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THE EXISTENCE OF THE SOLUTIONS FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS

LIU SHENGMAO (刘绳茂)*

Abstract

The author obtains the existence of the local and global solution for a class of semilinear evolution equations in a Hilbert spcae, and uses the results to prove the existence of the solution for semilinear parabolic partial differential equations in R^n.

§ 1. Introduction

In this paper, we consider the semilinear evolution equations in a Hilbert space H in the following form:

$$\begin{cases} u'(t)+A(t)u=f(t, u), \ t\in (0, T), \\ u(0)=u_0. \end{cases} \tag{1.1}$$

Here $A(t)=\partial\phi^t$ is the subdifferential of a family of lower semecontinuous convex functions ϕ^t defined on H. Many authors investigated the equations (1.1), but the assumtions for $f(t, u)$ were severe, for example, [1] assumed $f(t, u)$ to be nearly monotone, The papers [2, 3, 4] assume $f(t, u)$ to be compact in $[0, T]\times H$. In this paper, we give some conditions for ϕ^t and assume $f(t, u)$ to be continuous and bunded in $[0, T]\times H$. We prove the existence of the local and global solutions of (1.1). In particular' we use these results to the following equations:

$$\begin{cases} u'(t)-\Delta u=g(t, u)+\left(\int_\Omega u^2(x, t)dx\right)^m, \ (x, t)\in\Omega\times(0, T], \\ u=q(x, s), \ (x, t)\in\partial\Omega\times(0, T], \\ u(0)=u_0(x). \end{cases} \tag{1.2}$$

Here $\Omega\subset R^n$ is a bounded and open set with smooth boundary $\partial\Omega$, and $m\geqslant 0$.

§ 2. The Main Results

We begin with some general assumptions for ϕ^t and $f(t, u)$.

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* Branch of Beijing University of Iron and Steel Technology, Beijing, China.

(I) $\exists c_1>0$, such that $\phi^t(t)\geqslant c_1\|u\|^2$, $\forall u\in D(\phi^t)$.

(II) $\exists b\in W^{1,2}(0, T)$ with $|b'|_{L^2(0,T)}\leqslant a_0$, such that

$$\phi^s(u)-\phi^t(u)\leqslant |b(t)-b(s)|(\phi^t(u)+1),\ 0\leqslant s,\ t\leqslant T,$$

(III) the level set $\{u\in H,\ \phi^t(u)\leqslant r\}$ is compact in H for each $t\in[0, T]$ and $r\in R^+$.

(IV) $f(t, u)$: $[0, T]\times H\to H$ is continuous. In addition, for each $N>0$, there exists M_M such that $\|f(t, u)\|\leqslant M_M$, $\forall\|u\|\leqslant N$.

The main results are the following

Theorem 2.1 *Under the assumptions* (I)—(IV) *and* $u_0\in D(\phi^0)$, *there exists* $T_0>0$ *sufficieutly small such that problem* (1.1) *has at least one solution* $u\in W^{1,2}(0, T_0; H)$.

Theorem 2.2. *Let the general assumtsons* (I)—(IV) *be satisfied. Further, assume that there exists* $a>0$, $b\in R$, $L>0$, *such that* $M_M\leqslant LN$, $(f(t, u), u)\leqslant a\|u\|^2+b$. *Then the problem* (1.1) *has at least one solution* $u\in W^{1,2}(0, T; H)$.

Theorem 2.3. *If* $g\in C(\partial\Omega\times[0, T])$, $|g(t, u)|\leqslant c_3|u|+c_4$, $c_3>0$, $c_4\in R$, *then problem* (1.2) *has solution* $u\in W^{1,2}(0, T; L^2(\Omega))$ *if* $m\leqslant\frac{1}{2}$. *In addition, if* $m>\frac{1}{2}$, *there exists* $T_0>0$ *small enough such that problem* (1.2) *has solution* $u\in W^{1,2}(0, T_0; L^2(\Omega))$.

§ 3. Froor of The Main Results

Let ϕ^t_λ be convex approximation of ϕ^t, $J^t_\lambda=(I+\partial\phi^t_\lambda)^{-1}$. From the assumtions (I)—(IV), we have $\phi^t\in\mathscr{A}(c_1; a_0)$ by the definition 1.2 in [5].

Lemma 3.1 ([5], Lemmas 2.2, 2.3). (1) *The function* Φ: $L^2(0, T; H)\to R$ *given by* $\Phi(u)=\int_0^T\phi^t(u(t))\,dt$ *for* $u\in L^2(0, T; H)$ *is proper* 1. s. c *and convex function in* $L^2(0, T; H)$; *besides, the convex approximation of* Φ *is given by* $\Phi_\lambda(u)=\int_0^T\phi^t{}_\lambda(u(t))dt$.

