Chin. Ann. of Math.
13B: 4 (1992), 485—506.

OSCILLATIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS——GENERAL LINEARIZED OSCILLA TIONS RESULTS

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Abstract

It is shown that for a wide class of the scalar and vector functional equations of the delay and the neutral type a positive solution exists if and only if the linearized approximating equations are non-oscillatory or, equivalently, the characteristic equations of these linear equations have a real root.

§ 1. Introduction and Notations

In the applications one often faces the following problem: The biological model or the physical system under investigation is presented by a system of differential equations and we ask about the oscillatory properties of the solutions around a steady state of the system.

If the equations are linear with delays then there have been several papers dealing with necessary, sufficient and also necessary and sufficient conditions for the oscillation of all solutions via the characteristic equations (see. e. q. [10, 2, 16]) or without the characteristic equations (see. e. g. [3, 5, 7, 17, 21] and the references therein).

Recently the interest is growing to study nonlinear delay differential equations, whose solutions exhibit an oscillatory behavior. The main reason is that the delay differential equations which play an important role in the applications are nonlinear, and—for instance—in the biological applications the delay equations give better description of the fluctuations in the population than the ordinary ones (see, e. g. [20] and the references therein).

One of the most plausible ideas is to investigate the oscillations in nonlinear equations via linear approximations similar to the stability analysis of perturbed linear systems.

Manuscript received May 17, 1989.

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Such linearized oscillation results for the delay logistic equations without neutral term were initiated by Kulenovic, Ladas and Meimaridou[14] and by Kulenovic and Ladas[13] while for the neutral delay logistic equation were initiated by Győri[8,9]. For some further general results we refer to [8] and [15] in the constant delay case while for piecewise constant arguments we refer to [6] and [11].

In this paper, on the basis of the above mentioned results we prove some general oscillation theorems for a wide class of nonlinear delay equations. Our results are essentialy new because they give necessary and sufficient conditions both for the oscillations in scalar equations with nonlinear neutral term and with mixed monotone right sides and also for the vector case.

As is customary, we say that a function z: $[\alpha, \infty)\to R$, $(\alpha\ R)$, is oscillatory if $z(t)$ has arbitrarily large zeros and nonoscillatory if $z(t)$ is eventually positive or eventually negative. A vector valued function $z=(z_1, \cdots, z_n)^T$: $[\alpha, \infty)\to R^n$ is called positive if $z_i(t)>0$, $(1\leqslant i\leqslant n,\ t\geqslant\alpha)$, and is called eventually positive (negative) if $z_i(t)$ is eventually positive (negative) for all $1\leqslant i\leqslant n$. By using vector notation, we express the fact that $z(t)$ is positive by writing $z(t)>0$, $t\geqslant\alpha$. For two vectors u, $v\in R^n$, $u>v$ means that $u_i>v_i$, $(1\leqslant i\leqslant n)$, and $u\geqslant v$ means that $u_i\geqslant v_i$, $(1\leqslant i\leqslant n)$. A function z: $[\alpha, \infty)\to R^n$ is called slightly positive if $z(t)\geqslant 0$, $(t>\alpha)$, and there is a component $z_i(t)$ of $z(t)$ which is positive on $[\alpha, \infty)$.

In section 2 of this paper we prove some results for the existence of nonnegative, slightly positive and positive solutions of the general functional equation

$$\frac{d}{dt}[x(t)-b(t, x(\cdot))]=-a(t, x(\cdot)), \tag{1.1}$$

where $a(t, x(\cdot))$ and $b(t, x(\cdot))$ are so called Volterra-type functionals defined, as follows:

Let t_{-1}, t_0 and D be such that $-\infty<t_{-1}\leqslant t_0<\infty$ and $D\subset C([t_{-1}, \infty), R^n)$. Then a functional c: $[t_0, \infty)\times D\to R^n$ is called a Volterra-type functional if for any (t, x), $(t, y)\in[t_0, \infty)\times D$,

$$c(t, x(\cdot))=c(t, y(\cdot)),$$

where $x(u)=y(u)$, $(t_{-1}\leqslant u\leqslant t)$. We say that $c(t,x(\cdot))$ is continuous if for all $x\in D$ the function $c(t, x(\cdot))$ is continuous on $[t_0, \infty)$ and for all $T\geqslant t_0$ and x, $y\in D$

$$\max_{t_0\leqslant t\leqslant T}|c(t, x(\cdot))-c(t, y(\cdot))|\to 0$$

when $x(t)$ tends to $y(t)$ in the max norm in $[t_{-1}, T]$.

In this paper we will use monotone vector norms. A vector norm $|\cdot|$ is called monotone if x, $y\in R^n_+$ and $x\leqslant y$ imply $|x|\leqslant|y|$.

We say that $c(t, x(\cdot))$ satisfies a Charatheodory condition if for all $x\in D$ the function $c(t, x(\cdot))$ is locally bounded and locally integrable and furthermore, for all $T \geqslant t_0$ and $x, y \in D$

$$\operatorname*{ess\,sup}_{t_0 < t < T} |c(t, x(\cdot)) - c(t, y(\cdot))| \to 0$$

when $x(t)$ tends to $y(t)$ in the max norm in $[t_{-1}, T]$.

In section 2 we also give some conditions so that every slightly positive solution of (1.1) satisfies

$$\lim_{t \to +\infty} x(t) = 0 \text{ and } \int_{t_0}^{\infty} x(t)\,dt < \infty.$$

In section 3 we give necessary and sufficient condition so that a linear scalar neutral autonomous differential inequality has a positive solution and so that a system of autonomous linear delay equations has a slightly positive solution. Unfortunately we did not succeed in proving these results for neutral system. The reason of this failure is that we could only prove the following conjecture in the scalar neutral and in nonneutral system cases (see Proposition 3.1 and 3.2 in Section 3 of this paper).

Conjecture 1. 1. *Consider the n-dimensional neutral equation*

$$\frac{d}{dt}[v(t) - Cv(t-r)] = B[v(t-\hat{\sigma}) - v(t-\sigma)] - Av(t-\tau), \tag{1.2}$$

where $\tau > 0$, $\sigma > \hat{\sigma} > 0$ *and* $r \geqslant 0$ *are constants,* $A, B, C \in R_+^{n\times n}$ *are n by n constant matrices such that* $Ae \neq 0$ *if* $e \in R_+^n$ *and* $e \neq 0$ *and the spectral radius of the matrix* $B(\sigma - \hat{\sigma}) + C$ *satisfies the condition*

$$\rho((\sigma - \hat{\sigma})B + C) < 1. \tag{1.3}$$

Then Equation (1.2) *has a slightly positive solution on some interval* $[t_0, \infty)$ *if and only if thereexist* $\lambda_0 \leqslant 0$ *and* $e_0 \in R_+^n$ *such that*

$$\det(\lambda_0 I - \lambda_0 C e^{-\lambda_0 r} - B(e^{-\lambda_0 \hat{\sigma}} - e^{-\lambda_0 \sigma}) + Ae^{-\lambda_0 \tau}) = 0 \tag{1.4}$$

and

$$(\lambda_0 I - \lambda_0 C e^{-\lambda_0 r} - B(e^{-\lambda_0 \hat{\sigma}} - e^{-\lambda_0 \sigma}) + Ae^{-\lambda_0 \tau})e_0 = 0 \text{ and } |e_0| = 1. \tag{1.5}$$

In the nonneutral system case, i. e. when $C = 0$, we could use theasymptotic representation of the solutions of (1.2) via the characteristic roots as $t \to +\infty$. Unfortunately we do not know of such representation in the neutral case.

Our main linearized oscillations theorems are proved in section 4 where we give necessary, sufficient and also necessary and sufficient conditions for oscillations in the nonlinear scalar and vector equations.

To demonstrate the] main idea and some results of this paper we give the following special corollary of our general results:

Corollary 1. 1. *Consider the scalar neutral delay differential equation*

$$\frac{d}{dt}[x(t) - g(t, x(t-r))] = h(x(t-\hat{\sigma})) - h(x(t-\sigma)) - f(t, x(t-\tau)), \tag{1.6}$$

where $\tau \geqslant 0$, $\sigma \geqslant \hat{\sigma} \geqslant 0$ *and* $r \geqslant 0$ *are constants,* f, g: $[t_0, \infty) \times R$ *and* h: $R \to R$ *are continuous functions such that* $f(t, x) > 0$, $g(t, x) > 0$ *and* $h(x) > 0$, $(t \geqslant t_0, x > 0)$.

