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ON THE CAUCHY PROBLEM FOR A SINGULAR INTEGRODIFFERENTIAL EQUATION

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Abstract

The arthors establish the existence and uniqueness of global solutions to the Cauchy problem for the third-order Benjamin-Ono equation by the so-called continuation method.

§0. Introduction

In this paper we shall discuss the Cauchy problem for the third-order Benjamin-Ono equation,

$$\text{(I)} \qquad \begin{cases} u_t = -\dfrac{\partial}{\partial x}(u^3 + 3uHu_x + 3H(uu_x) - 4u_{xx}), \\ u(x,0) = \varphi(x), \end{cases}$$

where H denotes the Hilbert transform $Hu = P.V.\frac{1}{\pi}\int \frac{u(y)}{y-x}dy$.

The Lax hierarchy of the BO equation is given by

$$\text{(II)} \qquad u_t = -\frac{\partial}{\partial x}(\frac{\delta I_n}{\delta u}) = -\frac{\partial}{\partial x}K_n, \quad n = 3,4,5,\cdots,$$

where I_n is the nth conserved quantity of the BO equation and $\frac{\delta I_n}{\delta u} = K_n$ denotes the functional derivative defined by

$$\frac{\partial}{\partial \varepsilon}I_n(u+\varepsilon V)|_{\varepsilon=0} = \int \left(\frac{\delta I_n(u)}{\delta u}\right) v dx = \int K_n(u) v dx.$$

The first few I_n and K_n are :

$$I_1 = \int u dx, \qquad K_1 = 1,$$

$$I_2 = \int \frac{u^2}{2} dx, \qquad K_2 = u,$$

$$I_3 = \int (\frac{u^3}{3} + uHu_x)dx, \quad K_3 = u^2 + 2Hu_x,$$

$$I_4 = \int (\frac{u^4}{4} + \frac{3}{2}u^2 Hu_x + 2(u_x)^2 dx, \quad K_4 = u^3 + 3uHu_x + 3H(uu_x) - 4u_{xx},$$

$$I_5 = \int \{\frac{u^5}{5} + [\frac{4}{3}u^3 Hu_x + u^2 H(uu_x)] + [2u(Hu_x)^2 + 6u(u_x)^2 - 4u_{xx}Hu_x]\}dx,$$

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$$I_6 = \int \{\frac{u^6}{6} + [\frac{5}{4}u^4 Hu_x + \frac{5}{3}u^3 Huu_x] + \frac{5}{2}[5u^2(u_x)^2 + u^2(Hu_x)^2 + 2u(Hu_x)H(uu_x)] - 10[(u_x)^2 Hu_x + 2uu_{xx}Hu_x] + 8(u_{xx})^2\}dx.$$

For $n = 3$, equation (II) becomes the following usual BO equation

$$u_t = -\frac{\partial}{\partial x}(u^2 + 2Hu_x).$$

For $n = 4$, equation (II) reduces to equation (I). In [1,7], the authors discussed the Cauchy problem for the BO equation. The purpose of the present paper is to establish the existence and uniqueness of global solutions to problem (I). For this purpose we first consider the following parabolic regularized problem:

$$\text{(III)} \qquad \begin{cases} u_t = -\mu u_x^4 - \frac{\partial}{\partial x}(u^3 + 3uHu_x + 3H(uu_x) - 4u_{xx}), \\ u(x,0) = \varphi(x), \end{cases}$$

in suitable Banach spaces and solve it there. Then the solution of (I) is obtained by the limiting process of the vanishing of the coefficient $\mu \to 0$ through the solution of (III). In this paper we use the same notations as in [1].

Our results are as follows.

Theorem. *Let integer $s \geq 3$ and $\varphi \in H^s(R)$ with $\|\varphi\|_0 < \frac{\sqrt{2}}{3}$. Then for any $T > 0$ there exists a unique $u \in C([0,T]; H^s)$ such that $u_t \in C([0,T]; H^{s-3})$ and (I) is satisfied.*

Corollary 0.1. *Under the assumptions of the theorem above, there exists a unique $u \in C([0,\infty); H^S)$ such that $u_t \in C([0,\infty); H^{S-3})$ and (I) is satisfied.*

Corollary 0.2. *Let $H^\infty = \bigcap_S H^S$ and $\|\varphi\|_0 < \frac{\sqrt{2}}{3}$. Then there exists a unique $u \in C([0,\infty); H^\infty)$ such that $u_t \in C([0,\infty); H^\infty)$ and (I) is satisfied.*

§1. Preliminary Results

In this part we list serveral well-known results to be used in the sequel.

