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ON THE HARNACK INEQUALITY FOR HARMONIC FUNCTIONS ON COMPLETE RIEMANNIAN MAINFOLDS**

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Abstract

First it is shown that on the complete Riemannian manifold with nonnegative Ricci curvature $\overline{M}$ the Sobolev type inequality $\|\nabla u\|_2 \geq C_{n,\alpha}\|u\|_{2\alpha}$ $(\alpha \geq 1)$ for all $u \in H_1^2(\overline{M})$ holds if and only if $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_n r^n$ and $\alpha = \frac{n}{n-2}$. Let M be a complete Riemannian manifolds which is uniformly equivalent to $\overline{M}$, and assume that $V_x(r) \geq C_n r^n$ on $\overline{M}$. Then it is prioved that the John-Nirenberg inequality holds on M. Finally, based on the Sobolev inequality and John-Nirenberg inequality, the Harnack inequality for harmonic functions on M is obtained by the method of Moser, and consequently some Liouville theorems for harmonic functions and harmonic maps on M are proved.

§1. Introduction

Kendall[5,8] used stochastic methods to prove that if M is any manifold on which every boundary harmonic function is constant, N is a complete Riemannian manifold and ϕ : $M \to N$ is a harmonic map with image in a geodesically small disc, then ϕ is constant.

Combining this theorem and Tau's result[18] one obtains the Liouville theorem for harmonic maps proved by Yu[19]. By this theorem and Moser's results[14] one can also obtain the Liouville theorem for harmonic maps proved by Hildebrandt, Jost and Widman[7]. They proved that if (R^m, g) is uniformly equivalent to R^m, N is a complete Riemannian manifold with the sectional curvature $K_N \leq B < \infty$, and $\phi : (R^m, g) \to N$ is a harmonic map with $\phi(M)$ contained in a geodesically small disc, then ϕ is constant.

Kendall's result shows that the Liouville theorem of harmonic functions implies some Liouville theorem of harmonic maps. It is well known that the Liouville theorem of harmonic functions can be obtained by the Harnack inequality for harmonic functions. There are mainly two ways to obtain the Harnack inequality for harmonic function on complete Riemannian manifold. One is Moser's method[2,3,6,14]. In this case, one uses the Sobolev inequality and John-Nirengberg inequality which is not true in general on complete Riemannian manifold. The other is Yau's method[9,11,15,18]. The idea of Yau's method is to estimate the gradient of positive harmonic function. In general the gradient estimate depends on the Ricci curvature of the complete Riemannian manifold.

Suppose $\overline{M} \triangleq (M, h)$ is a complete Riemannian manifold with nonnegative Ricci curvature, $M \triangleq (M, g)$ is uniformly equivalent to $\overline{M}$, that is $c \cdot h_x(Y,Y) \leq g_x(Y,Y) \leq C \cdot h_x(Y,Y)$

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for all $x \in M$, $Y \in T_xM$. In this paper, we prove that if $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_n r^n$, where $B_x(r) = \{y \in M | \mathrm{dist}(y,x) < r\}$, then there is no non-constant bounded harmonic function on M, Therefore, we generalize the Moser's result[14] and by the Kendall's result we also generalize the result of Hildebrandt-Jost-Widman.

In section 2, we consider the Sobolev type inequality on M. In section 3, we consider the Pioncaré inequality and John-Nirengberg inequality on M. Finally, in section 4 we prove the Harnack inequality for harmonic function on M and particularly we obtain the Liouville theorem for harmonic functions and harmonic maps.

In whole paper, $C_{a,b,\cdots}$, denotes a positive constant depending on $a, b, \cdots$, which may be different in different places.

§2. Sobolev Inequality

Strichartz[16] proved the following L^2 boundedness of Riesz transform.

Lemma 2.1. *Let M be a complete Riemannian manifold. Suppose $f \in L^2(M)$. Then $(-\Delta)^{1/2} f$ is in $L^2(M)$ if and only if $|\nabla f|$ is in $L^2(M)$ and $\|(-\Delta)^{1/2} f\|_2 = \|\nabla f\|_2$.*

In [16], Strichartz also introduced the Riesz potential on complete Riemannian manifold M,

$$((-\Delta)^{-\frac{1}{2}\alpha} f)(x) = \Gamma(\frac{\alpha}{2})^{-1} \int_M \int_0^\infty t^{\frac{1}{2}\alpha - 1} H(x,y,t) dt f(y) dy,$$

where $H(x,y,t)$ is the heat kernel on M.

The author[10] proved the following boundedness of Riesz potential.

Lemma 2.2. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose that for any $x \in M$, $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_n r^n$, where $B_x(r) = \{y \in M | \mathrm{dist}(y,x) < r\}$ and $\mathrm{Vol}(B_x(r))$ is the volume of $B_x(r)$. Then the Riesz potential $(-\Delta)^{-\frac{1}{2}\alpha}$ $(0 < \alpha < n)$ is of type (p,q), that is,*

$$\|(-\Delta)^{-\frac{1}{2}\alpha} f\|_q \leq C_{n,p,q} \|f\|_p \quad \text{for all } f \in L^p(M),$$

where $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

In this section, we will use the two results and the estimate of the heat kernel proved by Cheng-Li-Yau[3] and Li-Yau[13] to prove the following result, which was proved by Varopoulos[17] when $\alpha = 1$ using totally different ideas.