Then

(a) *if there are constants* $a>0$, $b\geqslant 0$ *and* $c\geqslant 0$ *and* $\delta_0>0$ *such that* $b+c<1$, *moreover*

$$f(t, x)\leqslant ax,\ g(t, x)\leqslant bx \text{ and } h(x)\leqslant cx,\ (t\geqslant t_0,\ 0\leqslant x<\delta_0), \tag{1.7}$$

and the linear equation

$$\frac{d}{dt}[u(t)-bu(t-r)]=c[u(t-\hat{\sigma})-u(t-\sigma)]-au(t-\tau) \tag{1.8}$$

has a positive solution then Equation. (1.6) *has a positive solution on* $[t_0-\gamma, \infty)$, $(\gamma=\max\{\tau, \sigma, r\})$;

(b) *if for all* $s>0$ *small enough there exists a* $\delta_s>0$ *such that*

$$f(t, x)\geqslant(1-s)ax,\ g(t, x)\geqslant(1-s)bx \text{ and } h(x)\geqslant(1-s)cx,\ (t\geqslant t_0,\ 0\leqslant x<\delta_s), \tag{1.9}$$

where the nonnegative constants a, b *and* c *satitfy* $a>0$ *and* $b+c<1$, *then the existence of an eventually positive solution of* (1.6) *implies that* (1.8) *has a positive solution or equivalently the characteristic equation*

$$\lambda(1-be^{-\lambda r})=c(e^{-\lambda\hat{\sigma}}-e^{-\lambda\sigma})-ae^{-\lambda\tau} \tag{1.10}$$

has a negative root.

One can see from the general results of this paper that a suitable modified version of statement (a) of Corollary 1.1 is true in n dimensional system case, too. But statement (b) of Corollary 1.1 could be proved for systems under the restrictions that $g(t, x)$ and $h(x)$ are zero functions.

The main ideas of the proof of the fundamental theorem about the existence of a positive solution is based on the following observations.

Equation (1.6) is equivalent to the following equation

$$\frac{d}{dt}\left[x(t)-g(t, x(t-r))-\int_{t-\sigma}^{t-\hat{\sigma}} h(x(s)ds\right]=-f(t, x(t-\tau)) \tag{1.11}$$

and Equation. (1.8) has the folloving equivalent form

$$\frac{d}{dt}\left[u(t)-bu(t-r)-c\int_{t-\sigma}^{t-\hat{\sigma}} u(s)ds\right]=-au(t-\tau). \tag{1.12}$$

But one can see that in that forms of Equations (1.11) and (1.12) it is hard to compare their positive solutions because of the negative signs and of the neutral terms. Therefore we investigate the next more reasonable functional equation

$$x(t)=g(t, x(t-r))+\int_{t-\sigma}^{t-\hat{\sigma}} h(x(s))ds+\int_t^{\infty} f(s, x(s-\tau))ds \tag{1.13}$$

and functional inequality

$$u(t)\geqslant bu(t-r)+c\int_{t-\sigma}^{t-\hat{\sigma}} u(s)ds+\int_t^{\infty} au(s-\tau)ds. \tag{1.14}$$

Namely, for instance, if (1.14) has a positive solution on $[t_0-\gamma, \infty)$ and (1.7) holds then the functionals

$$a(t, x(\cdot)) = f(t, x(t-\tau)),\ b(t, x(\cdot)) = g(t, x(t-r)) + \int_{t-\sigma}^{t-\hat{\sigma}} h(x(s))ds$$

and

$$\alpha(t, u(\cdot)) = au(t-\tau),\ \beta(t, u(\cdot)) = bu(t-r) + c\int_{t-\sigma}^{t-\hat{\sigma}} u(s)ds$$

satisfy the inequalities

$$0 \leqslant a(t, x(\cdot)) \leqslant \alpha(t, u(\cdot)) \text{ and } 0 \leqslant b(t, x(\cdot)) \leqslant \beta(t, u(\cdot)),$$

for all $t \geqslant t_0$ and $x \in \{y \in C([t_0-\gamma, \infty), R): 0 \leqslant y(t) \leqslant u(t),\ t \geqslant t_0-\gamma\}$.

In that case the Shauder's fixed point theorem is applicable to prove that (1.13) has a positive solution on $[t_0-\gamma, \infty)$. But a positive solution of (1.13) is a positive solution of Equation (1.12) and of its equivalent form (1.8). Therefore in this way we could conclude statement (a) of Corollary 1.1.

§ 2. Two Fundamental Theorems of Linearized Oscillation

Consider the following Volterra-type neutral differential equation

$$\frac{d}{dt}[v(t) - b(t, v(\cdot))] = -a(t, v(\cdot)),\ t \geqslant t_0. \tag{2.1}$$

Let $-\infty < t_{-1} \leqslant t_0 < \infty$. We will need throughout this paper the following hypotheses:

(H_1) a, b: $[t_0, \infty) \times C_u \to R_+^n$ are Volterra-type functionals such that $a(t, v(\cdot))$ is continuous and $b(t, v(\cdot))$ satisfies the Charatheodory condition, where

$$C_u := \{y \in C([t_{-1}, \infty), R^n): 0 \leqslant y(t) \leqslant u(t),\ t \geqslant t_{-1}\}$$

and the function $u(t)$ is continuous, slightly positive and bounded on $[t_{-1}, \infty)$;

(H_2) there exist Volterra-type functionals α, β: $[t_0, \infty) \times C_u \to R_+^n$ such that for all $t \geqslant t_0$ and $x \in C_u$

$$a(t, v(\cdot)) \leqslant \alpha(t, u(\cdot)) \text{ and } b(t, v(\cdot)) \leqslant \beta(t, u(\cdot)), \tag{2.2}$$

moreover

$$\int_{t_0}^{\infty} \alpha(s, u(\cdot))ds < \infty \text{ and } \beta(t, u(\cdot)) \to 0, \text{ as } t \to +\infty; \tag{2.3}$$

(H_3) $u(t)$ satisfies the inequality

$$u(t) \geqslant \beta(t, u(\cdot)) + \int_t^{\infty} \alpha(s, u(\cdot))ds, \text{ on } [t_0, \infty), \tag{2.4}$$

and

$$u(t) \geqslant u(t_0),\ t_{-1} \leqslant t \leqslant t_0. \tag{2.5}$$

Now we state our fundamental theorem about the existence of a nonnegative solution of Equation (2.1).

Theorem 2.1. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied and* $\phi \in C([t_{-1}, t_0], R^n)$ *is a givenfunction such that*

$$0 \leqslant \phi(t) \leqslant u(t) - u(t_0),\ t_{-1} \leqslant t \leqslant t_0. \tag{2.6}$$

Then Equation (2.1) *has at least one solution* $v\in C([t_{-1},\infty), R^n)$ *such that*

$$0\leqslant v(t)\leqslant u(t),\ t_{-1}\leqslant t<\infty, \tag{2.7}$$

and

$$v(t)=\phi(t)+c,\ t_{-1}\leqslant t\leqslant t_0, \tag{2.8}$$

where $C\in R^n_+$ *is a suitable vector.*

Proof Denote by BC the Banach space of continuous bounded functions on $[t_{-1},\infty)$, with the norm $\|v\|=\sup\limits_{t_{-1}\leqslant t<\infty}|v(t)|$, $v\in BC$.

Using the fact, that $u(t)$ is a continuous and bounded function on $[t_{-1},\infty)$, it can easily be seen, that C_u is a bounded, closed and convex subset of BC.

On the other hand, by (2.2) we have

$$0\leqslant\int_{t_0}^{\infty}a(s,v(\cdot))ds\leqslant\int_{t_0}^{\infty}\alpha(s,u(\cdot))ds<\infty, \tag{2.9}$$

for all $v\in C_u$.

Define the operator T for any $v\in C_u$ by

$$(Tv)(t)=\begin{cases} b(t,v(\cdot))+\int_t^{\infty}a(s,v(\cdot))ds, & t\geqslant t_0,\\ \phi(t)+b(t_0,v(\cdot))+\int_{t_0}^{\infty}a(s,v(\cdot))ds, & t_{-1}\leqslant t\leqslant t_0.\end{cases} \tag{2.10}$$

The operator T is defined for every $v\in C_u$ because of (2.9) and $(Tv)(t)$ is a continuous function on $[t_{-1},\infty)$ because of $\phi(t_0)=0$.

Now we show that $T(C_u)\subset C_u$ and T is continuous and compact on C_u.

(i) $T(C_u)\subset C_u$.

Let $v\in C_u$ be arbitrarily fixed. Then by virtue of (2.2) and (2.6), from (2.10) it follows that

$$(Tv)(t)\leqslant\begin{cases} \beta(t,u(\cdot))+\int_t^{\infty}\alpha(s,u(\cdot))ds, & t\geqslant t_0,\\ u(t)-u(t_0)+\beta(t_0,u(\cdot))+\int_{t_0}^{\infty}\alpha(s,u(\cdot))ds, & t_{-1}\leqslant t\leqslant t_0.\end{cases}$$

But since $u(t)$ is a solution of (2.4), the last inequality yields

$$(Tv)(t)\leqslant u(t),\ t_{-1}\leqslant t<\infty,$$

that is, $T(C_u)\subset C_u$.

(ii) T is continuous.

Let $v_0\in C_u$ be arbitrary fixed, and let $v_n\in C_u$, $(n>1)$, be a sequence, such that $|v_n-v_0|\to 0$, as $n\to+\infty$. Note that the Volterra-type functionals $a(t,v(\cdot))$ and $b(t,v(\cdot))$ are such that $a(t,v(\cdot))$ satisfies the Charatheodory condition and $b(t,v(\cdot))$ is continuous.