Lemma 1.1 (Gagliardo-Nirenberg inequenality). *For $u \in H^m(R)$ and $p > 2$, we have*

$$\|D^d u\|_{L^p} \leq 2^{(p-2)/(2p)}|D^k u|^\lambda |u|^{1-\lambda},$$

where $\lambda = (d + \frac{1}{2} - \frac{1}{p})/k, \quad 1 \leq d < k \leq m.$

Lemma 1.2 (properties of Hilbert transform).

$$\int uH(v)dx = -\int H(u)vdx,$$

$$\int H(u)H(v)dx = \int uvdx,$$

$$H(uH(v) + vH(u)) = H(u)h(v) - uv.$$

Lemma 1.3. *$\int K_m(u)(K_n(u))_x dx = 0$ for all m and n.*

See [8] for its proof.

§2. Local Existence in $H^S(R)$

This section is devoted to the local existence of solutions to problem (I). Putting $f =$

$\frac{\partial}{\partial x}(u^3+3uHu_x+3H(uu_x))$ and taking Fourier transform on both sides of (III) yield

$$\begin{cases} \frac{d\hat{u}(\xi)}{dt} = -\mu\xi^4\hat{u}(\xi) - 4i\xi^3\hat{u}(\xi) - \hat{f}(\xi), \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi). \end{cases} \tag{2.1}$$

From (2.1) we have

$$u(x,t) = S_\mu(t)\varphi(x) - \int_0^t S_\mu(t-t')f(x,t)dt',$$

where $\widehat{S_\mu(t)v} = E_\mu(\xi,t)\hat{V}(\xi)$ and $E_\mu(\xi,t) = \exp[-(\mu\xi^4+4i\xi^3)t]$.

With the notations above we have the following theorem:

Theorem 2.1. *Let $\lambda \in [0,\infty)$. Then*

(a) $S_\mu(t)$, for all $t>0$ and $s \in R$, satisfies

$$\|S_\mu(t)\varphi\|_{S+\lambda} \le M_\lambda(1+(4\mu t)^{-\lambda/4})\|\varphi\|_S$$

for all $\varphi \in H^S$, where M_λ is a positive constant depending only on λ. Moreover the mapping $t \in (0,\infty)$, $S_\mu(t)\varphi$ is continuous with respect to the topology of $H^{S+\lambda}$.

b) $S_\mu(t) : H^S \to H^h$ is a strongly continuous contractive semigroup of operators with one parameter.

Theorem 2.2. *Let $\mu > 0$ be fixed, $\varphi \in H^s$ and $s \ge 1$. Then there exist a $T_s > 0$ depending on s, $\|\varphi\|$ and μ, and a unique function $u_\mu(t) \in C([0,T_s];H^s)$ satisfying the following integrodifferential equation*

$$u(t) = S_\mu(t)\varphi - \int_0^t S(t-t')\partial_x(u^3+3uHu_x+3H(uu_x))dt'.$$

Moreover, u_μ is the unique solution to problem (III) such that $u_\mu \in C([0,T_s];H^s)$ and $\partial_t u_\mu \in C([0,T_s];H^{s-4})$.

Theorem 2.3. *Let $u_\mu \in C([0,T];H^s)$, $s \ge 1$, be the solution to problem (III). Then $u_\mu \in C((0,T];H^r)$ and $\partial_t u_\mu \in C((0,T];H^q)$ for all $r \ge s$ and $q \ge s-4$. Furthermore, for every fixed $t \in (o,T]$, $u_\mu(t)$ is a continuously infinite differentiable function of x with the derivative of any order tending to zero as $|x| \to \infty$.*

Theorem 2.4. *Let $s > 2$ be an integer and $u_\mu(t) \in C([0,T];H^s)$ be the solution of problem (III) with $\mu \in (0,1)$. Then $u_\mu(t)$ can be extended to an interval $[0,T']$ independent of $\mu \in (0,1)$. Furthermore there exists a monotone positive-valued function $\rho(t) \in C([0,T'];R)$ such that $\|u_\mu(t)\|_s \le \rho(t)$ for $t \in [0,T']$ and $\rho(0)$ depends only on $\|\varphi\|_s$.*

The proofs of all the above theorems are similar to those of the counterparts in [1].