Theorem 2.1. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Then the Sobolev type inequality*

$$\|\nabla u\|_2 \geq C_{n,\alpha} \|u\|_{2\alpha} \quad (\alpha \geq 1) \tag{2.1}$$

holds for all $u \in H_1^2(M)$ if and only if for all $x \in M$, $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_n r^n$ and $\alpha = \frac{n}{n-2}$.

Proof. We first prove that the condition is sufficient. Note that[16] if $f \in L^2(M)$, then the definition of Riesz potential $(-\Delta)^{-\frac{1}{2}\alpha} f$ agrees with that given by spectral theory. We therefore have $u = (-\Delta)^{-1/2}(-\Delta)^{1/2} u$ for all $u \in H_1^2(M)$.

By Lemma 2.1, we have

$$\|(-\Delta)^{1/2} u\|_2 = \|\nabla u\|_2. \tag{2.2}$$

By Lemma 2.2, we have

$$\|u\|_{2n/(n-2)} \le C_n \|(-\Delta)^{1/2} u\|_2. \tag{2.3}$$

Clearly (2.2) and (2.3) imply (2.1) with $\alpha = \frac{n}{n-2}$.

Now we prove the condition is necessary.

Suppose (2.1) holds for all $u \in H_1^2(M)$. Assume that x is a fixed point in M and t is a fixed positive constant. We set $u(y) = H(x,y,t)$ where $H(x,y,t)$ is the heat kernel on M.

$$\left(\int_M H^{2\alpha}(x,y,t)dy\right)^{1/\alpha} \le C_{n,\alpha} \int_M |\nabla H(x,y,t)|^2 dy. \tag{2.4}$$

If $\alpha = 1$,

$$\int_M H^2(x,y,t)dy \le C_n \int_M |\nabla H(x,y,t)|^2 dy. \tag{2.5}$$

By the estimate of the heat kernel proved by Li-Yau[13], we have

$$H(x,y,2t) \ge C_n \frac{1}{V_x(\sqrt{2t})}. \tag{2.6}$$

By the estimate of the heat kernel proved by Cheng-Li-Yau[3], we have

$$\int_M |\nabla H(x,x,t)|^2 dy \le \frac{C_n}{t} H(x,x,t). \tag{2.7}$$

In [13], Li-Yau also proved that

$$H(x,y,t) \le C_n \frac{1}{V_x(\sqrt{t})}. \tag{2.8}$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain

$$\frac{1}{V_x(\sqrt{2t})} \le C_n \frac{1}{tV_x(\sqrt{t})},$$

that is,

$$\frac{V_x(\sqrt{2t})}{V_x(\sqrt{t})} \ge C_n t. \tag{2.9}$$

By Bishop comparsion theorem[1,2,15], we have

$$\frac{V_x(\sqrt{2t})}{V_x(\sqrt{t})} \le (\sqrt{2})^n. \tag{2.10}$$

Clearly (2.9) is contrary to (2.10). Therefore, we may assume $\alpha > 1$.

By Hölder's inequality, we have

$$\int_M H^2(x,y,t) \le \left(\int_M H^{2\alpha}(x,y,t)dy\right)^{\frac{1}{2\alpha}} \left(\int_M H^{2\alpha'}(x,y,t)dy\right)^{\frac{1}{2\alpha'}}, \tag{2.11}$$

where $\frac{1}{\alpha'} = 2 - \frac{1}{\alpha}$ and $\frac{1}{2} < \alpha' < 1$.

Substituting (2.11) into (2.4), we obtain

$$\left(\int_M H^{\alpha}(x,y,t)dy\right)^2 \le C_{n,\alpha} \int_M |\nabla H(x,y,t)|^2 dy \left(\int_M H^{2\alpha'}(x,y,t)dy\right)^{\frac{1}{\alpha'}}. \tag{2.12}$$

Noting that $\int_M H(x,y,t)dy \le 1$, by (2.8) we have

$$\left(\int_M H^{2\alpha'}(x,y,t)dy\right)^{\frac{1}{\alpha'}} \le C_{n,\alpha}\left(\frac{1}{V_x(\sqrt{t})}\right)^{\frac{2\alpha'-1}{\alpha'}}. \tag{2.13}$$

Substituting (2.13), (2.7) and (2.6) into (2.12), we obtain

$$\frac{1}{V_x^2(\sqrt{2t})} \leq C_{n,\alpha}\frac{1}{tV_x(\sqrt{t})}\Big(\frac{1}{V_x(\sqrt{t})}\Big)^{\frac{2\alpha'-1}{\alpha'}},$$

that is,

$$\frac{V_x^{2\alpha'}(\sqrt{t})}{V_x^{2\alpha'}(\sqrt{2t})} \leq C_{n,\alpha}\frac{1}{t^{\alpha'}V_x^{\alpha'-1}(\sqrt{t})}. \tag{2.14}$$

By (2.10) and (2.14), we have

$$V_x(\sqrt{t}) \geq C_{n,\alpha}t^{\frac{\alpha'}{1-\alpha'}}. \tag{2.15}$$

Applying Bishop comparison theorem, one obtains

$$V_x(\sqrt{t}) \leq C_n t^{n/2}. \tag{2.16}$$

So $t^{\frac{n}{2}} \geq C_{n,\alpha}t^{\frac{\alpha'}{1-\alpha'}}$ for all $t > 0$.