Therefore

$$a(t,v_0(\cdot))=\lim_{n\to\infty}a(t,v_n(\cdot)),\ \text{a. e. on}\ [t_0,\infty).$$

Moreover by (2.9), it follows that

$$0 \leqslant \int_{t_0}^{\infty} a(s, v_n(\cdot))ds \leqslant \int_{t_0}^{\infty} \alpha(s, u(\cdot))ds \leqslant \infty, \ n > 1.$$

Thus, on the basis of Lebesque's integral theorem, we obtain

$$\lim_{n\to+\infty} \int_{t_0}^{\infty} a(s, v_n(\cdot))ds = \int_{t_0}^{\infty} a(s, v_0(\cdot))ds.$$

On the other hand since $b(t, v(\cdot))$ is a continuous functional, we have

$$\sup_{t\geqslant t_0} |b(t, v_n(\cdot)) - b(t, v_0(\cdot))| \to 0, \text{ as } n \to +\infty.$$

Therefore $|Tv_n - Tv_0| \to 0$, as $n \to +\infty$, which means that operator T is continuous.

(iii) T is a compact operator on C_u.

According to one of Levitan's theorems[18], if for any real number $\varepsilon > 0$ the interval $[t_{-1}, \infty)$ can be divided into a finite number of subintervals $\{I_k\}_{k=0}^{N}$, so that

$$\max_{t_1, t_2 \in I_k} |(Tv)(t_2) - (Tv)(t_1)| < \varepsilon,$$

for every $v \in C_u$ and for $0 \leqslant k \leqslant N$, then operator T is compact.

Let $\varepsilon > 0$ be an arbitrarily fixed real number. Then by (2.3) and (2.9), we obtain

$$\int_{s_1}^{\infty} a(s, v(\cdot))\, ds \leqslant \int_{s_1}^{\infty} \alpha(s, u(\cdot))ds < \varepsilon/2 \tag{2.11}$$

and

$$0 \leqslant b(t, v(\cdot)) \leqslant \beta(t, u(\cdot)) < \varepsilon/2, \ t \geqslant s_0, \tag{2.12}$$

for all $v \in C_u$ and for a fixed $s_0 > t_0$ large enough.

Thus (2.10) yields

$$\max_{t_1, t_2 \in I_0} |(Tv)(t_2) - (Tv)(t_1)| \leqslant \max_{t_1, t_2 \in I_0} \{|(Tv)(t_2)| + |(Tv)(t_1)|\} < \varepsilon$$

on the interval $I_0 = [s_0, \infty)$ for every function $v \in C_u$.

Furthermore for $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $t_1, t_2 \in [t_0, s_0]$ and $|t_1 - t_2| \leqslant \delta$ then

$$\left| \int_{t_1}^{t_2} a(s, v(\cdot))ds \right| < \left| \int_{t_1}^{t_2} \alpha(s, u(\cdot))ds \right| < \varepsilon/2, \ v \in C_u$$

and

$$\sup \{|b(t_2, v(\cdot)) - b(t_1, v(\cdot))| : v \in C_u\} < \varepsilon/2.$$

This yields that if $t_1, t_2 \in [t_0, s_0]$ and $|t_1 - t_2| < \delta$ then

$$|(Tv)(t_2) - (Tv)(t_1)| < \varepsilon, \text{ for every } v \in C_u.$$

This also means that the interval $[t_0, s_1]$ can be divided into subintervals $I_k (k = 1, 2, \cdots, N)$, the lengths of which are less than δ, and

$$\max_{t_1, t_2 \in I_k} |(Tv)(t_2) - (Tv)(t_1)| < \varepsilon$$

for every $v \in C_u$ and $1 \leqslant k \leqslant N$, where N is a natural number depending on ε.

Such a division of the interval $[t_{-1}, t_0]$ can obviously be found, because

$$(Tv)(t) = \phi(t) + c \quad t_{-1} \leqslant t \leqslant t_0,$$

where $\phi(t)$ is a continuous function such that $\phi(t_0)=0$ and

$$c=b(t_0, v(\cdot))+\int_{t_0}^{\infty} a(s, v(\cdot))ds\geqslant 0 \tag{2.13}$$

is a constants vector for every fixed $v\in C_u$.

Since for an arbitrarily fixed $\varepsilon>0$ we could divide the interval $[t_{-1}, \infty)$ into a finite number of subintervals according to Levitan's Theorem, it follows that the operator T is compact on C_u.

As $C_u\subset BC$ is bounded, closed and convex, T is continuous, compact and $T(C_u)\subset C_u$, so according to Shauder's fixed point theorem[4], there exists $v\in C_u$, for which $v=Tv$. This means at the same time that

$$0\leqslant v(t)\leqslant u(t),\ t\geqslant t_{-1}, \tag{2.14}$$

and

$$v(t)=\begin{cases} b(t, v(\cdot))+\int_{t_0}^{\infty} a(s, v(\cdot))ds, & t\geqslant t_0, \\ \phi(t)+c, & t_{-1}\leqslant t\leqslant t_0, \end{cases} \tag{2.15}$$

where $C\in R_+^n$ is defined by (2.13).

If both sides of (2.15) are differentiated on the interval $[t_0, \infty)$, then we see that $v(t)$ is a solution of (2.1) on $[t_{-1}, \infty)$ with initial condition (2.8).

The proof of the theorem is complete.

In the next corollary we give a condition for the existence of a slightly positive solution.

Corollary 2.1. *Assume that the assumptions of Theorem 2.1 are satisfied. Furthermore assume that:*

(H_4) *there is an index $i_0\in\{1, \cdots, n\}$ such that $u_{i_0}(t)>u_{i_0}(t_0)>0$, $(t_{-1}\leqslant t<t_0)$, and for all $(t_1, x)\in[t_0, \infty)\times C_u$, the inequality $x_{i_0}(t)>0$, $(t_{-1}\leqslant t<t_1)$, implies*

$$b_{i_0}(t_1, x(\cdot))+\int_{t_0}^{\infty} a_{i_0}(s, x(\cdot))ds>0, \tag{2.16}$$

where u_{i_0}, x_{i_0}, a_{i_0} and b_{i_0} denote the i_0th components of u, x, a and b, respectively.

Then Equation (2.1) hasa slightly positive solution on $[t_{-1}, \infty)$.

Proof By Theorem 2.1, we have that for the initial function

$$\phi(t)=u(t)-u(t_0),$$

Equation (2.1) has at least one solution $v(t)$ on $[t_{-1}, \infty)$ such that (2.8) holds where $C\in R_+^n$ is defined by (2.13).

Now we show that $c_{i_0}>0$. Indeed from (2.13) and (2.16) we have

$$c_{i_0}=b_{i_0}(t_0, v(\cdot))+\int_{t_0}^{\infty} a_{i_0}(s, v(\cdot))ds>0.$$

Thus $v_{i_0}(t)=\phi_{i_0}(t)+c_{i_0}>0$, $t_{-1}\leqslant t\leqslant t_0$, and by using the same argument as above one can see that $v_{i_0}(t)>0$ for $t\geqslant t_{-1}$. The proof of the corollary is complete.

A similar result can be proved for the existence of a positive solution with minor changes in the proof of Corollary 2.1.

Corollary 2. 2. *Assume that the assumptions of Theorem 2.1 are satisfied. Furthermore assume that:*

(H_5) $u(t)>u(t_0)>0, (t_{-1}\leqslant t<t_0)$, *and for all* $(t_1, x)\in[t_0, \infty)\times C_u$, *the inequality* $x(t)>0$, $(t_{-1}\leqslant t<t_1)$, *implies that*

$$b(t_1, x(\cdot))+\int_{t_0}^{\infty} a(s, x(\cdot))ds>0. \tag{2.17}$$

Then Equation (2.1) *has a positive solution on* $[t_{-1}, \infty)$.

Now we will prove our second fundamental theorem which is very useful in the subsequent discussion of the qualitative properties of nonoscillatory solutions.

Theorem 2. 2. *Assume that* a, b: $[t_0, \infty)\times C([t_{-1}, \infty), D^n)\to R^n(D^n\subset R^n)$ *are Volterra-type functionals, and that there exist constants* $r>0$ *and* $T_0\geqslant t_1+r$, *and a matrix* $Q\in R_+^{n\times n}$ *such that for all* $y\in C([t_{-1}, \infty), D^n), y(t)\geqslant 0$, $(t\geqslant T_0-r)$ *we have*

$$a(t, y(\cdot))\geqslant 0 \text{ and } 0\leqslant b(t, y(\cdot))\leqslant A\max_{t-r\leqslant s\leqslant t} y(s), \ (t\geqslant T_0) \tag{2.18}$$

(where max is meant componentwise).

Assume further that the spectral radius of the matrix A satisfies the condition

$$\rho(A)<1. \tag{2.19}$$

Then

(a) *for every function* $x\in C([t_{-1}, \infty), D^n)$ *such that*

$$\frac{d}{dt}[x(t)-b(t, x(\cdot))]\leqslant -a(t, x(\cdot)), \ t\geqslant t_0, \tag{2.20}$$

and $x(t)\geqslant 0$, $(t\geqslant T_0-r)$, *one has that* $x(t)$ *is bounded on* $[t_{-1}, \infty)$, *moreover*

$$m_0=\lim_{t\to+\infty}(x(t)-b(t, x(\cdot)))\in R_+^n \text{ and } \int_{T_0}^{\infty} a(s, x(\cdot))ds<\infty; \tag{2.21}$$

(b) *if we also assume that for all eventually slightly positive function*

$$y\in C([t_{-1}, \infty), R^n),$$

$\liminf_{t\to+\infty}|y(t)|>0$ *implies that*

$$\lim_{t\to+\infty}\left|\int_{t_0}^{t} a(s, y(\cdot))ds\right|=+\infty,$$

then

$$\lim_{t\to+\infty} x(t)=\lim_{t\to+\infty} b(t, x(\cdot))=0.$$

Remark 2. 1. For the condition $\rho(A)<1$ of Theorem 2.2 to hold it is sufficient that $|A|<1$ where $|\cdot|$ may denotes any matrix norm. In the scalar case, i. e. when $n=1$, $\rho(A)$ is equal to the absolute value of the scalar value A.