With the above theorems, we proceed as in [1] to obtain

Theorem 2.5. *Let $\varphi \in H^s$, $\|\varphi\|_0 < \sqrt{2}/3$ and s be an integer > 2. Then there exists $T > 0$ depending on s and $\|\varphi\|_s$ and $u_0 \in C([0,T];H^s)$ such that $\partial_t u_0 \in C([0,T];H^{s-3})$ and u_0 satisfies problem (I) locally, i.e.,*

$$\begin{cases} \partial_t u_0 = -\frac{\partial}{\partial x}(u_0^3 + 3u_0 H\partial_x u_0 + 3H(u_0\partial_x u_0) - 4\partial_x^2 u_0), \quad 0 < t \le T', \\ u_0(x,0) = \varphi(x). \end{cases}$$

§3. Proof of the Theorem

The main difficulty in the study of the global existence by the continuation method is to establish certain delicate a priori estimsates. For the sake of brevity, we merely give the demonstration of the a priori estimates. The detailed proof of the Theorem is omitted. To begin with we state the following Theorem 3.1 and its corollaries, whose proofs are not very difficult.

Theorem 3.1. *Let $\mu > 0$ be fixed, $s > 2$ be an integer, $u_\mu \in C([0,T); H^s)$ be the solution of problem* (III). *Then $\|u_\mu(t)\|_s \le M$, $t \in [0,T)$, where M is a positive constant depending on T but not on μ.*

Corollary 3.1. *Let $\mu > 0$ be fixed, $s > 2$ be an integer. Then for any $T > 0$, there exists a solution $u_\mu \in C([0,T]; H^s)$ satisfying problem* (III).

Corollary 3.2. *Under the conditions of Corollary 3.2, problem* (III) *has a solution $u_\mu \in C([0,\infty); H^s)$.*

Theorem 3.2. *Suppose that integer $s > 2$, $\mu > 0$ and $u_\mu \in C([0,\infty); H^s)$ is the solution of* (III). *Then*

$$\|u_\mu(t)\|_0^2 \le \|\varphi\|_0^2 = M_0$$

holds uniformly for μ.

Theorem 3.3. *Under the assumptions of Theorem 3.2, for any $T > 0$,*

$$|u_\mu(t)|_1^2 \le M_1 \text{ and } |u(t)|_2^2 \le M_2$$

hold uniformly on $[0,T]$ for μ, where M_1, M_2 are positive constants depending on T and the size of φ.

Proof. The proofs of the two inequalities are similar. So we only give that of the first inequality as an example.

For simplity, we write $u_\mu = u$. Multiplying equation by K_4, integrating the resulting expression in x and using Lemma 1.3 we obtain

$$\frac{d}{dt} I_4(u(t)) = -(K_4(u), \mu\partial_x^4 u)_0 = -4\mu|u|_3^2 + (K_4'(u), \mu\partial_x^4 u)_0, \tag{3.1}$$

where $K_4(u) = -(u^3 + 3uHu_x + 3H(uu_x))$. By using Lemma 1.1 and Young inequality and integrating by parts we have

$$|(K_4(u), \mu\frac{\partial}{\partial x}^4 u)_0| \le 3\mu|u|_3^2 + C|u|_1^2 + C,$$

where C is a constant independing of μ. Now integrating (3.1) in t yields

$$2|u(t)|_1^2 + \mu\int_0^t |u|_3^2 dt \le C + \int (\frac{u^4}{4} + \frac{3}{2}u^2 Hu_x) dx + C\int_0^t |u|_1^2 dt, \tag{3.2}$$

where C depends on the size of φ and T. Using Lemma 1.1 and Young inequality we have

$$|\int \frac{3}{2}u^2 Hu_x dx| \le \frac{1}{2}|u|_1^2 + C_1,$$

$$|\int \frac{u^4}{4} dx| \le \frac{1}{2}|u|_1^2 + C_2,$$

where C_1 and C_2 are constants independent of μ. From (3.2) we get

$$|u(t)|_1^2 + \mu\int_0^t |u|_3^2 dt \le C + C\int_0^T |u|_1^2 dt.$$

Gronwall's inequality gives

$$|u(t)|_1^2 \le M_1, \quad \text{for any } t \in [0,T].$$

The second inequality of the theorem can be proved in the same way as above by using the conserved quantity $I_6(u)$.