(2) *For* u, $u^*\in L^2(0, T; H)$, $u^*\in\partial\Phi(u)$ *if and only if* $u^*(t)\in\partial\phi^t(u(t))$, *a, c, t.*

(3) *Let* (II) *be satisfied,* $u\in W^{1,1}(0, T; H)$, $\lambda>0$. *Then the funtion* $t\to\phi^t_\lambda(u(t))$ *is differentiable a, c,* $t\in[0, T]$, *and*

$$\phi^t_\lambda(u(t))-\phi^\tau_\lambda(u(t))\leqslant\int_s^t\frac{d}{d\tau}\phi^\tau_\lambda(u(\tau))\,d\tau,$$

$$\frac{d}{dt}\phi^t_\lambda(u(t))-(u'(t),\ \partial\phi^t_\lambda(u(t)))\leqslant|b'(t)|\,(2+\phi^\tau_\lambda(u(t)))\ \text{a, c, t.}$$

Making the approximation of $f(t, u)$ in the following

$$f_N(t, u)=\begin{cases} f(t, u) \text{ if } \|u\|\leqslant N,\\ f(t, u_N) \text{ if } \|u\|>N.\end{cases}$$

Here u_N is a fixed element in the set $\{u;\ \|u\|=N\}$. Thus f_N: $L^2(0, T; H)\to L^2(0, T; H)$ is continuous.

First, we discuss the equation:

$$\begin{cases} u'_\lambda(t)+\partial\phi^t_\lambda(u_\lambda)=f(t), \\ u_\lambda(0)=u_0. \end{cases} \tag{3.1}$$

Lemma 3.2. *If $f\in L^2(0, T; H)$, then equations (3.1) has unique solution $u\in W^{1,2}(0, T; H)$ and*

$$\int_0^t\left\|\frac{du_\lambda}{dt}\right\|^2dt+\phi^t_\lambda(u_\lambda(t))\leqslant c(1+\|f\|_{L^2(0,t;H)}), \tag{3.2}$$

$$\|J^\lambda_\lambda u_\lambda(t)\|^2+\|u_\lambda(t)\|^2\leqslant c(1+\|f\|_{L^2(0,t;H)}). \tag{3.3}$$

Moreover, if u_1, u_2 are the solutions about f_1, f_2 respectively, then

$$\|u_1(t)-u_2(t)\|\leqslant c\Big(1+\frac{T}{\lambda}e^{T/\lambda}\Big)\|f_1-f_2\|_{L^2(0,T;H)}. \tag{3.4}$$

Here c is independent of λ, u, f.

Proof Obviously, (3.1) is equivalent to the following equation

$$u_\lambda(t)=u_0+\int_0^t f(s)\,ds-\int_0^t\partial\phi^s_\lambda(u_\lambda(s))ds,$$

$\partial\phi^t_\lambda$ is Lipschitzian. Therefore, $u_\lambda\in W^{1,2}(0, T; H)$ exists.

Form the inner product of (3.1) with $u'_\lambda(t)$, integrating over $[0, t]$, and using Lemma 3.1, we get

$$\int_0^t\left\|\frac{du_\lambda}{dt}\right\|^2dt+\phi^t_\lambda(u_\lambda(t))-\phi^t_\lambda(u_0)$$
$$\leqslant\int_0^t\Big(f,\frac{du_\lambda}{dt}\Big)dt+\int_0^t|b'(t)|(\phi^t_\lambda(u_\lambda)+1)dt.$$

Using $\phi^t_\lambda(u_0)\leqslant\phi^t(u_0)\leqslant|b(t)-b(0)|(\phi^0(u)+1)$, Granwall's inenquality and (I), we obtain (3.2). From

$$\phi^t_\lambda(z)=(2\lambda)^{-1}\|z-J^t_\lambda z\|^2+\phi^t(J^t_\lambda z)\geqslant(2\lambda)^{-1}\|z-J^t_\lambda z\|^2+c_1\|J^t_\lambda Z\|^2$$
$$\geqslant c_3\|z\|^2\qquad(\text{if }(2\lambda)^{-1}\geqslant c_3).$$

we obtain (3.3). (3.4) is easy.