We therefore have $\frac{\alpha'}{1-\alpha'} = \frac{n}{2}$, that is,

$$\alpha = \frac{n}{n-2}. \tag{2.17}$$

By (2.15) and (2.17) we have $V_x(r) \geq C_n r^n$ for all $x \in M$.

Theorem 2.2. *Let $\overline{M}$ be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose M is a complete Riemannian manifold which is uniformly equivalent to $\overline{M}$. Then the Sobolev type inequality (2.1) holds for all $u \in H_1^2(M)$ if and only if, for all $x \in M$, $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_n r^n$ and $\alpha = \frac{n}{n-2}$.*

This theorem easily follows from Theorem 2.1, because that M is uniformly equivalent to $\overline{M}$ implies that the gradient and the volume element on M are also uniformly equivalent to those on $\overline{M}$.

§3. Poincaré Inequality and John-Nirenberg Inequality

In the following two sections, we suppose that $\overline{M} \triangleq (M, d\bar{s}^2)$ is a complete Riemannian manifold with nonnegative Ricci curvature, and we assume that $M = (M, ds^2)$ is a complete Riemannian manifold which is uniformly equivalent to $\overline{M}$, that is,

$$\lambda^2 d\bar{s}^2 \leq ds^2 \leq \Lambda^2 d\bar{s}^2, \tag{3.1}$$

where λ and Λ are positive constants. The gradient and the volume element on $\overline{M}$ are denoted by $\overline{\nabla}$ and $d\overline{V}$ respectively; the gradient and the volume element on M are denoted by ∇ and dV respectively. The geodesic distance from x to y on $\overline{M}$ and on M are respectively denoted by $\bar{\rho}(x,y)$ and $\rho(x,y)$. Clearly, ∇, dV and ρ are also uniformly equivalent to $\overline{\nabla}, d\overline{V}$ and $\bar{\rho}$ respectively. We set

$$B_x(r) = \{y \in M | \rho(x,y) < r\},\ \overline{B}_x(r) = \{y \in M | \bar{\rho}(x,y) < r\},$$

$$V_x(r) = \int_{B_x(r)} 1dV,\ \ \overline{V}_x(r) = \int_{\overline{B}_x(r)} 1d\overline{V}.$$

$V_x(r)$ and $\overline{V}_x(r)$ are obviously uniformly equivalent. In addition we assume $V_x(r) \geq C_n r^n$ for all $x \in M$.

Let x_0 be a fixed point in M. For $\mu \in (0,1)$ we define the linear operator V_μ on $L^1(B_{x_0}(R))$ by

$$(V_\mu f)(x) = \int_{B_{x_0}(R)} (\rho(x,y))^{n(\mu-1)} f(y) dV \tag{3.2}$$

and define the linear operator $\overline{V}_\mu$ on $L^1(\overline{B}_{x_0}(R))$ by

$$(\overline{V}_\mu f)(x) = \int_{\overline{B}_{x_0}(R)} (\bar{\rho}(x,y))^{n(\mu-1)} f(y) d\overline{V}. \tag{3.3}$$

The following lemma implies V_μ and $\overline{V}_\mu$ are well defined.

Lemma 3.1. *Suppose $0 \le \delta = \frac{1}{p} - \frac{1}{q} < \mu$, where $p \ge 1$ and $1 \le q \le \infty$. Then V_μ is a bounded linear operator from $L^p(B_{x_0}(R))$ into $L^q(B_{x_0}(R))$, $\overline{V}_\mu$ is a bounded linear operator from $L^p(\overline{B}_{x_0}(R))$ into $L^q(\overline{B}_{x_0}(R))$, and*

$$\|\overline{V}_\mu f\|_q \le \Omega_n^{1-\delta}(\frac{1-\delta}{\mu-\delta})^{1-\delta}(2R)^{n(\mu-\delta)}\|f\|_p, \tag{3.4}$$

$$\|V_\mu f\|_q \le ((\frac{\Lambda}{\lambda})^n\Omega_n)^{1-\delta}(\frac{1-\delta}{\mu-\delta})^{1-\delta}(2R)^{n(\mu-\delta)}\|f\|_p, \tag{3.5}$$

where Ω_n is a constant depending on n and Λ, λ are the constants in (3.1).

Proof. We set $h(x,y) = (\rho(x,y))^{n(\mu-1)}$, $\bar{h}(x,y) = (\bar{\rho}(x,y))^{n(\mu-1)}$,

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta \le 1. \tag{3.6}$$

Suppose x is a point in $\overline{B}_{x_0}(R)$.

$$\int_{\overline{B}_{x_0}(R)} \bar{h}^r(x,y) d\overline{V} \le \int_{\bar{B}_x(2R)} (\bar{\rho}(x,y))^{n(\mu-1)r} d\overline{V}.$$