Theorem 3.4. *Under the conditions of Theorem 3.3, there exists an* $M_3 > 0$ *such that*

$$|u_\mu(t)|_3^2 \le M_3$$

holds on $[0,T]$ *uniformly for* $\mu \in (0,1)$ *and* M_3 *does not depend on* $\mu \in (0,1)$.

Proof. The proof of the theorem is very long. We shall divide it into several steps.

(1) Setting

$$I_8' = \int (\partial_x^3 u)^2 dx + K\int (\partial_x^2 u)^2 H u_x dx + l\int (u_x \partial_x^3 u H u_x) dx + n\int u\partial_x^3 u H \partial_x^2 u dx,$$

we shall consider $I = \int \frac{\delta I_8'}{\delta u}\partial_x \frac{\delta I_4}{\delta u} dx$, where $\frac{\delta I_4}{\delta u} = u^3 + 3uHu_x + 3H(uU_x) - 4\partial_x^2 u$ and

$$\begin{aligned}\frac{\delta I_8'}{\delta u} =& -2\partial_x^6 u + 2K(\partial_x^2 u H u_x)_{xx} + K(H(\partial_x^2 u)^2)_x - L(\partial_x^3 u H u_x)_x \\ &+ L(u_x H u_x)_{xxx} + LH(u_x\partial_x^3 u)_x + N\partial_x^3 u H \partial_x^2 u \\ &- N(uH\partial_x^2 u)_{xxx} + NH(u\partial_x^3 u)_{xx}.\end{aligned}$$

The constants K, L, N appearing in I_8' will be determined later. I_8' here seems to play the role of "8th conservation quantity" of the BO equations. We then see that

$$\begin{aligned} I =& \int [-6\partial_x^4 u(uHu_x)_{xxx} - 6\partial_x^4 u H(uu_x)_{xxx}]dx \\ &+ 4K\int [2\partial_x^4 u(u_{xx}Hu_x)_x + \partial_x^4 u H(\partial_x^2 u)^2]dx \\ &+ 4L\int [-\partial_x^4 u(\partial_x^3 u H u_x) - \partial_x^4 u(u_x H u_x)_{xx} + \partial_x^4 u H(u_x\partial_x^3 u)]dx \\ &+ (-4N)\int [\partial_x^3 u \partial_x^3 u H\partial_x^2 u + \partial_x^4 u(uH\partial_x^2 u)_{xx} + \partial_x^4 u H(u\partial_x^3 u)_x]dx \\ &+ 6\int [-(u^2u_x)_{xx}\partial_x^4 u]dx + 6K\int [(u^2u_x)_{xx}u_{xx}Hu_x]dx \\ &- 6K\int (u_{xx}Hu_x)_x(uHu_x)_{xx}dx - 6K\int (u_{xx}Hu_x)_x H(uu_x)_{xx}dx \\ &+ 3K\int u^2u_x H(u_{xx})_x^2 dx + 3K\int (uHu_x)_x(H(\partial_x^2 u)^2)_x dx \\ &+ 3K\int H(uu_x)_x(H(\partial_x^2 u)^2)_x dx + 3L\int (u^2u_x)_x\partial_x^3 u H u_x dx \\ &+ 3L\int (uHu_x)_{xx}\partial_x^3 u H u_x dx + 3L\int H(uu_x)_{xx}\partial_x^3 u H u_x dx \\ &+ 3L\int (u^2u_x)_x(u_xHu_x)_{xx}dx + 3L\int (uHu_x)_{xx}(u_xHu_x)_{xx}dx \\ &+ 3L\int H(uu_x)_{xx}(u_xHu_x)_{xx}dx - 3L\int (u^2u_x)_x H(u_x\partial_x^3 u)dx \end{aligned}$$