To prove the Theorem 2.2 we need the next lemma which is a slight modified version of Lemma 2 in [10] with the same proof.

Lemma 2. 1. *Let* $A\in R_+^{n\times n}$ *be a nonnegative matrix,* $r>0$ *a real number,* $\alpha\in R^n$ *a nonnegative vector, and* x: $[T_0-r, \infty)\to R_+^n$ *a negative continuous function, and suppose that the vectorial inequality*

$$x(t)\leqslant d+A\max_{t-r\leqslant s\leqslant t} x(s), \ t\geqslant T_0 \tag{2.24}$$

holds (where $\leqslant$ and max are meant componentwise).

Assume that the spectral radius of the matrix A satisfies the condition (2.19). *Then there exist positive constants p, q and R such that*

$$|x(t)| \leqslant \max_{T_0-r\leqslant s\leqslant T_0+R} |x(s)| \exp(-p(t-R)) + q|d|, \quad t \geqslant T_0+R. \tag{2.25}$$

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2 (a) Since $x(t) \geqslant 0, (t \geqslant T_0 - r)$, and it satisfies (2.20), we have that $x(t) - b(t, x(\cdot))$ is a monotone nonincreasing function on $[T_0, \infty)$, because of (2.18). Therefore there exists a vector $d \in R^n_+$ such that

$$x(t) \leqslant d + b(t, x(\cdot)), \quad t \geqslant T_0,$$

that is by virtue of (2.18), we obtain that $x(t)$ satisfies the inequality (2.24). Thus by Lemma 2.1, we obtain that $x(t)$ is bounded on $[t_{-1}, \infty)$. But using the same arguments, one can see that the limit

$$m = \lim_{t\to+\infty} (x(t) - b(t, x(\cdot)))$$

is a nonnegative vector. Therefore, from (2.20) by integration, it follows that

$$m - (x(T_0) - b(T_0, x(\cdot))) \leqslant -\lim_{t\to+\infty} \int_{T_0}^{t} a(s, x(\cdot))ds,$$

which means that (2.21) holds. The proof of statement (a) is complete.

(b) First we show that $m=0$. Otherwise there is an index $i \in \{1, \cdots, n\}$ such that $m_i > 0$. Moreover there is a $T_i > T_0$ such that

$$x_i(t) - b_i(t, x(\cdot)) > m_i/2, \quad t \geqslant T_i.$$

But $b_i(t, x(\cdot)) \geqslant 0 \quad (t \geqslant T_i)$, and consequently $x_i(t) > m_i/2$ and $|x(t)| > m_i/2$ for all $t \geqslant T_i$. Therefore by virtue of (2.22), we obtain

$$\left| \int_{T_0}^{\infty} a(s, x(\cdot))\, ds \right| = +\infty,$$

which contradicts (2.21). Thus $m=0$.

Since $m=0$, we have a function $f: [t_{-1}, \infty) \to R^n$ such that $\lim_{t\to+\infty} f(t) = 0$ and because of (2.18)

$$0 < x(t) = b(t, x(\cdot)) + f(t) \leqslant A \max_{t-r<s<t} x(s) + f(t), \quad t \geqslant T_0. \tag{2.26}$$

On the other hand $M = \limsup_{t\to+\infty} x(t)$ is a nonnegative vector and (2.26) yields $0 < M < AM$. Since $\rho(A) < 1$, we obtain $M=0$ and the proof of the theorem is complete.

§ 3. Linear Systems of Differential Inequalities and Equations

In this section we give some equivalent statements about the existence of the slightly positive as well as slightly positive and non-increasing solutions of the inequalities

$$\frac{d}{dt}\left[x(t)-\sum_{m=1}^{M}C_m x(t-r_m)\right]\leqslant\sum_{j=1}^{J}B_j[x(t-\hat{\sigma}_j)-x(t-\sigma_j)]-\sum_{k=1}^{K}A_k x(t-\tau_k), \tag{3.1}$$

and

$$x(t)\geqslant\sum_{m=1}^{M}C_m x(t-r_m)+\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}x(s)ds+\int_t^{\infty}\sum_{k=1}^{K}A_k x(s-\tau_k)ds \tag{3.2}$$

as well as the equation

$$\frac{d}{dt}\left[v(t)-\sum_{m=1}^{M}C_m v(t-r_m)\right]=\sum_{j=1}^{J}B_j[v-\hat{\sigma}_j)-v(t-\sigma_j)]-\sum_{k=1}^{K}A_k v(t-\tau_k), \tag{3.3}$$

where we assume that

(i) $\tau_k\geqslant 0$ $(1\leqslant k\leqslant K)$, $\sigma_j\geqslant\hat{\sigma}_j\geqslant 0$ $(1\leqslant j\leqslant J)$, and $r_m\geqslant 0$ $(1\leqslant m\leqslant M)$ are given constants, $\tau=\max\limits_{1<k<K}\tau_k$, $\sigma=\max\{\max\limits_{1\leqslant j\leqslant J}\hat{\sigma}_j,\ \max\limits_{1\leqslant m\leqslant M}r_m\}$ and $t_{-1}=t_0-\max\{\tau,\sigma\}$;

(ii) A_k $(1\leqslant k\leqslant K)$, B_j $(1\leqslant j\leqslant J)$, and C_m $(1\leqslant m\leqslant M)$, are nonnegative n by n matrices such that

$$\left(\sum_{k=1}^{K}A_k\right)e\neq 0,\ \text{if } e\in R_+^n \text{ and } e\neq 0, \tag{3.4}$$

and the spectral radius of the matrix $\sum\limits_{m=1}^{M}C_m+\sum\limits_{j=1}^{J}B_j(\sigma_j-\hat{\sigma}_j)$ satisfies the condition

$$\rho\left(\sum_{m=1}^{M}C_m+\sum_{j=1}^{J}B_j(\sigma_j-\hat{\sigma}_j)\right)<1.$$

At the end of this section we will also consider the characteristic equation of (3.2):

$$\det\left(\lambda\left(I-\sum_{m=1}^{M}C_m e^{-\lambda r_m}\right)-\sum_{j=1}^{J}B_j(e^{-\lambda\hat{\sigma}_j}-e^{-\lambda\sigma_j})+\sum_{k=1}^{K}A_k e^{-\lambda\tau_k}\right)=0$$

under the conditions (i) and (ii).

First we prove the following useful lemma.

Lemma 3.1. *Assume that conditions* (i) *and* (ii) *are satisfied. If a continuous function* x: $[t_{-1},\infty)\to R^n$ *satisfies* (3.1) *on* $[t_0,\infty)$ *and it is eventually slightly positive then* $x(t)$ *is bounded on* $[t_{-1},\infty)$ *and there is a* $T_0\geqslant t_0$ *such that* $x(t)$ *satisfies* (3.2) *for all* $t\geqslant T_0$.

Proof Since $x(t)$ satisfies (3.1), we obtain that $x(t)$ is a solution of the inequality

$$\frac{d}{dt}\left[x(t)-\sum_{m=1}^{M}C_m x(t-r_m)-\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}x(s)ds\right]\leqslant-\sum_{k=1}^{K}A_k x(t-\tau_k), \tag{3.7}$$

for all $t\geqslant t_0$.

Set

$$a(t,x(\cdot))=\sum_{k=1}^{K}A_k x(t-\tau_k)\ \text{and}\ b(t,x(\cdot))=\sum_{m=1}^{M}C_m x(t-r_m)+\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}x(s)ds$$

for all $(t,x)\in[t_0,\infty)\times C([t_{-1},\infty),R^n)$. Then the assumption of Theorem 2.2 about $a(t,x(\cdot))$ and $b(t,x(\cdot))$ are satisfied and the inequality (2.20) reduces to the inequality (2.30). Therefore by Theorem 2.2, we obtain

$$m_0=\lim_{t\to+\infty}[x(t)-b(t,\ x(\cdot))]\geqslant 0 \text{ and } \int_t^\infty \sum_{k=1}^K A_k x(s-\tau_k)ds<\infty.$$

Thus by integrating (3.7) from t to $+\infty$, we have

$$x(t)\geqslant m_0+\sum_{m=1}^M C_m x(t-r_m)+\sum_{j=1}^J B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j} x(s)ds+\int_t^\infty \sum_{k=1}^K A_k x(s-\tau_k)ds.$$

Therefore (3.7) is satisfied and the proof of the lemma is complete.

Now we prove the main theorem of this section.