By Bishop comparison theorem, we have

$$\begin{aligned}\int_{\overline{B}_x(2R)} (\bar{\rho}(x,y))^{n(\mu-1)r} d\overline{V} &\le \omega_{n-1}\int_0^{2R} \bar{\rho}^{n(\mu-1)r+n-1} d\bar{\rho}\\ &= \omega_{n-1}\frac{(2R)^{n(1+\mu-1)r)}}{n(1+(\mu-1)r)},\end{aligned}$$

where $\omega_{n-1} = $ Area (S^{n-1}). So

$$\|\bar{h}\|_r \le \Omega_n^{1-\delta}(\frac{1-\delta}{\mu-\delta})^{1-\delta}(2R)^{n(\mu-\delta)}. \tag{3.7}$$

Note that $B_{x_0}(R) \subset \overline{B}_{x_0}(\lambda^{-1}R)$. So

$$\begin{aligned}\int_{B_{x_0}(R)} h^r(x,y) dV &\le \Lambda^n \int_{\overline{B}_{x_0}(\lambda^{-1}R)} (\lambda\bar{\rho}(x,y))^{n(\mu-1)r} d\overline{V}\\ &\le \omega_{n-1}\frac{(2R)^{n(1+(\mu-1)r)}}{n(1+(\mu-1)r)}(\frac{\Lambda}{\lambda})^n.\end{aligned}$$

Then

$$\|h\|_r \le \Omega_n^{1-\delta}(\frac{1-\delta}{\mu-\delta})^{1-\delta}(2R)^{n(\mu-\delta)}(\frac{\Lambda}{\lambda})^{n(1-\delta)}. \tag{3.8}$$

Using Hölder's inequality, one obtains

$$|V_\mu f| \le \{\int_{B_{x_0}(R)} h^r(x,y)|f(y)|^p dV\}^{1/q}\{\int_{B_{x_0}(R)} h^r(x,y)dV\}^{1-1/p} \cdot \{\int_{B_{x_0}(R)} |f(y)|^p dV\}^\delta \tag{3.9}$$

and

$$|\overline{V}_\mu f| \le \{\int_{\overline{B}_{x_0}(R)} \bar{h}^r(x,y)|f(y)|^p d\overline{V}\}^{1/q}\{\int_{\overline{B}_{x_0}(R)} \bar{h}^r(x,y)d\bar{V}\}^{1-1/p} \cdot \{\int_{\overline{B}_{x_0}(R)} |f(y)|^p d\overline{V}\}^\delta \tag{3.10}$$

By Minkowski's inequality one obtains

$$\|V_\mu f\|_q \le \sup_{B_{x_0}(R)} \{\int_{B_{x_0}(R)} h^r(x,y)dV\}^{1/r}\|f\|_p, \tag{3.11}$$

$$\|\overline{V}_\mu f\|_q \le \sup_{\overline{B}_{x_0}(R)} \{\int_{B_{x_0}(R)} h^r(x,y)dV\}^{1/r}\|f\|_p. \tag{3.12}$$

Clearly, (3.7) and (3.12) yield (3.4); (3.8) and (3.11) yield (3.5).

We define $M^p(B_{x_0}(R)) = \{f \in L^1(B_{x_0}(R))|$ there is $K > 0$ such that $\int_{B_{x_0}(R)\cap B(r)} |f(y)|dy \le K \cdot r^{n(1-1/p)}$ for all ball $B(r) \subset M\}$,

$$\|f\|_{M^p(B_{x_0}(R))} = \inf\{r^{-n(1-1/p)} \cdot \int_{B_{x_0}(R)\cap B(r)} |f(y)|dy\}.$$

Lemma 3.2. *Suppose $\delta = \frac{1}{p} < \mu$. Then for almost all $x \in B_{x_0}(R)$*

$$|V_\mu f(x)| \le (\frac{1-\delta}{\mu-\delta})(2R)^{n(\mu-\delta)}\|f\|_{M^p(B_{x_0}(R))} \tag{3.13}$$

if $f \in M^p(B_{x_0}(R))$.

Proof. We assume $f(x) = 0$ if $x \in B^c_{x_0}(R) = M\backslash B_{x_0}(R)$. Since

$$f \in M^p(B_{x_0}(R)), \quad V(\rho) = \int_{B_{x_0}(\rho)} |f(y)|dy \le \|f\|_{M^p(B_{x_0}(R))}\rho^{n(1-1/p)}, \tag{3.14}$$

$$|V_\mu f(x)| \le \int_{B_{x_0}(R)} (\rho(x,y))^{n(\mu-1)}|f(y)|dV \le \int_{B_x(2R)} (\rho(x,y))^{n(\mu-1)}|f(y)|dV.$$

We choose the geodesic spherical coordinates about x. Since the measure of $\mathrm{Cut}(x)$ is 0, we may ignore it and assume that $dV = \sqrt{g(\rho,\theta)}d\theta d\rho$ where $\theta \in S^{n-1}$ and $d\theta$ is the Haar measure on S^{n-1}. Then

$$\begin{aligned} |V_\mu f(x)| &\le \int_0^{2R} \rho^{n(\mu-1)} \int_{S^{n-1}} |f(\rho,\theta)|\sqrt{g(\rho,\theta)}d\theta d\rho \\ &= \int_0^{2R} \rho^{n(\mu-1)}V'(\rho)d\rho \\ &= (2R)^{n(\mu-1)}V(2R) + n(1-\mu)\int_0^{2R} V(\rho)\cdot\rho^{n(\mu-1)-1}d\rho. \end{aligned} \tag{3.15}$$

Obviously (3.14) and (3.15) yield (3.13).