Lemma 3.3.

$$\|J^t_\lambda(u(t)-u^s_\lambda u(s)\|^2\leqslant 2\lambda|b(t)-b(s)|(\phi^t_\lambda(u(t))+\phi^s_\lambda(u(s))+2)$$
$$+2\|u(t)-u(s)\|^2,\ \forall u\in L^2(0, T; H). \tag{3.5}$$

Proof Let $y(t)=J^t_\lambda u(t)$. Then $y(t)=u(t)-\lambda\partial\phi^t_\lambda(u(t))$,

$$y(t)-y(s)=\lambda(\partial\phi^s_\lambda(u(s))-\partial\phi^t_\lambda(u(t)))+u(t)-u(s).$$

$$\|y(t)-y(s)\|^2$$
$$\leqslant\lambda(\phi^s(y(t))-\phi^s(y(s))+\phi^t(y(s))-\phi^t(y(t)))+(u(t)-u(s), y(t)-y(s))$$
$$\leqslant\lambda|b(t)-b(s)|(\phi^t(y(t))+\phi^s(y(s))+2)+\|u(t)-u(s)\|\,\|y(t)-y(s)\|.$$

Here we have used $\partial\phi^t_\lambda(u)=\partial\phi^t(J^t_\lambda v)$ (see [5]).

Now we consider the equations

$$\begin{cases} u'_\lambda(t)+\partial\phi^t_\lambda(u_\lambda)=f_N(t,\ J^t_\lambda u_\lambda), \\ u_\lambda(0)=u_0. \end{cases} \tag{3.6}$$

We define operation K: $C(0,\ T;\ H)\rightarrow C(0,\ T;\ H)$, $w\rightarrow Kw=J^t_\lambda u_\lambda$, where u_λ is the unique solution of the equations

$$\begin{cases} u^1_\lambda(t)+\partial\phi^t_\lambda(u_\lambda)=f_N(t,\ w), \\ u_\lambda(0)=u_0. \end{cases} \tag{3.7}$$

Obviously, K's fixed point is the solution of equations (3.6). From (3.3), we have

$$|J^t_\lambda u_\lambda\|^2\leqslant c(1+\|f_N(t,\ w)\|_{L^2(0,T;H)})\leqslant c(1+TM_N)\triangleq C_N.$$

Define $X=\{u\in C(0,\ T;\ H);\ \|J^t_\lambda u\|^2\leqslant C_N\}$. Then $K(X)\subset X$. X is a nonempty, bounded, closed and convex subset in $C(0,\ T;\ H)$. Using (3.4) and continutity of f_N, we know K: $C(0,\ T;\ H)\rightarrow C(0,\ T;\ H)$ is continuous. We show that K is compact in the following.

For fixed $t\in[0,\ T]$, using (3.2) we have

$$\phi^t(J^t_\lambda u_\lambda(t))\leqslant\phi^t_\lambda(u_\lambda(t))\leqslant c(1+TM_N).$$

Therefore $X_1=\{J^t_\lambda u_\lambda(t);\ u\in X\}$ is relatively compact in H. In addtion,

$$\|u_\lambda(t)-u_\lambda(s)\|\leqslant\left\|\int_s^t\frac{du_\lambda}{dt}dt\right\|\leqslant|t-s|^{1/2}\int_0^T\left\|\frac{du_\lambda}{dt}\right\|^2dt\leqslant c|t-s|^{1/2},$$

Combining Lemma 3.3 and (3.2), (3.3), we know that X_1 is equicontinuous in H. According to Arzelà-Ascoli theorem we may conclude that $K(X)$ is relatively compact in $C(0,\ T;\ H)$. Therefore, K: $X\rightarrow X$ is compact.

In terms of the Shauder fixed point theorem, the operator K has at least one fixed point. Therefore, equation (3.6) has solution u_λ, and

$$\int_0^t\left\|\frac{du_\lambda}{dt}\right\|^2dt+\phi^t_\lambda(u_\lambda(t))\leqslant c\,(1+t^{1/2}M_N), \tag{3.8}$$

$$\|J^t_\lambda u_\lambda(t)\|+\|u_\lambda(t))\|\leqslant c(c(1+t^{1/2}M_N). \tag{3.9}$$

Lemma 3.4. *There exist* $T_0\in(0,\ T)$, $\lambda_0>0$, *such that the following equation*

$$\begin{cases} u^1_\lambda(t)+\partial\phi^t_\lambda(u_\lambda)=f(t,\ J^t_\lambda u_\lambda), \\ u_\lambda(0)=u_0 \end{cases} \tag{3.10}$$

has solution $u_\lambda\in W^{1,2}(0,\ T_0;\ H)$, *if* $0<\lambda\leqslant\lambda_0$.