Theorem 3. 1. *Assume that conditions* (i) *and* (ii) *are satisfied. Then the following statements are equivalent*:

(a_1) *inequality* (3.1) *has a slightly positive solution on an interval* $[t_1, \infty)$;

(a_2) *inequality* (3.1) *has a slightly positive and non-increasing solution on an interval* $[t_2, \infty)$:

(a_3) *inequality* (3.2) *has a slightly positive solution in an interval* $[t_3,\infty)$;

(a_4) *equation* (3.3) *has a slightly positive solution on an interval* $[t_4, \infty)$;

(a_5) *equation* (3.3) *has a slightly positive and non-increasing solution on an interval* $[t_5, \infty)$.

Proof By virtue of Lemma 3.1 one can see that if one of the statements (a_1), (a_2), (a_4) and (a_5) holds then (a_3) is also satisfied.

Now assume that (a_3) is satisfied, that is, there exists a slightly positive solution $x(t)$ of (3.2) on an interval $[t_3-\gamma, \infty)$, where $t_3\in R$ and $\gamma=\max\{\tau, \sigma\}$ is defined by condition (i).

Let $u(t)$ be defined by

$$u(t)=\int_t^\infty x(s)ds,\quad t\geqslant t_3-\gamma. \tag{3.8}$$

Then $u(t)$ is a continuous and slightly positive function on $[t_3-\gamma, \infty)$ and $\dot{u}(t)=-x(t)\leqslant 0$, $t\geqslant t_3$.

On the other hand, from (3.2), we obtain

$$\frac{d}{dt}\Big[u(t)-\sum_{m=1}^M C_m u(t-r_m)\Big]\leqslant\sum_{j=1}^J B_j[u(t-\hat{\sigma}_j)-u(t-\sigma_j)]-\sum_{k=1}^K A_k u(t-\tau_k), \tag{3.9}$$

for all $t\geqslant t_3$.

This means that $u(t)$ is a slightly positive and non-increasing solution of (2.27) on $[t_3-\gamma, \infty)$. This means that if (a_3) is satisfied then (a_1) and (a_2) hold, too. By using the same argument as above, one can see that if $x(t)$ is a slightly positive solution of (3.3) on an interval $[t_4, \infty)$ then $u(t)$ defined in (3.8) is a slightly positive and non-increasing solution of (3.8). Therefore (a_4) implies (a_5).

It remains only to show that (a_3) implies (a_4). If (a_3) holds, that is, (3.3) has a slightly positive solution on $[t_3, \infty)$ which is denoted by $x(t)$, then $u(t)$ by (3.8) satisfies inequality (3.9) on $[t_3, \infty)$. But in that case by virtue of Lemma 3.1 we have

$$u(t)\geq\sum_{m=1}^{M}C_m u(t-r_m)+\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}u(s)ds+\int_t^{\infty}\sum_{k=1}^{K}A_k u(s-\tau_k)ds,\ t\geq t_3,\tag{3.10}$$

moreover

$$u(t)\geq u(t_3),\ t_3-\gamma\leq t\leq t_3,\tag{3.11}$$

and there is an index $i_0\in\{1, 2, \cdots n\}$

$$u_{i_0}(t)>u_{i_0}(t_3),\ t_3-\gamma\leq t<t_3.\tag{3.12}$$

Set

$$a(t, v(\cdot))=\sum_{k=1}^{K}A_k v(t-\tau_k)\quad\text{and}\quad b(t, v(\cdot))=\sum_{m=1}^{M}C_m v(t-r_m)+\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}v(s)ds,$$

for all $(t, v)\in[t_0, \infty)\times C([t_3-\gamma, \infty), R^n)$ and

$$\alpha(t,u(\cdot))=\sum_{k=1}^{K}A_k u(t-\tau_k)\ \text{and}\ \beta(t, u(\cdot))=\sum_{m=1}^{M}C_m u(t-r_m)+\sum_{j=1}^{J}B_j\int_{t-\sigma_j}^{t-\hat{\sigma}_j}u(s)ds,\ t\geq t_0.$$

Then from conditions (i) and (ii), it follows that the assumptions about $a(t, v(\cdot))$ and $b(t, v(\cdot))$ as well as those about $\alpha(t, u(\cdot))$ and $\beta(t, u(\cdot))$ are satisfied in Theorem 2.1 and in Corollary 2.1. Thus by virtue of Corollary 2.1 we have that Equation (3.3) has a slightly positive solution on $[t_3-\gamma, \infty)$. The proof of the theorem is complete.

The oscillations in the linear delay differential equations and systems have been the object of intensive analysis in numerus papers and notably in [1, 2, 3, 5, 7, 16, 17, 21].

The results in this direction can be classified into two groups: results which are proved without the characteristic equation and results which are proved through the characteristic equation. The following fundamental theorem about the characteristic equation was proved by Arino and Győri in [2]:

A system of linear homogeneous neutral differential equations has a nonoscillatory solution if and only if its characteristic equation has a real root.

By using this general result we prove two usefull statements.

Proposition 3. 1. *Consider the scalar equation*

$$\frac{d}{dt}\Big[v(t)-\sum_{m=1}^{M}C_m v(t-r_m)\Big]=\sum_{j=1}^{J}b_j[v(t-\hat{\sigma}_j)-v(t-\sigma_j)]-\sum_{k=1}^{K}a_k v(t-\tau_k),\tag{3.13}$$

where the constants τ_k, σ_j, $\hat{\sigma}_j$ and r_m satisfy condition (i), *moreover $a_k>0$ $(1\leq k\leq K)$, $b_j\geq 0$ $(1\leq j\leq J)$ and $c_m\geq 0$ $(1\leq m\leq M)$ are such that*

$$\sum_{k=1}^{K}a_k>0,\tag{3.14}$$

and

$$\sum_{m=1}^{M}C_m+\sum_{j=1}^{J}b_j(\sigma_j-\hat{\sigma}_j)<1.\tag{3.15}$$

Then Equation (3.13) *has a positive solution on some interval* $[t_0, \infty)$ ***if*** *and only* ***if*** *the characteristic equation*

$$\lambda\Big(1-\sum_{m=1}^{M} C_m e^{-\lambda r_m}\Big)-\sum_{j=1}^{J} b_j\,(e^{-\lambda\hat{\sigma}_j}-e^{-\lambda\sigma_j})+\sum_{k=1}^{K} a_k e^{-\lambda\tau_k}=0 \tag{3.16}$$

has a negative root.

Proof From the above mentioned general theorem it follows that (3.16) has a real root if (3.13) has a positive solution on an interval $[t_0, \infty)$. But under our conditions it is easily seen that every real solution of (3.16) is negative.

Now we assume that Equation (3.16) has a negative root, say λ_0. Then the function

$$v(t)=e^{-\lambda_0 t}$$

is a positive solution of (3.13). The proof of the proposition is complete.

Corollary 3.1. *Assume that the conditions of Proposition 3.1 are satisfied and for all $\varepsilon>0$ small enaugh there are $T_\varepsilon\in R$ and a continuous function $v_\varepsilon:[T_\varepsilon-r, \infty)\to(0,\infty)$ such that*

$$\frac{d}{dt}\Big[v_\varepsilon(t)-(1-\varepsilon)\sum_{m=1}^{M} C_m v_\varepsilon(t-r_m)\Big]$$
$$=(1-\varepsilon)\sum_{j=1}^{J} b_j[v_\varepsilon(t-\hat{\sigma}_j)-v_\varepsilon(t-\sigma_j)]-(1-\varepsilon)\sum_{k=1}^{K} a_k v_\varepsilon(t-\tau),\ t\geqslant T_\varepsilon. \tag{3.17}$$

Then Equation (3.13) has a positive solution or equivalently the character istic equation (3.16) has a negative root.

Proof Since for any $\varepsilon>0$ small enough, (3.17) has a positive solution on an interval $[T_\varepsilon, \infty)$, by virtue of Proposition 3.1, we have that the equation

$$\lambda(1-(1-\varepsilon)\sum_{m=1}^{M} C_m e^{-\lambda r_m})-(1-\varepsilon)\sum_{j=1}^{J} b_j(e^{-\lambda\hat{\sigma}_j}-e^{-\lambda\sigma_j})+(1-\varepsilon)\sum_{k=1}^{K} a_k e^{-\lambda\tau_k}=0$$

has a negative root. Then as $\varepsilon\to 0$, we obtain that Equation (3.16) has a nonpositive root. But $\lambda=0$ is not root of (3.16). Therefore (3.16) has a negative root. The proof of the corollary is complete.