$$
\begin{aligned}
&- 3L\int(uHu_x)_{xx}H(u_x\partial_x^3u)dx - 3L\int H(uu_x)_{xx}H(u_x\partial_x^3u)dx\\
&+ 3N\int(u^2u_x)\partial_x^3uH\partial_x^2udx + 3N\int(uHu_x)_x\partial_x^3uH\partial_x^2udx\\
&+ 3N\int H(uu_x)_x\partial_x^3uH\partial_x^2udx - 3N\int(u^2u_x)_{xx}(uH\partial_x^2u)_xdx\\
&+ 3N\int(uHu_x)_{xx}(uH\partial_x^2u)_{xx}dx + 3N\int H(uu_x)_{xx}(uH\partial_x^2u)_{xx}dx\\
&- 3N\int(u^2u_x)_{xx}H(u\partial_x^3u)dx + 3N\int(uHu_x)_{xx}H(u\partial_x^3u)_xdx\\
&+ 3N\int H(uu_x)_{xx}H(u\partial_x^3u)_xdx\\
=&A_1 + A_2 + \cdots + A_{29}.
\end{aligned}
$$

For A_1, we have

$$
\begin{aligned}
A_1 =&6\int[\partial_x^4uuH\partial_x^4u + 3\partial_x^4uu_xH\partial_x^3u + 3\partial_x^4uu_{xx}Hu_{xx} + \partial_x^4u\partial_x^3uHu_x\\
&+ \partial_x^4uH(u\partial_x^4u) + 4\partial_x^4uH(u_x\partial_x^3u) + 3\partial_x^4uH(u_{xx})^2]dx\\
=&6\int[-1.5\partial_x^4uH(u_{xx})^2 + 7\partial_x^4uH(u_x\partial_x^3u) - 3.5(\partial_x^3u)^2Hu_{xx}\\
&+ 1.5\partial_x^4uH(u_{xx})^2 + 3\partial_x^4uH(u_{xx})^2]dx\\
=&- 42\int\partial_x^4uH(u_x\partial_x^3u)dx + 21\int(\partial_x^3u)^2Hu_{xx}dx.
\end{aligned}
$$

For A_2, we have

$$
\begin{aligned}
A_2 &= 4K\int[2\partial_x^4u\partial_x^3uHu_x + 2\partial_x^4uu_{xx}Hu_{xx} + \partial_x^4uH(u_{xx})^2]dx\\
&= 4K\int[-3(\partial_x^3u)^2Hu_{xx} - \partial_x^4uH(u_{xx})^2 + \partial_x^4uH(u_{xx})^2]dx\\
&= -12K\int(\partial_x^3u)^2Hu_{xx}dx.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
A_3 =&4L\int[-\partial_x^4u\partial_x^3uHu_x + \partial_x^4uH(u_x\partial_x^3u) - \partial_x^4u\partial_x^3uHu_x\\
&- 2\partial_x^4uu_{xx}Hu_{xx} - \partial_x^4uu_xHu_{xxx}]dx\\
=&12L\int(\partial_x^3u)^2Hu_{xx}dx - 6L\int(u_{xx})^2H\partial_x^4udx
\end{aligned}
$$

and

$$
\begin{aligned}
A_4 =& - 4N\int[(\partial_x^3u)^2Hu_{xx} + \partial_x^4uu_{xx}Hu_{xx} + 2\partial_x^4uu_xH\partial_x^3u\\
&+ \partial_x^4uuH\partial_x^4u + \partial_x^4uH(u\partial_x^4u) + \partial_x^4uH(u_x\partial_x^3u)]dx\\
=& - 6N\int(\partial_x^2u)^2H\partial_x^4udx - 12N\int\partial_x^4uH(u_x\partial_x^3u)dx
\end{aligned}
$$

Therefore,

$$A_1 + A_2 + A_3 + A_4 = -(12N+42)\int \partial_x^4 u H(u_x\partial_x^3 u)dx - (6L+6N)\int (u_{xx})^2 H\partial_x^4 u dx + (-12K+21+12L)\int (\partial_x^3 u)^2 H u_{xx} dx$$

Considering the algebraic equation

$$\begin{cases} 12N+42=0, \\ 6L+6N=0, \\ -12K+12L+21=0, \end{cases}$$

we get $N=-3.5,\ L=3.5,\ K=21/4$.

