$,

$$M=\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}+\begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix}+\cdots,\ 1>\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_v\geqslant 0,\ v=[n/2],$$

the direct sum stops at $\begin{pmatrix} 0 & \lambda_v \\ -\lambda_v & 0 \end{pmatrix}$ if n is even, the direct sum stops at $+0$ if n is odd. $\lambda_j=\tanh x_j$, $j=1, \cdots, v$.

If all components of z are zero except $z_{rs}=1$, we denote this vector by e_{rs}. Consider $z_0=z_{jk}e_{jk}+z_{pq}e_{pq}$. The corresponding $Z_0=z_{jk}(E_{jk}-E_{kj})+z_{pq}(E_{pq}-E_{qp})$. If z_{jk}, z_{pq} are not on the same row or same column at Z_0, then j, k, p, q are mutually unequal to each other, $z_0\in R_{III}$ if and only if $|z_{jk}|<1$, $|z_{pq}|<1$. Set $\{z_0\in R_{III}\}$ is a polydisc. By Lemma 2.3,

$$|d^{(mn)}_{(jk)(pq)}|\leqslant 1/2.$$

If z_{jk}, z_{pq} are on the same row or same column at z_0, we only need to consider the case of same row, i. e., $j=p$. Then j, k, q are mutually unequal to eachother. $z_0\in R_{III}$ if and only if $1-|z_{jk}|^2-|z_{pq}|^2>0$. Set $\{z_0\in R_{III}\}$ is a ball. By Lemma 2.4,

$$|d^{(jq)}_{(jk)(jq)}|<\sqrt{2}/2.$$

Using all these estimations and Lemma 2.1 we get the upper bound of $c_3(S)$

$$c_3(S)\leqslant(n(n-1)/2-(2n-4)-1)/2+\sqrt{2}(2n-4)/2+1$$
$$=(n(n-1)/2+1)/2+(\sqrt{2}-1)(n-2)$$

Now we consider the lower bound of $c_3(S)$.

We know that the holomorphic automorphisms of R_{III} are

$$W=Q(Z-P)(I+\bar{P}Z)^{-1}Q^{-1},$$

where P, Q satisfy

$$P=-P'=P^{(n)}=(p_{ij}),\ \text{i. e.,}\ I+P\bar{P}>0,$$

$$Q=Q^{(n)},\ Q(I+Q\bar{P})\bar{Q}'=I.$$

The holomorphic automorphisms of R_{III} are convex mappings since R_{III} is convex. Obviously

$$W+QP\bar{Q}^{-1}=Q(Z-P+P+P\bar{P}Z)(I+\bar{P}Z)^{-1}\bar{Q}^{-1}$$
$$=Q(I+P\bar{P})Z(I+\bar{P}Z)^{-1}\bar{Q}^{-1}=\bar{Q}'^{-1}Z(I+\bar{P}Z)^{-1}\bar{Q}^{-1}.$$

Let $F(Z)=Z(I+\bar{P}Z)^{-1}$. Then $F(Z)$ is a nomalized biholomorphic convex mapping which maps R_{III} into $C^{n(n-1)/2}$. The (ks) entry of

$$F(Z)=(F^{(ks)}(Z))=Z(I+\bar{P}Z)^{-1}=Z-Z\bar{P}Z+\cdots$$

is

$$F^{(ks)}(Z)=z_{ks}-\sum_{i,j}z_{ki}\bar{p}_{ij}z_{js}+\cdots.$$

Take $P=\begin{pmatrix}0&\lambda\\-\lambda&0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0&\lambda\\-\lambda&0\end{pmatrix}\dotplus\cdots$. Then the direct sum stops at $\begin{pmatrix}0&\lambda\\-\lambda&0\end{pmatrix}$ if n is even, and stops at $\dotplus 0$ if n is odd. We have

$$\frac{1}{2}\sum_{(ks)}\frac{\partial^2F^{(ks)}}{\partial z_{2r-1,2r}\,\partial z_{ks}}=(1-n)\lambda.$$

Hence $c_3(S)\geqslant(n-1)$.

Let $\lambda\to 1$. Then

$$F_3(Z)=Z(I+P_1Z)^{-1}=\lim_{\lambda\to 0}Z(I+\bar{P}Z)^{-1},\ P_1=\lim_{\lambda\to 1}P$$

is a normalized biholomorphic convex mapping which maps R_{III} into $C^{n(n-1)/2}$. The determinant of the Jacobian of F_3 is $\det(I+P_1Z)^{-n+1}$. Take

$$Z=\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0&\lambda_v\\-\lambda_v&0\end{pmatrix}\dotplus\cdots.$$

Then

$$\det J_{F_3}(Z)=\left(\prod_{j=1}^{v}(1-\lambda_j)^2\right)^{-n+1},\ v=[n/2].$$

Finally, we get the estimation of $|\det J_{F_3}(Z)|$

$$\left(\prod_{j=1}^{v}(1+\lambda_j)^2\right)^{-n+1}\leqslant|\det J_{F_3}(Z)|\leqslant\left(\prod_{j=1}^{v}(1-\lambda_j)^2\right)^{-n+1}.$$

The estimation is precise, i. e., the equalities holds at some points.

We conclude it as

Theorem 5.1. *If $f(z)$: $R_{III}\to C^{n(n-1)/2}$ is a normalized bipolomorphic convex mapping, then*

$$\frac{\prod_{j=1}^{v}(1-\lambda_j)^{c_3(S)-n+1}}{\prod_{j=1}^{v}(1+\lambda_j)^{c_3(S)+n-1}}\leqslant|\det J_f(Z)|\leqslant\frac{\prod_{j=1}^{v}(1+\lambda_j)^{c_3(S)-n+1}}{\prod_{j=1}^{v}(1-\lambda_j)^{c_3(S)+n-1}},$$

where $Z=UMU'$, $U\in U_n$, $M=\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\dotplus\cdots$, $1>\lambda_1\geqslant\cdots\geqslant\lambda_v\geqslant 0$, $v=[n/2]$. the dirct sum stops at $\begin{pmatrix}0&\lambda_v\\-\lambda_v&0\end{pmatrix}$ if n is even and stops at $\dotplus 0$ if n is cdd, and $c_3(S)$ is a constant which satisfies

$$n-1\leqslant c_3(S)\leqslant(n(n-1)/2+1)/2+(\sqrt{2}-1)(n-2).$$

Conjecture 3. $c_3(S)=n-1$, and $F_3(Z)=Z(I-UZ)^{-1}$, $U=-U'\in U_n$ are extremal mappings.

If the conjecture is true, then

$$\prod_{j=1}^{v}(1+\lambda_j)^{-2n+2}\leqslant|\det J_f(z)|\leqslant\prod_{j=1}^{v}(1-\lambda_j)^{-2n+2}.$$

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