Proposition 3.2. *Consider the n-dimensional delay equation*

$$\dot{v}(t)=\sum_{j=1}^{J} B_j[v(t-\hat{\sigma}_j)-v(t-\sigma_j)]-\sum_{k=1}^{K} A_k v(t-\tau_k), \tag{3.18}$$

where $\sigma_j>\hat{\sigma}_j>0$ $(1\leqslant j\leqslant J)$ and $\tau_k\geqslant 0$ $(1\leqslant k\leqslant K)$ are constants,

$$\gamma=\max\{\max_{1\leqslant j\leqslant J}\sigma_j,\ \max_{1\leqslant k\leqslant K}\tau_k\},$$

and $B_j\in R_+^{n\times n}$ $(1\leqslant j\leqslant J)$ and $A_k\in R_+^{n\times n}$ $(1\leqslant k\leqslant K)$ are given matrices such that

$$\Big(\sum_{k=1}^{K} A_k\Big)e\neq 0 \text{ if } e\in R_+^n \text{ and } e\neq 0, \tag{3.19}$$

and

$$\rho\Big(\sum_{j=1}^{J} B_j(\sigma_j-\hat{\sigma}_j)\Big)<1. \tag{3.20}$$

Then Equation (3.18) has a slightly positive solution on some interval $[t_0, \infty)$ if and only if the characteristic equation

$$\det\Big(\lambda I-\sum_{j=1}^{J} B_j(e^{-\lambda\hat{\sigma}_j}-e^{-\lambda\sigma_j})+\sum_{k=1}^{K} A_k e^{-\lambda\tau_k}=0 \tag{3.21}$$

has a negative root λ_0 *and the eigenvalue-equation*

$$\left(\lambda_0 I-\sum_{j=1}^{J}B_j(e^{-\lambda_0\hat{\sigma}_j}-e^{-\lambda_0\sigma_j})+\sum_{k=1}^{K}A_k e^{-\lambda_0\tau_k}\right)e=0 \tag{3.22}$$

has an eigenvector solution e_0 *such that*

$$e_0>0 \text{ and } |e_0|=1. \tag{3.23}$$

Proof If λ_0 is a negative root of (3.21) and e_0 is a solution of (3.22) such that (3.23) holds, then it is clear that $x(t)=e_0e^{-\lambda_0 t}$ is a slightly positive and non-increasing solution of (3.18).

Now assume that equation (3.21) has no realroot. Then from [12] (see also [19]) we know that the solutions of (3.18) can be written in the form

$$e^{-at}\sum_{j=1}^{N}p_j(t)\cos(\alpha_j t+\beta_j)+o(t^k e^{-at}) \tag{3.24}$$

in which $a+i\alpha_j$ is a root of the characteristic equation, $p_j(t)$ are some polynomials and k denotes the greatest power of the polynomials $p_j(t)$. Since $\sum_{j=1}^{N}c_j\cos(\alpha_j t+\beta_j)$ has a zero mean value, it is oscillatory function if $\sum_{j=1}^{N}|\alpha_j|>0$ and $c_j\in R^n$ $(1\leqslant j\leqslant N)$ are such that $\sum_{j=1}^{N}|c_j|>0$. But in that case $\sum_{j=1}^{N}c_j\cos(\alpha_j t+\beta_j)$ is almost periodic, therefore the function defined in (3.24) is oscillatory.

Now assume that (3.18) has a slightly positive solution. Then by Theorem 3.1 we have that (3.18) has a solution $x(t)$ such that $x(t)$ is slightly positive and non-increasing on some interval $[T_0, \infty)$. But in that case Equation (3.21) has a real root λ_0 such that

$$x(t)=e^{\lambda_0 t}p(t)+o(t^k e^{\lambda_0}), \text{ as } t\to+\infty, \tag{3.25}$$

where $p(t)=\sum_{j=0}^{K}e_{k-j}t^j$ is a polynomial of degree k.

Since $x(t)$ is a slightly positive and non-increasing function, (3.25) yields that $\lambda_0<0$ and e_0 is a nonnegative vector such that $|e_0|>0$. But from [12] we know that $e_0e^{\lambda_0 t}$ is a solution of (3.18), which means that e_0 is a solution of (3.22).

Now we show that $\lambda_0<0$. Otherwise $\lambda_0=0$ and $x(t)=e_0$ is a solution of (3.18). But this means that $\sum_{k=1}^{K}A_k e_0=0$, that is $e_0=0$, which is a contradiction. The proof of Proposition 3.2 is complete.

Corollary 3. 2. *Assume that the conditions of Proposition* 3.2 *are satisfied and for all* $\varepsilon>0$ *small enough there are* $T_\varepsilon\in R$ *and a continuous and slightly positive function* $v_\varepsilon: [T_\varepsilon-\gamma, \infty)\to R_+^n$ *such that*

$$\dot{v}_\varepsilon(t)=(1-\varepsilon)\sum_{j=1}^{J}B_j[v_\varepsilon(t-\hat{\sigma}_j)-v_\varepsilon(t-\sigma_j)]-(1-\varepsilon)\sum_{k=1}^{K}A_k v(t-\tau_k),\ t\geqslant T_\varepsilon.$$

Then Equation (3.18) *has a slightly positive solution on an interval* $[T_0, \infty)$.

Proof If there exists a slightly positive and continuous function $v_\varepsilon: [T_\varepsilon-$

$\gamma, \infty) \to R_+^n$ which satisfies (3.25), then by virtue of Proposition 3.2 we have that there are a $\lambda_\varepsilon < 0$ and an $e_\varepsilon \in R_+^n$ such that

$$\det\left(\lambda_\varepsilon I - (1-\varepsilon)\sum_{j=1}^{J} B_j(e^{-\lambda_\varepsilon \hat{\sigma}_j} - e^{-\lambda_\varepsilon \sigma_j}) + (1-\varepsilon)\sum_{k=1}^{K} A_k e^{-\lambda_\varepsilon \tau_k}\right) = 0,$$

and

$$\left(\lambda_\varepsilon I - (1-\varepsilon)\sum_{j=1}^{J} B_j(e^{-\lambda_\varepsilon \hat{\sigma}_j} - e^{-\lambda_\varepsilon \sigma_j}) + (1-\varepsilon)\sum_{k=1}^{K} A_k e^{-\lambda_\varepsilon \tau_k}\right) e_\varepsilon = 0,$$

moreover $|e_\varepsilon| = 1$.

But in that case $\lambda_0 = \lim_{\varepsilon \to 0} \lambda_\varepsilon$ exists and it is a nonpositive solution of (3.21). On the toher hand, since $e_\varepsilon > 0$ and $|e_\varepsilon| = 1$, there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$ and $e_0 = \lim_{n\to+\infty} e_{\varepsilon_n}$ exists. But one can see that e_0 satisfies (3.22) and (3.23). Therefore Proposition 3.2 completes the proof of the corollary.

§ 4. Necessary and Sufficient Conditions for Oscillations in Asymptotically Linear Equations

In this section we give some conditions for the oscillations in non-linear scalar and vector equations via some corresponding linear equations and their characteristic equations.

At first consider the scalar differential equation

$$\frac{d}{dt}[x(t) - h(t, x(t-r_1), \cdots, x(t-r_M)]$$
$$= \sum_{j=1}^{J}[g_j(x(t-\hat{\sigma}_j) - g_j(x(t-\sigma_j))] - f(t, x(t-\tau_1, \cdots, x(t-\tau_K)), \qquad (4.1)$$

where we state the following hypotheses which would be used as indicated in each result:

(A_1) $\tau_k \geq 0$ $(1 \leq k \leq K)$, $\sigma_j \geq \hat{\sigma}_j \geq 0$ $(1 \leq j \leq J)$, and $r_m \geq 0$ $(1 \leq m \leq M)$ are given constants, $\tau = \max_{1<k<K} \tau_k$, $\sigma = \max\{\max_{1<j<J} \sigma_j, \max_{1<m<M} r_m\}$ and $t_{-1} = t_0 - \max\{\tau, \sigma\}$;

(A_2) $f: [t_0, \infty) \times R^k \to R$, $g_j: R \to R$ $(1 \leq j \leq J)$, and $h: [t_0, \infty) \times R^M \to R$ are continuous functions such that

$f(t, x) \geq 0$ $((t, x) \subset [t_0, \infty) \times R_+^K)$, $g_j(y) \geq 0$ $(1 \leq j \leq J, y \in R_+)$ and $h(t, z) \geq 0$ $((t, z) \in [t_0, \infty) \times R_+^n)$, moreover

$$\int_{t_0}^{\infty} \min\{f(t, x_1, \cdots, x_n): x_k \geq \delta, 1 \leq k \leq K\} dt = +\infty, \qquad (4.2)$$

for all $\delta > 0$;

(A_3) there are constants $a_k \geq 0$ $(1 \leq k \geq K)$, $b_j \geq 0$ $(1 \leq j \leq J)$, $c_m \geq 0$ $(1 \leq m \leq M)$, and $\delta_0 > 0$ such that

$$f(t, x_1, \cdots, x_K) \leq \sum_{k=1}^{K} a_k x_k, \; t \geq t_0, \; x_k \in [0, \delta_0), \; 1 \leq k \leq K, \qquad (4.3)$$

and

$$g_j(y) \leqslant b_j \cdot y,\ y \in [0, \delta_0),\ 1 \leqslant j \leqslant J, \tag{4.4}$$

and

$$h(t, z_1, \cdots, z_M) \leqslant \sum_{m=1}^{M} c_m z_m,\ z_m \in [0, \delta_0),\ 1 \leqslant m \leqslant M, \tag{4.5}$$

moreover

$$\sum_{k=1}^{K} a_k > 0 \text{ and } \sum_{m=1}^{M} c_m + \sum_{j=1}^{J} b_j(\sigma_j - \hat{\sigma}_j) < 1; \tag{4.6}$$

(A_4) for all $\varepsilon > 0$ small enough there exists a $\delta_\varepsilon > 0$ such that

$$f(t, x_1, \cdots, x_k) \geqslant (1-\varepsilon) \sum_{k=1}^{K} a_k x_k,\ t \geqslant t_0,\ x_k \in [0, \delta_\varepsilon) 1 \leqslant k \leqslant K, \tag{4.7}$$

and

$$g_j(y) \geqslant (1-\varepsilon) b_j y,\ y \in [0, \delta_\varepsilon),\ 1 \leqslant j \leqslant J,$$

and

$$h(t, z_1, \cdots, z_M) \geqslant (1-\varepsilon) \sum_{m=1}^{M} c_m z_m,\ z_m \in [0, \delta),\ 1 \leqslant m \leqslant M,$$

where the constants a_k, b_j and c_m satisfy (4.6).