For A_5, we have

$$\begin{aligned} A_5 =& 6\{\int -uu_x(\partial_x^3 u)^2dx + 2\int[-u_x^2u_{xx}\partial_x^3u - u(u_{xx})^2\partial_x^3 u - uu_x(\partial_x^3u)^2]dx \\ &+\int[-\partial_x^3u\partial_x^3uu_xu - \partial_x^3uu_{xx}(u_x)^2 - 2\partial_x^3u(u_x)^2u_{xx} \\ &- \partial_x^3u(u_{xx})^2 - \partial_x^3u(u_x)^2u_{xx}]dx\} \\ \le& C_5|u|_3^2 + C_5'. \end{aligned}$$

In the sequel we denote by $P_j(u)$ the polynomial of $u, \partial_x u, \cdots, \partial_x^j u$ and in different conditions $P_j(u)$ may be different but $P_j(u)$ is always some polynomial of $u, \partial_x u, \cdots, \partial_x^j u$ with the highest power of $\partial_x^j u$ less than 2. If $\partial_x^j u H \partial_x^j u$ apperas as a multiplier in some term of $P_j(u)$, we also say the power of $\partial_x^j u$ is 2. Under concrete condition, $P_j(u)$ will be obviously known.

We are now about to estimate $A_6, \cdots, A_{29}$ as follows.

$$\begin{aligned} A_6 =& \frac{63}{2}\int (u^2u_x)_{xx}u_{xx}Hu_xdx \\ =& \frac{63}{2}\int u^2u_{xxx}u_xHu_xdx + \frac{63}{2}\int P_2(u)dx \\ \le& C_6|u|_3^2 + C_6'. \end{aligned}$$

$$\begin{aligned} A_7 =& -\frac{63}{2}\int (u_{xx}Hu_x)_x(uHu_x)_{xx}dx \\ =& -\frac{63}{2}\{\int \partial_x^3uHu_xuH\partial_x^3udx + 2\int \partial_x^3u(Hu_x)u_xHu_{xx}dx \\ &+ \int \partial_x^3u(Hu_x)u_{xx}Hu_xdx + \int u_{xx}Hu_{xx}uH\partial_x^3udx + \int P_2(u)dx\} \\ \le& \frac{63}{2}\{||u_x||_{L^\infty}||u||_{L^\infty}|u|_3^2 + 2||Hu_x||_{L^\infty}||u_x||_{L^\infty}|u|_2|u|_3 \\ &+ ||Hu_x||_{L^\infty}^2|u|_3|u|_2 + ||Hu_{xx}||_{L^\infty}||u_{xx}||_{L^\infty}|u|_0|u|_3 + \int P_2(u)dx\} \\ \le& C_7|u|_3^2 + C_7'. \end{aligned}$$

$$\begin{aligned} A_8 &= -\frac{63}{2}\int (u_{xx}Hu_x)_x H(uu_x)_{xx}dx \\ &= -\frac{63}{2}\int u_{xxx}(Hu_x)H(uu_x)_{xx}dx - \frac{63}{2}\int u_{xx}(Hu_{xx})H(uu_x)_{xx}dx \\ &\le C_8|u|_3^2 + C_8'. \end{aligned}$$

$$A_9 = \frac{63}{4}\int u^2u_x(H(u_{xx})^2)dx \le C_9|u|_3^2 + C_9'$$

$$A_{10} = \frac{63}{4}\int uH\partial_x^3uH(u_{xx})^2dx + \int P_2(u)dx \le C_{10}|u|_3^2 + C_{10}'.$$

$$A_{11} \le C_{11}|u|_3^2 + C_{11}'.$$

$$\cdots\cdots$$

$$A_{24} \le C_{24}|u|_3^2 + C_{24}'.$$

$$A_{27} \le C_{27}|u|_3^2 + C_{27}'.$$

$$\begin{aligned} A_{25} =& \frac{21}{2}\int u(H\partial_x^3u)uH\partial_x^4udx + \int_R (P_3(u) + P_2(u))dx \\ \le& \frac{21}{2}\left|\int uu_x(H\partial_x^3u)^2dx\right| + \int P_3(u) + P_2(u))dx \\ \le& C_{25}|u|_3^2 + C_{25}'. \end{aligned}$$

$$A_{29} \le C_{29}|u|_3^2 + C_{29}'.$$

$$\begin{aligned} A_{26} + A_{28} =& -\frac{21}{2}\Big[\int H(u\partial_x^3u)uH\partial_x^4udx + \int uH\partial_x^3uH(u\partial_x^4u)dx\Big] + \int P_3(u)dx \\ =& -\frac{21}{2}\Big[\int -H(u\partial_x^4u)uH\partial_x^3udx + \int H(u\partial_x^4u)uH\partial_x^3udx \\ &+ \int P_3(u)dx\Big] + \int P_3(u)dx \\ \le& C_{26}|u|_3^2 + C_{26}'. \end{aligned}$$