Lemma 3.3. *Suppose $f \in M^p(B_{x_0}(R))$ $(\infty > p > 1)$, $g = V_\mu f$, $\mu = 1/p$. Then there exist positive constants C_1 and C_2 which depend on n and p such that*

$$\int_{B_{x_0}(R)} \exp(\frac{g}{C_1 K})dV \le C_2(2\frac{\Lambda}{\lambda}R)^n, \tag{3.16}$$

where $K = \|f\|_{M^p(B_{x_0}(R))}$.

Proof. For any $q \ge 1$, by Hölder's inequality one obtains

$$|g(x)| \le (V_{\mu/q}|f|)^{1/q}(V_{\mu+\mu/q}|f|)^{1-1/q}. \tag{3.17}$$

Using Lemma 3.2, we have

$$V_{\mu+\mu/q}|f| \le (p-1)q(2R)^{n/(pq)}K. \tag{3.18}$$

Using Lemma 3.1, we have

$$\int_{B_{x_0}(R)} V_{\mu/q}|f|dV \le \Omega_n \cdot p \cdot q \cdot K(2R)^{n(1-1/p+1/(pq))}(\frac{\Lambda}{\lambda})^n. \tag{3.19}$$

Therefore

$$\int_{B_{x_0}(R)} |g|^q dV \le p' \cdot \Omega_n \cdot \{(p-1)qK\}^q(2\frac{\Lambda}{\lambda}R)^n, \tag{3.20}$$

where $p' = \frac{p}{p-1}$. So

$$\int_{B_{x_0}(R)} \sum_{m=0}^{N} \frac{|g|^m}{m!(C_1K)^m}dV \le p' \cdot \Omega_n \cdot (2\frac{\Lambda}{\lambda}R)^n \sum_{m=0}^{N}(\frac{p-1}{C_1})^m\frac{m^m}{m!}$$

If $(p-1)e < C_1$, then

$$\int_{B_{x_0}(R)} \exp(\frac{g}{C_1K})dV \le C_2(2\frac{\Lambda}{\lambda}R)^n,$$

that is (3.16).

Lemma 3.4. *Suppose $u \in H_1^1(B_{x_0}((2\frac{\Lambda}{\lambda}+1)R))$. Then for almost all $x \in B_{x_0}(R)$*

$$|u(x) - u_{B_{x_0}(R)}| \le C_{n,\lambda,\Lambda} \int_{B_{x_0}((2\frac{\Lambda}{\lambda}+1)R)} (\rho(x,y))^{1-n} \cdot |\nabla u(y)|dV, \tag{3.21}$$

where

$$u_{B_{x_0}(R)} = \frac{1}{V_{x_0}(R)}\int_{B_{x_0}(R)} u(y)dV.$$

Proof. Clearly we may assume $u \in C^1(B_{x_0}((2\frac{\Lambda}{\lambda}+1)R))$. For any $x, y \in B_{x_0}(R)$, assume $\bar{r} : [0, \bar{\rho}(x,y)] \to M$ is a geodesic segment from x to y in $\overline{M}$.

$$u(x) - u(y) = -\int_0^{\bar{\rho}(x,y)} \frac{du(\bar{r}(t))}{dt}dt,$$

$$V_{x_0}(R)(u(x) - u_{B_{x_0}(R)}) = -\int_{B_{x_0}(R)}\int_0^{\bar{\rho}(x,y)} \frac{du(\bar{r}(t))}{dt}dtdV. \tag{3.22}$$

Clearly

$$\bar{r}(t) \subset B_x(2\frac{\Lambda}{\lambda}R) \subset B_{x_0}((2\frac{\Lambda}{\lambda}+1)R).$$

We set $\overline{V}(y)=|\overline{\nabla}u(y)|$. Then

$$V_{x_0}(R)|u(x)-u_{B_{x_0}(R)}|\le\int_{B_{x_0}(R)}\int_0^{\bar\rho(x,y)}\overline{V}(\bar r(t))dtdV$$
$$\le\Lambda^n\int_{B_x(\frac{2}{\lambda}R)}\int_0^{\bar\rho(x,y)}\bar V(\bar r(t))dtd\overline{V}. \tag{3.23}$$

We choose the geodesic spherical coordinates about x in $\bar M$, and assume

$$d\overline{V}=\sqrt{\bar g(\bar\rho,\theta)}d\theta d\bar\rho.$$

Then

$$|u(x)-u_{B_{x_0}(R)}|\le\frac{\Lambda^n}{V_{x_0}(R)}\int_0^{\frac{2}{\lambda}R}\int_0^{\bar\rho}\overline{V}(r)dr\int_{S^{n-1}}\sqrt{\bar g(\bar\rho,\theta)}d\theta d\bar\rho.$$

By the Bishop comparison theorem, we know $\frac{\sqrt{\bar g(\bar\rho,\theta)}}{\bar\rho^{n-1}}$ is a decreasing function. Since $r\le\bar\rho$, we have

$$|u(x)-u_{B_{x_0}(R)}|\le\frac{\Lambda^n}{V_{x_0}(R)}\int_0^{\frac{2}{\lambda}R}\int_0^{\bar\rho}\int_{S^{n-1}}r^{1-n}\overline{V}(r)\sqrt{\bar g(r,\theta)}d\theta dr\bar\rho^{n-1}d\bar\rho$$
$$\le\frac{\Lambda^n(2R/\lambda)^n}{nV_{x_0}(R)}\int_{\overline{B}_x(2R/\lambda)}|\overline{\nabla}u(y)|(\bar\rho(x,y))^{1-n}d\overline{V}$$
$$\le(\frac{\Lambda}{\lambda})^n(\frac{2\Lambda R}{\lambda})^n\frac{1}{nV_{x_0}(R)}\int_{B_{x_0}((2\frac{\Lambda}{\lambda}+1)R)}(\rho(x,y))^{1-n}|\nabla u(y)|dV.$$

That is (3.21).