In the next theorem we give a sufficient condition for the existence of a positive solution.

Theorem 4.1. *Assume that assumptions* (A_1), (A_2) *and* (A_3) *are satisfied and that the linear equation*

$$\frac{d}{dt}\left[x(t) - \sum_{m=1}^{M} c_m x(t-r_m)\right] = \sum_{j=1}^{J} b_j [x(t-\hat{\sigma}_j) - x(t-\sigma_j)] - \sum_{k=1}^{K} a_k x(t-\tau_k) \tag{4.10}$$

has a positive solution on an interval $[T_0, \infty)$. *Then Equation* (4.1) *has a positive solution on* $[t_0, \infty)$.

Proof Since Equation (4.10) is autonomous, one can see that if Equation (4.10) has a positive solution on an interval $[T_0, \infty)$ then it has a positive solution on $[t_{-1}, \infty)$, too. Therefore, by Theorem 3.1 we obtain that there exists a positive function $u_0 \in C([t_{-1}, \infty), R_+)$ such that

$$u_0(t) > u_0(t_0)\ (t_{-1} \leqslant t < t_0), \text{ and } \lim_{t \to +\infty} u_0(t) = 0, \tag{4.11}$$

and

$$u_0(t) \geqslant \beta(t, u_0(\cdot)) + \int_t^{\infty} \alpha(s, u_0(\cdot)) ds,\ t \geqslant t_0, \tag{4.12}$$

where

$$\alpha(t, u_0(\cdot)) = \sum_{k=1}^{K} a_k u_0(t-\tau_k),\ t \geqslant t_0, \tag{4.13}$$

and

$$\beta(t, u_0(\cdot)) = \sum_{m=1}^{M} c_m u_0(t-r_m) + \sum_{j=1}^{J} b_j \int_{t-\sigma_j}^{t-\hat{\sigma}_j} u_0(s) ds,\ t \geqslant t_0. \tag{4.14}$$

Let c be a positive constant such that the function $u(t)$ defined by $u(t) = cu_0(t)$ $(-\gamma \leqslant t < \infty)$ satisfies the inequality $0 < u(t) \leqslant \delta_0 (-\gamma \leqslant t < \infty)$. In that case (4.11) and (4.12) yield

$$u(t)>u(t_0)\ (t_{-1}\leqslant t<t_0),\ \text{and}\ \lim_{t\to+\infty}u(t)=0,\tag{4.15}$$

and

$$u(t)\geqslant\beta(t,\ u(\cdot))+\int_t^\infty\alpha(s,\ u(\cdot))ds,\ t\geqslant t_0.\tag{4.16}$$

Now define the functionals $a(t, x(\cdot))$ and $b(t, x(\cdot))$ by

$$a(t,\ x(\cdot))=f(t,\ x(t-\tau_1),\ \cdots,\ x(t-\tau_k)),\tag{4.17}$$

and

$$b(t,\ x(\cdot))=h(t,\ x(t-r_1),\ \cdots,\ x(t-r_M))+\sum_{j=1}^{J}\int_{t-\sigma_j}^{t-\hat{\sigma}_j}g_j(x(s))ds,\tag{4.18}$$

for all $(t, x)\in[t_0, \infty)\times C_u$, where C_u is defined in our hypothesis (H_1) in section 2.

Then in Corollary 2.1 the assumptions about $a(t, x(\cdot))$, $b(t, x(\cdot))$, $\alpha(t, u(\cdot))$ and $\beta(t, u(\cdot))$ are satisfied. Therefore by this corollary, it follows that Equation (4.1) has at least one positive solution on $[t_{-1}, \infty)$. The proof of the theorem is complete.

Now we give necessary condition for the existence of a positive solution of (4.1).

Theorem 4.2. *Assume that conditions* (A_1), (A_2) *and* (A_4) *are satisfied and Equation* (4.1). *has an eventually positive solution on* $[t_{-1}, \infty)$. *Then Equation* (4.10) *has a positive solution.*

Proof Assume that x_0: $[t_{-1}, \infty)\to R$ is an eventually positive solution of (4.1). Then $x_0(t)$ is a solution of the equation

$$\frac{d}{dt}[x(t)-b(t,\ x(\cdot))]=-a(t,\ x(\cdot)),\tag{4.19}$$

where $a(t, x(\cdot))$ and $b(t, x(\cdot))$ are defined in (4.17) and (4.18), respectively.

But under our hypotheses $a(t, x(\cdot))$ and $b(t, x(\cdot))$ satisfy the conditions in Theorem 2.2. Therefore by Theorem 2.2, we obtain

$$\lim_{t\to+\infty}x_0(t)=\lim_{t\to+\infty}b(t,\ x_0(\cdot))=0\ \text{and}\int_{T_0}^\infty a(s,\ x_0(\cdot))ds<\infty,\tag{4.20}$$

where $T_0\geqslant t_0$ is defined such that $x_0(t)>0$, $t\geqslant T_0-\max\{\tau, \sigma\}$.

In that case from (4.19), by integrating from t to $+\infty$, we obtain that $x_0(t)$ satisfies the equation

$$x_0(t)=b(t,\ x_0(\cdot))+\int_t^\infty a(s,\ x_0(\cdot))ds,\ t\geqslant T_0.\tag{4.21}$$

Now let $\varepsilon>0$ be an arbitrarily small fixed number. Then (4.20) yields that there exists a $T_\varepsilon\geqslant T_0$ such that $0<x_0(t)<\delta_\varepsilon$, $t\geqslant T_\varepsilon-\max\{\tau, \sigma\}$, where δ_ε is defined in condition (A_4). Thus by virtue of (4.7), (4.8) and (4.9), we have

$$a(t,\ x_0(\cdot))\geqslant(1-\varepsilon)\sum_{k=1}^{K}a_kx_0(t-\tau_k),\ t\geqslant T_\varepsilon,$$

and

$$b(t, x_0(\cdot)) \geqslant (1-\varepsilon) \sum_{m=1}^{M} c_m x_0(t-r_m) + (1-\varepsilon) \sum_{j=1}^{J} b_j \int_{t-\sigma_j}^{t-\hat{\sigma}_j} x_0(s)ds, \quad t \geqslant T_\varepsilon,$$

where we used the definitions (4.17) and (4.18) of $a(t, x(\cdot))$ and $b(t, x(\cdot))$, respectively.

Therefore from (4.21), it follows that

$$x_0(t) \geqslant (1-\varepsilon)\Big[\sum_{m=1}^{M} c_m x_0(t-r_m) + \sum_{j=1}^{J} b_j \int_{t-\sigma_j}^{t-\hat{\sigma}_j} x_0(s)ds + \int_t^\infty \sum_{k=1}^{K} b_k x_0(s-\tau_k)ds, \quad t \geqslant T_\varepsilon.$$

But by virtue of Theorem 3.1, we obtain that the equation

$$\frac{d}{dt}\Big[v(t) - (1-\varepsilon)\sum_{m=1}^{M} c_m v(t-r_m)\Big]$$
$$= (1-\varepsilon)\sum_{j=1}^{J} b_j[v(t-\hat{\sigma}_j) - v(t-\sigma_j)] - (1-\varepsilon)\sum_{k=1}^{K} a_k x(t-\tau_k)$$

has an eventually positive solution.

Since $\varepsilon>0$ is arbitrarily small, by Corollary 3.1, we obtain that Equation (4.10) has a positive solution on some interval $[T_0, \infty)$. The proof of the theorem is complete.

From Theorems 4.1 and 4.2 it immediately follows a necessary and sufficient condition for the existence of an eventually positive solution of (4.1):

Theorem 4.3. *Assume that conditions* (A_1), (A_2), (A_3) *and* (A_4) *are satisfied. Then Equation* (4.1) *has an eventually positive solution on* $[t_{-1}, \infty)$ *if and only if Equation* (4.10) *has an positive solution or eqvivalently the characteristic equation*

$$\lambda\Big(1-\sum_{m=1}^{M} c_m e^{-\lambda r_m}\Big) = \sum_{j=1}^{J} b_j [e^{-\lambda\hat{\sigma}_j} - e^{-\lambda\sigma_j}] - \sum_{k=1}^{K} a_k e^{-\lambda\tau_k} \tag{4.22}$$

has a negative root.