Finally we arrive at

$$|I| \le C|u|_3^2 + C'.$$

(2)

$$\begin{aligned} |u|_3^2 =& I_8' + \frac{21}{4}\int(\partial_x^2u)^2Hu_xdx - \frac{7}{2}\int u_x\partial_x^3uHu_xdx + \frac{7}{2}\int u\partial_x^3uH\partial_x^2udx \\ \le& I_8' + C_1 + C_2|u|_3 + C_3|u|_3 \\ \le& I_8' + \frac{1}{2}|u|_3^2 + C_4. \end{aligned}$$

Therefore,

$$|u|_3^2 \le 2I_8' + C_4.$$

(3) In this part we study

$$\frac{d}{dt}I_8' = -\mu\int\frac{\delta I_8'}{\delta u}\partial_x^4udx - \int\frac{\delta I_8'}{\delta u}\partial_x\frac{\delta I_4}{\delta u}dx.$$

We now estimate $-\int \frac{\delta I_8'}{\delta u}\partial_x^4 u dx$ as follows:

$$\begin{aligned}
-\int \frac{\delta I_8'}{\delta u}\partial_x^4 u dx \le & -2|u|_5^2 + 2K|\int(\partial_x^2 u H u_x)_{xx}\partial_x^4 u dx| \\
& + K|\int (H(\partial_x^2 u)^2)_x \partial_x^4 u dx| + L|\int(\partial_x^3 u H u_x)_x \partial_x^4 u dx| \\
& + L|\int (u_x H u_x)_{xxx}\partial_x^4 u dx| + |N|\int (uH\partial_x^2 u)_{xxx}\partial_x^4 u dx| \\
& + |N||\int H(u\partial_x^3 u)_{xx}\partial_x^4 u dx| \\
= & -2|u|_5^2 + B_1 + B_2 + \cdots + B_6.
\end{aligned}$$

$$\begin{aligned}
|B_1| \le & 2K|\int \partial_x^3 u(Hu_x)\partial_x^5 u dx| + 2K|\int \partial_x^2 u(Hu_{xx})\partial_x^5 u dx| \\
\le & 2\sqrt{2}K(|u|_1^{1/2}|u|_2^{1/2}|u|_3|u|_5 + |u|_2^{3/2}|u|_3^{1/2}|u|_5) \\
\le & j^{-1}|u|_5^2 + C_5|u|_3^2 + C_6,
\end{aligned}$$

where j is a positive integer to be determined later.

$$\begin{aligned}
|B_2| = & K|\int (H(u_{xx})^2)\partial_x^5 u dx| \le \sqrt{2}K|u|_2^{3/2}|u|_3^{1/2}|u|_5 \\
\le & j^{-1}|u|_5^2 + C_7|u|_3^2 + C_8.
\end{aligned}$$

$$|B_3| = L|\int \partial_x^3 u(Hu_x)\partial_x^5 u dx| \le j^{-1}|u|_5^2 + C_9|u|_3^2.$$

$$|B_4| = L|\int (u_x H u_x)_{xx}\partial_x^5 u dx| \le j^{-1}|u|_5^2 + C_{10}|u|_3^2 + C_{10}'.$$

$$\begin{aligned}
|B_5| = & |N||\int (uHu_{xx})_{xx}\partial_x^5 u dx| \\
\le & |N||\int uH\partial_x^4 u\partial_x^5 u dx| + |N||\int 2u_x H\partial_x^3 u\partial_x^5 u dx| + |N||\int u_{xx}Hu_{xx}\partial_x^5 u dx| \\
\le & |N|(||u||_{L^\infty}|u|_4|u|_5 + 2||u_x||_{L^\infty}|u|_3|u|_5 + ||u_{xx}||_{L^\infty}|u|_2|u|_5).
\end{aligned}$$