Now we can prove the following Poincaré inequality and John-Nirenberg inequality.

Theorem 3.1. *Suppose $u\in H_1^1(B_{x_0}((2\frac{\Lambda}{\lambda}+1)R))$. Then*

$$||u-u_{B_{x_0}(R)}||_{L^p(B_{x_0}(R))}\le C_{n,\lambda,\Lambda}R||\nabla u||_{L^p(B_{x_0}((2\frac{\Lambda}{\lambda}+1)R))}.$$

Proof. Lemma 3.4 and Lemma 3.1 yield this theorem.

Theorem 3.2. *Suppose $u\in H_1^1(B_{x_0}((2\frac{\Lambda}{\lambda}+1)R))$, and assume that there exists a positive constant K such that for all metric ball $B(r)$ in M*

$$\int_{B_{x_0}((2\frac{\Lambda}{\lambda}+1)R)\cap B(r)}|\nabla u|dV\le K\cdot r^{n-1}. \tag{3.24}$$

Then there exist positive constants $\mu_{n,\lambda,\Lambda}$ and $C_{n,\lambda,\Lambda}$, which depend on n,λ,Λ, such that

$$\int_{B_{x_0}(R)}\exp(\frac{\mu_{n,\lambda,\Lambda}}{K}|u-u_{B_{x_0}(R)}|)dx\le C_{n,\lambda,\Lambda}R^n. \tag{3.25}$$

Proof. Using Lemma 3.4 we have

$$\int_{B_{x_0}(R)}\exp(\frac{\mu_{n,\lambda,\Lambda}}{K}|u-u_{B_{x_0}(R)}|)dx$$
$$\le\int_{B_{x_0}(R)}\exp(\frac{\mu_{n,\lambda,\Lambda}}{K}C_{n,\lambda,\Lambda}V_{1/n}|\nabla u|)dx$$
$$\le\int_{B_{x_0}((2\frac{\Lambda}{\lambda}+1)R)}\exp(\frac{\mu_{n,\lambda,\Lambda}}{K}C_{n,\lambda,\Lambda}V_{1/n}|\nabla u|)dx,$$

where

$$V_{1/n}|\nabla u|(x)=\int_{B_{x_0}((2\frac{\Lambda}{\lambda}+1)R)}(\rho(x,y))^{1-n}|\nabla u(y)|dy. \tag{3.26}$$

Obviously the theorem follows from (3.24), (3.26) and Lemma 3.3.

§4. Harnack Inequality and Its Applications

We first use the Moser's method to prove the following Harnack inequality for Harmonic function on M.

Theorem 4.1. *Suppose x_0 is a fixed point in M. If $u \in H_1^2(B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R))$, $u \geq 0$ and $\Delta u = 0$, then*

$$\sup_{B_{x_0}(R)} u \leq C_{n,\lambda,\Lambda} \inf_{B_{x_0}(R)} u. \tag{4.1}$$

We set $\phi(p,r) = (\frac{1}{V_{x_0}(r)}\int_{B_{x_0}(r)} \bar{u}^p dV)^{1/p}$, where $\bar{u} = u+A$, $A > 0$, and set $V = \bar{u}^k$, $k \neq \frac{1}{2}$.

Lemma 4.1. *Suppose u satisfies the hypotheses of Theorem 4.1. then for any $\varphi \in C_0^\infty(B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R))$*

$$\begin{aligned}&(\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} (\varphi v)^{2n/(n-2)} dV)^{(n-2)/n}\\ \leq& C_{n,\Lambda,\lambda}(\frac{2k}{2k-1})^2 \int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} |\nabla\varphi|^2 V^2 dV.\end{aligned}$$

Proof. Since $\varphi \in C_0^\infty(B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R))$ and $\Delta\bar{u} = 0$, we have

$$\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \varphi^2 \bar{u}^{2k-1}\Delta\bar{u} dV = 0$$

and

$$\begin{aligned}&(2k-1)\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \varphi^2\bar{u}^{2k-2}|\nabla u|^2 dV\\ =&-2\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \varphi\bar{u}^{2k-1}\nabla\varphi\nabla\bar{u} dV\\ \leq& 2(\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \varphi^2\bar{u}^{2k-2}|\nabla\bar{u}|^2 dV)^{1/2}(\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \bar{u}^{2k}|\nabla\varphi|^2 dV)^{1/2}.\end{aligned}$$

We therefore have

$$\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} |\varphi\cdot\nabla V|^2 dV \leq 4(\frac{2k}{2k-1})^2 \int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} |\nabla\varphi|^2 V^2 dV. \tag{4.2}$$

Applying Sobolev inequality one has

$$\begin{aligned}&(\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} (\varphi\cdot V)^{\frac{2n}{n-2}} dV)^{\frac{n-2}{n}}\\ \leq& C_{n,\Lambda,\lambda}\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} |\nabla(\varphi\cdot V)|^2 dV\\ \leq& C_{n,\Lambda,\lambda}\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} (|\nabla\varphi V)|^2 + |\varphi\cdot\nabla V|^2) dV.\end{aligned} \tag{4.3}$$

Lemma 4.1 follows from (4.2) and (4.3).