Proof It is enough to prove that there exist $N>0$, $T_0\in(0,\ T]$, λ_0, such that $\|J^t_\lambda u_\lambda(t)\|\leqslant N$, if $0<\lambda\leqslant\lambda_0$, $t\leqslant T_0$. But

$$\|u_\lambda(t)\|\leqslant\|u_0\|+t^{1/2}\int_0^t\left\|\frac{du_\lambda}{dt}\right\|^2dt\leqslant\|u_0\|+ct^{1/2}(1+t^{1/2}M_N),$$

$$\|J^t_\lambda u_\lambda(t)\|\leqslant\|u_\lambda(t)\|+\|u_\lambda(t)-J^t_\lambda u_\lambda(t)\|\leqslant\|u_\lambda(t)\|+(2\lambda\,\varphi^t_\lambda(u_\lambda(t)))^{1/2}$$
$$\leqslant\|u_0\|+2c\,\lambda(1+t^{1/2}M_N)^{1/2}+ct^{1/2}(1+t^{1/2}M_N).$$

First we choose fixed $N\geqslant2\|u_0\|$, then we choose T_0 and λ_0, such that

$$\|J^t_\lambda u_\lambda(t)\|\leqslant N.$$

Proof of Theorem 2.1 From (3.8), (3.9), assunlption (III) and 3.2, there exist some subsequence of λ (for simplicity denoted again by λ) and $u(t)$, such that

$$J^t_\lambda u_\lambda(t)\to u(t) \text{ in } C(0, T_0; H),$$
$$u_\lambda(t)\to u(t) \text{ in } C(0, T_0; H).$$

Because of $\|\partial\phi^t_\lambda(u_\lambda)\|_{L^2(0,T;H)}\leqslant c$, there exist $\eta\in L^2(0, T_0; H)$, such that

$$\partial\Phi_\lambda(u_\lambda)\to\eta, \text{ in } L^2(0, T_0; H) \text{ (using the sign of Lemma 3.1)}.$$

Therefore $\eta=\partial\Phi(u)=\int_0^{T_0}\partial\phi^t(u)dt$. $u'_\lambda(t)\to u'(t)$ in $L^2(0, T_0; H)$. Taking limit from (3.10), we see that u is the solution of (1.1).

Proof of Theorem 2.2 We must prove that we can choose $T_0=T$ in Lemma 3.4. (Without loss of generality, we assume $0\in D(\partial\phi^t)$). Forming inner product of (3.6) with $u_\lambda(t)$, and integrating over $(0, t)$, we get

$$\|u_\lambda(t)\|^2+\int_0^t\phi^t_\lambda(u_\lambda)dt\leqslant c+\int_0^t(f_N(t, J^t_\lambda u_\lambda), u_\lambda)dt,$$

$$\begin{aligned}\|u_\lambda(t)\|^2+c_1\int_0^t\|J^t_\lambda u_\lambda\|^2dt &\leqslant c+\int_0^t(f_N(t, J^t_\lambda u_\lambda), u_\lambda-J^t_\lambda u_\lambda)dt+\int_0^t(f_N(t, J^t_\lambda u_\lambda), J^t_\lambda u_\lambda)dt\\ &\leqslant c+\int_0^t M_N(2\lambda\phi^t_\lambda(u_\lambda))^{1/2}dt+\int_0^t(c_1+L)\|J^t_\lambda u_\lambda\|^2dt,\end{aligned}$$

We get $$\|J^t_\lambda u_\lambda(t)\|\leqslant[c+c\lambda^{1/2}M_N^{3/2}][1+Te^{(c+c\lambda^{1/2}M_N^{3/2}+L)}].$$

Here we have used $\|u_\lambda(t)-J^t_\lambda u_\lambda(t)\|\leqslant 2\lambda\phi^t_\lambda(u_\lambda(t))$. Therefore we fix $N>0$ large enough, and then choose $\lambda_0>0$ small enough such that $\|J^t_\lambda u_\lambda(t)\|\leqslant N$, if $\lambda\leqslant\lambda_0$,

Proof of Theorem 2.3 Let $\psi(t)$ be the solution of equation

$$\begin{cases}\Delta\psi(t)=0 \text{ in } \Omega,\\ \psi(t)=y(t) \text{ in } \partial\Omega.\end{cases}$$

Define $$\phi(u)=\begin{cases}\int_\Omega|\nabla u(x)|^2dx, & u\in H^1_0(\Omega),\\ +\infty, & u\in L^2(\Omega)\setminus H^1_0(\Omega).\end{cases}$$

Then ϕ: $L^2(\Omega)\to+\infty$ is proper convex lower semicontinuous function, and

$$\partial\phi(u)=-\Delta u, \text{ if } u\in D(\partial\phi)=\{u; u\in A^2(\Omega)\cap H^1_0(\Omega)\}.$$

We define $\phi^t(u)=\phi(u-\psi(t))$, Equation (1.2) is equivalent to the equation in $L^2(\Omega)$:

$$\begin{cases}u'(t)+\partial\phi^t(u)=F(t, u),\\ u(0)=u_0.\end{cases}$$

Here $F(t, u)=g(t, u)=\left(\int_\Omega u^2dt\right)^m$, and $F(t, u)$ satisfies (I)——(IV).

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