Now we apply Theorem 4.3 to the equation

$$\dot{x}(t) = g(x(t-\sigma)) - f(x(t-\tau)), \tag{4.23}$$

where we assume

(i) τ, σ are some nonnegative constants, $\gamma = \max\{\tau, \sigma\}$;

(ii) $f, g: R \to R$ are continuous functions such that $f(x) > g(x) > 0\ (x>0)$, and for all $\varepsilon>0$ small enough thereis a $\delta_\varepsilon>0$ such that

$$(1-\varepsilon)px \leqslant f(x) \leqslant px \text{ and } (1-\varepsilon)qx \leqslant g(x) \leqslant qx, \quad x \in [0, \delta_\varepsilon),$$

where the constants p and q satisfy the conditions

$$0 \leqslant q < p \text{ and } 0 \leqslant q(\tau-\sigma) < 1.$$

Then Equation (4.23) has an eventually positive solution on $[t_0-\gamma, \infty)$ if the equation

$$\lambda = qe^{-\lambda\sigma} - pe^{-\lambda\tau}$$

has a negative root.

Proof Since Equation (4.23) is equivalent to the following equation

$$\dot{x}(t) = g(x(t-\sigma)) - g(x(t-\tau)) - [f(x(t-\tau)) - g(x(t-\tau))],$$

by Theorem 4.3 the proof is completed.

Now we consider the differential system

$$\dot{x}(t)=\sum_{j=1}^{J}[g_j(x(t-\hat{\sigma}_j))-g(x(t-\sigma_j))]-f(t,\ x(t-\tau_1),\ \cdots,x(t-\tau_K)),\qquad(4.24)$$

where we state the following assumptions which would be used as indicated in each result

(B_1) $\tau_k\geqslant 0$ $(1\leqslant k\leqslant K)$ and $\sigma_j\geqslant\hat{\sigma}_j\geqslant 0$ $(1\leqslant j\leqslant J)$ are given constants, $\tau=\max\{\max_{1\leqslant k\leqslant K}\tau_k,\ \max_{1\leqslant j\leqslant J}\sigma_j\}$ and $t_{-1}=t_0-\tau$;

(B_2) $f:[t_0,\ \infty)\times R^{nN}\to R^n$ and $g_j:R^n\to R^n$ $(1\leqslant j\leqslant J)$ are continuous functions such that

$$f(t,\ x_1,\ \cdots,\ x_k)\geqslant 0 \text{ and } g_j(x_1)\geqslant 0 \quad (1\leqslant j\leqslant J,\ x_k\in R^n_+,\ 1\leqslant k\leqslant K),$$

and

$$\left|\int_{t_0}^{\infty}\min\{f(t,\ x_1,\ \cdots,\ x_k):x_k\geqslant c,\ 1\leqslant k\leqslant K\}\,dt\right|=+\infty,$$

for all $c\in R^n_+$ such that $|c|>0$;

(B_3) there are nonnegative n by n matrices A_k and B_j and a positive vector $\delta_0\in R^n$ such that

$$f(t,\ x_1,\ \cdots,\ x_K)\leqslant\sum_{k=1}^{K}A_kx_k,\ t\geqslant t_0,\ 0\leqslant x_k\leqslant\delta_0,\ 1\leqslant k\leqslant K,$$

and

$$g_j(y)\leqslant B_jy,\ 0\leqslant y\leqslant\delta_0,\ 1\leqslant j\leqslant J,$$

moreover

$$\left(\sum_{k=1}^{K}A_k\right)e\neq 0 \text{ if } e\in R^n_+ \text{ is such that } e\neq 0,\qquad(4.25)$$

and the spectral radius of the matrix $\sum_{j=1}^{J}B_j$ satisfies the condition

$$\rho\left(\sum_{j=1}^{J}B_j\right)<1;$$

(B_4) for all $\varepsilon>0$ small enough there exists a $\delta_\varepsilon\in R^n_+$ such that $\delta_\varepsilon>0$ and

$$f(t,\ x_1,\ \cdots,\ x_k)\geqslant(1-\varepsilon)\sum_{k=1}^{K}A_kx_k,\ t\geqslant t_0,\ 0\leqslant x_k<\delta_\varepsilon,\ 1\leqslant k\leqslant K,$$

and

$$g_j(y)\geqslant(1-\varepsilon)B_jy,\ 0\leqslant y<\delta_\varepsilon,\ 1\leqslant j\leqslant J,$$

where the constant matrices A_k and B_j satisfy (4.25) and (4.26).

By using Proposition 3.2 and Corollary 3.2, a repetition of the proof of Theorems 4.1 and 4.2, with appropriate changes, proves the following theorems.

Theorem 4.4. *Assume that conditions* (B_1), (B_2) *and* (B_3) *aresatisfied. Assume further that the linear equation*

$$\dot{x}(t)=\sum_{j=1}^{J}B_j[x(t-\hat{\sigma}_j)-x(t-\sigma_j)]-\sum_{k=1}^{K}A_kx(t-\tau_k)\qquad(4.27)$$

has a slightly positive solution on an interval $[T_0,\ \infty)$. *Then Equation* (4.24) *has* *a slightly positive selution on* $[t_0-\tau, \infty)$.

Theorem 4. 5. *Assume that conditions* (B_1), (B_2) *and* (B_4) *are satisfied. Assume further that Equation* (4.24) *has a slightly positive solution on* $[t_0-\tau, \infty)$. *Then Equation* (4.27) *has a slightly positive solntion on an interval* $[T_0, \infty)$.

Theorem 4. 6. *Assume that conditions* (B_1), (B_2), (B_3) *and* (B_4) *are satisfied. Then Equation* (4.24) *has a slightly positive solution on* $[t_0-\tau, \infty)$ *if and only if Equation* (4.27) *has a slightly positive solution on an interval* $[T_0, \infty)$ *or equivalently therex exist a*$^{\hat{\sigma}}$*negative number* λ_0 *and a vector* $e_0\in R_+^n$ *such that*

$$\Big((\lambda_0 I-\sum_{j=1}^{J} B_j\,(e^{-\lambda_0\hat{\sigma}_j}-e^{-\lambda_0\sigma_j})+\sum_{k=1}^{K} A_k e^{-\lambda_0\tau_k}\Big)e_0=0$$

and $|e_0|>0$.

References

[1] Arino, O., Győri, I., Jawhari, A., Osclllation criteria in delay equations, *J. Diff. Eqs.*, **53** (1984), 115—123.

[2] Arino, O., Győri, I., Necessary and sufficient condition for oscillation of neutral differential systems with several delays, *J. Diff. Eqs.* (to appear).

[3] Arino, O., Ladas, G., Sficas, Y. G., On oscillations of some retarded differential equations, *SIAM J. Math. Anal.*, **18** (1987), 64—73.

[4] Cronin, J., Fixed Points and Topological Degree in Linear Analysis, Math. Surveys 11, Am. Math. Soc., Providence, R. I., 1964.

[5] Ferreira, J., Győri, I., Oscillatory behavior in linear retarded functional differential equations, *J. Math. Anal. Appl.*, **128** (1987), 332—346.

[6] Gopalsamy, K., Győri, I., Ladas, G., Oscillations of a class of delay equations with continuous and piecewise constant arguments, *Funkcialaj Ekvacioj* (to appear).

[7] Győri, I., Oscillations of retarded differential equations of the neutral and the mixed type, *J. Math. Anal. Appl.* (to appear).

[8] Győri, I., Oscillation and comparison results in neutral differential equations with application to the delay logistic equation, in Proceedings of the Conference on Mathematical Problems in Population Dynamics, Oxford, MS, 1986 (to appear).

[9] Győri, I., Oscillation of neutral delay differential equations arising in populationdynamics, in Proceedings of Equadiff'87 (Xanthi, Greece) (to appear).

[10] Győri, I., Eller, J., Modelling thrombopoiesis regulation-II Mathematical investigation of the model, *Comput. Math. Appl.*, **14** (1987), 849—859.

[11] Győri, I., Ladas, G., Linearized oscillations for equations with piecewise constant arguments, *Differential and Integral Equations*, **2** (1988), 123—131.

[12] Hale, J., Theory of Functional Differential Equations, Applied Mathematical Sciences 3, Springer-Verlag, New York, 1977.

[13] Kulenovic, M. R. S., Ladas, G., Linearized oscillation in population dynamics, *Bull. Math. Biol.*, **49** (1987), 617—627.

[14] Kulenovic, M. R. S., Ladas, G., Meimaridou, A., On oscillation of non-linear delay equations, *Quart. Appl. Math.*, **45** (1987), 155—164.

[15] Ladas, G., Linearized oscillations for neutral equations, Proceeding of Equadiff'87 (Xanthi, Greece) (to appear).

[16] Ladas, G., Sficas, Y. G., Stavroulakis, I. P., Necessary and sufficient conditions for oscillations, *Amer. Math. Monthly*, **90** (1983), 637—640.

[17] Ladas, G., Stavroulakis, I. P. Oscillations caused by several retarded and advanced arguments, *J. Diff. Eqs.*, **44** (1982), 134—152.

[18] Levitan, B. M., Almost Periodic Functions (in Russian), "Gostekhizdat", Moscow, 1953.

[19] Myskis, A. D., Linear Differential Equations with Retarded Argument (in Russian), Moscow, "Mauka", 1972.

[20] Nisbet, P. M., Gurney, W. S., Modeling Fluctuating Populations, John Willey and Sons Ltd., New York, 1982.

[21] Zhang, B. G., A survey of the oscillation of solutions to first order differential equations with deviting arguments, *Ann. Diff. Eqs.*, 2 (1986), 65—86.