Lemma 4.2. *Suppose u satisfies the hupotheses of Theorem 4.1. For any $p > 1$,*

$$\phi(\infty, R) \leq C_{n,\Lambda,\lambda,p}\phi(p, 2R). \tag{4.4}$$

For any $0 < p_0 < p < \beta = \frac{n}{n-2}$,

$$\phi(p, 2R) \leq C_{n,\Lambda,\lambda;p,p_0}\phi(p_0, 3R). \tag{4.5}$$

For any $p<0$,

$$\phi(p,3R)\le C_{n,\Lambda,\lambda,p}\phi(-\infty,R). \tag{4.6}$$

Proof. Let

$$\varphi(x)=\begin{cases}1, & x\in B_{x_0}(h'R),\\ 0, & x\bar{\in}B_{x_0}(h'R),\end{cases}$$

$$0\le\varphi(x)\le 1,\quad |\nabla\varphi|\le\frac{C}{(h-h')R},$$

where $0<h'<h\le 2h'$, C is an absolute constant. By Lemma 4.1 one has

$$\Big(\int_{B_{x_0}(h'R)}V^{2\beta}dV\Big)^{1/\beta}\le C_{n,\Lambda,\lambda}\Big(\frac{2k}{2k-1}\Big)^2\Big(\frac{1}{h-h'}\Big)^2\frac{1}{R^2}\int_{B_{x_0}(hR)}V^2dV.$$

Let $p=2k\ne 1$. The above inequality yields

$$\begin{aligned}&\Big(\frac{1}{V_{x_0}(h'R)}\int_{B_{x_0}(h'R)}\bar{u}^{p\beta}dV\Big)^{1/\beta}\\ \le&C_{n,\Lambda,\lambda}\Big(\frac{p}{p-1}\Big)^2\frac{1}{R^2}\Big(\frac{1}{h-h'}\Big)^2\frac{1}{V_{x_0}(hR)}\int_{B_{x_0}(hR)}\bar{u}^pdV\frac{(hR)^n}{(h'R)^{n-2}}\\ \le&C_{n,\Lambda,\lambda}\Big(\frac{p}{p-1}\Big)^2\Big(\frac{h}{h-h'}\Big)^2\frac{1}{V_{x_0}(hR)}\int_{B_{x_0}(hR)}\bar{u}^pdV.\end{aligned} \tag{4.7}$$

So, if $p>0$,

$$\phi(\beta p,h'R)\le C_{n,\Lambda,\lambda}^{1/p}\Big(\frac{h}{h-h'}\Big)^{2/p}\Big(\frac{p}{p-1}\Big)^{2/p}\phi(p,hR); \tag{4.8}$$

if $p<0$,

$$\phi(p,hR)\le C_{n,\Lambda,\lambda}^{-1/p}\Big(\frac{h}{h-h'}\Big)^{-2/p}\Big(\frac{p}{p-1}\Big)^{-2/p}\phi(p,h'R). \tag{4.9}$$

Choosing $h=h_v=1+2^{-v}$, $h'=h_{v+1}$, $p_v=\beta^v p$ and using (4.8), we have

$$\begin{aligned}\phi(p_{v+1},h_{v+1}R)&\le C_{n,\Lambda,\lambda}^{1/p_v}\Big(\frac{h_v}{h_v-h_{v+1}}\Big)^{2/p_v}\Big(\frac{p_v}{p_v-1}\Big)^{2/p_v}\phi(p_v,h_vR)\\ &\le C_{n,\Lambda,\lambda,p}^{1/p_v}2^{v\cdot 1/p_v}\phi(p_v,h_vR)\\ &\le (C_{n,\Lambda,\lambda,p})^{\frac{1}{p}\cdot\sum_{j=0}^{v}\beta^{-j}}\cdot 2^{\frac{1}{p}\cdot\sum_{j=0}^{v}\frac{1}{\beta^j}}\phi(p,2R).\end{aligned}$$

Letting $v\to\infty$, we have (4.4).

Similarly one can obtain (4.5) and (4.6).

Proof of Theorem 4.1. Applying Lemma 4.2, we know that it suffices to prove there is a positive constant p_0 such that

$$\phi(p_0,3R)\le C_{n,\Lambda,\lambda}\cdot\phi(-p_0,3R). \tag{4..10}$$

For this purpose, we set $W=\log\bar{u}$ and assume $2r<R$, $B(r)\cap B_{x_0}(3(2\frac{\Lambda}{\lambda}+1)R)\ne\emptyset$, where $B(r)$ is any metric ball in M. Cleraly $B(r)\subset B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)$.

Suppose $\eta(x)\in C_0^\infty(B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R))$,

$$\eta(x)=\begin{cases}1, & x\in B(r),\\ 0, & x\bar{\in}B(2r),\end{cases}$$

$0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{C}{r}$, where C is an absolute constant.

We choose $\phi(x) = \frac{1}{\bar{u}}\eta^2(x)$. Then

$$\begin{aligned} 0 &= \int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \nabla\varphi\nabla\bar{u}dV \\ &= -\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \eta^2|\nabla W|^2 dV + 2\int_{B_{x_0}(6(\frac{\Lambda}{\lambda}+1)R)} \eta\nabla\eta\cdot\nabla W dV. \end{aligned}$$

So $\int_{B(r)} |\nabla W|^2 dV \le C_n r^{n-2}$ and therefore $\int_{B(r)} |\nabla W| dV \le C_n r^{n-1}$.

Similarly one obtains

$$\int_{B_{x_0}(3(2\frac{\Lambda}{\lambda}+1)R)} |\nabla W| dV \le C_{n,\lambda,\Lambda} R^{n-1}.$$

So $\int_{B_{x_0}(3(2\frac{\Lambda}{\lambda}+1)R)\cap B(r)} |\nabla W| dV \le C_n r^{n-1}$ for all $r > 0$.

Applying Theorem 3.2, we know that there exist positive constants p_0 and $C_{n,\lambda,\Lambda}$ which depend on n, λ, Λ such that

$$\int_{B_{x_0}(3R)} e^{p_0|W-W_0|} dV \le C_{n,\lambda,\Lambda} R^n,$$

where

$$W_0 = \frac{1}{V_{x_0}(3R)} \cdot \int_{B_{x_0}(3R)} W dV,$$

So

$$\int_{B_{x_0}(3R)} e^{p_0 W} dV \cdot \int_{B_{x_0}(3R)} e^{-p_0 W} dV \le C_{n,\lambda,\Lambda} R^{2n}, \tag{4.11}$$

Clearly (4.10) follows from (4.11).

The following Corollary easily follows from Theorem 4.1.

Corollary 4.1. *Let Ω be a domain in M. Suppose $u \in H_1^2(\Omega)$, $u \ge 0$ and $\Delta u = 0$. If $\Omega' \subset\subset \Omega$, then there exists a positive constant $C(n, \lambda, \Lambda, \Omega', \Omega)$ depending on $n, \lambda, \Lambda, \Omega', \Omega$ such that*

$$\sup_{\Omega'} u \le C(n, \lambda, \Lambda, \Omega', \Omega) \inf_{\Omega'} u.$$

One can also obtain strong maximum principle for harmonic functions in Ω.

Now, we consider the global behavior of harmonic function on M. Suppose $u(x)$ is a harmonic function on M. We set

$$M(r) = \max_{\rho(x_0,x)=r} u(x), \quad \mu(r) = \min_{\rho(x_0,x)=r} u(x),$$

where x_0 is a fixed point in M. By the maximum principle we know that $M(r)$ is an increasing function and $\mu(r)$ is a decreasing function.

Theorem 4.2. *If $u(x)$ is a non-constant harmonic function on M, then there exist positive constants $C_{n,\lambda,\Lambda}$ and α which depend on n, λ, Λ such that $W(r) = M(r) - \mu(r) \ge C_{n,\lambda,\Lambda} r^\alpha$.*

Proof. We consider the functions $M(2r) - u(x)$ and $u(x) - \mu(2r)$ in $B_{x_0}(r)$. Using Theorem 4.1 we have

$$\begin{aligned} M(2r) - \mu(r) &\le C_{n,\lambda,\Lambda}(M(2r) - M(r)), \\ M(r) - \mu(2r) &\le C_{n,\lambda,\Lambda}(\mu(r) - \mu(2r)). \end{aligned}$$

So $W(2r) \ge \theta W(r)$, where $\theta = \frac{C_{n,\lambda,\Lambda}+1}{C_{n,\lambda,\Lambda}-1}$.

Therefore

$$W(2^v r) \geq 2^{v \log_2 \theta} W(r) \tag{4.12}$$

for all $v = 1, 2, \cdots$. Clearly this theorem follows from (4.12).

Corollary 4.2. *There is no non-constant boundd harmonic function on M.*

Theorem 4.2 obviously implies this corollary. In fact, we have the following result.

Corollary 4.3. *There is no non-constant positive harmonic function on M.*

Proof. Suppose $u(x)$ is a positive harmonic function on M. $u_0 = \inf_{x \in M} u(x)$. Then there exist x_n $(n = 1, 2, \cdots) \in M$ such that $u(x_n) \to u_0$. Since $W(x) = u(x) - u_0$ is also a positive harmonic function, for all $R > 0$, we have

$$\sup_{B_{x_0}(R)} W(x) \leq C_{n,\lambda,\Lambda} \cdot \inf_{B_{x_0}(R)} W(x),$$

where x_0 is a fixed point in M. Letting R tend to ∞ yields $W \equiv 0$. So $u \equiv u_0$.

Applying Kendall's result and Corollary 4.2 one has the following Liouville theorem for harmonic maps.

Corollary 4.4. *Suppose N is a complete Riemannian manifold. If $\phi : M \to N$ is a harmonic map with image in a geodesically small disc, then ϕ is constant.*

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