By using the inequality $|u|_4 \le C\varepsilon|u|_5 + C\varepsilon^{-1}|u|_3$ and taking $\varepsilon = j^{-1}(1 + C||u||_{L^\infty})|N|$, we have

$$|B_5| \le 2j^{-1}|u|_5^2 + C_{11}|u|_3^2 + C_{11}'.$$

Similarly, we know that

$$|B_6| \le 2j^{-1}|u|_5^2 + C_{12}|u|_3^2 + C_{12}'.$$

Therefore,

$$-\int \frac{\delta I_8'}{\delta u}\partial_x^4 u dx \le -2|u|_5^2 + 8j^{-1}|u|_5^2 + C_{13}|u|_3^2 + C_{13}'.$$

Taking $j = 4$, we get

$$-\int \frac{\delta I_8'}{\delta u}\partial_x^4 u dx \le C_{13}|u|_3^2 + C_{13}'.$$

So

$$\frac{d}{dt}I_8'(\mu) \le (\mu C_{13} + C)|u|_3^2 + \mu C_{13}' + C'.$$

If $\mu \in (0,1)$ we know that

$$I_8'(u) \le |I_8'(\varphi)| + C_{14}T + \int_0^t C_{15}|u|_3^2 dt$$

holds on $[0,T]$ uniformly for $\mu \in (0,1)$. Considering this inequality and the result of (2), we conclude that

$$|u|_3^2 \le C_{16} + \int_0^t C_{17}|u|_3^2 du,$$

where C_{16}, C_{17} are constants independent of μ. By using Gronwall's inequality, we know that

$$|u|_3^2 \le C_{16}e^{C_{17}t} \le C_{16}e^{C_{17}T} = M_3.$$

This completes the proof.

Up to now we are ready to prove the following general theorem about the uniform boundedness for $||u_\mu(t)||_s$. We shall write $u = u_\mu$ if no confusion occures. In showing the following Theorem 3.5 we employ a special technique.

Theorem 3.5. *Let $s > 2$ be an integer, $0 < \mu < 1$ and $u_\mu \in C([0,\infty); H^s)$ be the solution of problem* (II). *Then for any $T > 0$ there exists a positive constant M_s' such that $||u_\mu(t)||_s^2 \le M_s'$ holds on $[0,T]$ uniformly for $\mu \in (0,1)$, where M_s' is a positive constant independent of μ.*

Proof. Under the assumptions of the theorem, from the results of the theorems proved above we deduce that

$$|u|_0^2 \le M_0,\ |u|_1^2 \le M_1,\ |u|_2^2 \le M_2,\ |u|_3^2 \le M_3,$$

where $M_0,\ M_1,\ M_2,\ M_3$ are positive constants independent of $\mu \in (0,1)$. Therefore

$$||u||_3^2 \le M_0 + M_1 + M_2 + M_3 = M_3',$$

where M_3' is obviously a positive constant independent of $\mu \in (0,1)$, that is, for $s = 3$ we have proved the theorem. Our next task is to inductively prove that, for $s > 3$, $||u_\mu(t)||_s^2 \le M_s'$ holds on $[0,T]$ uniformly for $\mu \in (0,1)$ with M_s' positive and independent of μ. For this purpose first we would like to display in detail the proof for the uniform boundedness of $|u|_4^2$, for its proof owns generality in the induction. Now we consider

$$\frac{d}{dt}(|u|_4^2 + \frac{9}{2}\int u\partial_x^3 uH\partial_x^4 udx).$$

We estimate it as follows.

$$\begin{aligned}\frac{d}{dt}(|u|_4^2) =& -2\mu|u|_6^2 + 2\int \partial_x^5 u\partial_x^4 u^3 dx \\ &+ 6[\int \partial_x^5 u\partial_x^4(uHu_x)dx + \int \partial_x^5 u\partial_x^4(H(uu_x))dx] \\ =& -2\mu|u|_6^2 + E_1 + E_2.\end{aligned}$$

$$
\begin{aligned}
E_1 =& 2\int \partial_x^5 u \partial_x^4 u^3 dx = -C_1 \int (\partial_x^4 u)^2 u u_x dx + \int P_4(u) dx \\
\leq& C_2 |u|_4^2 + C_3. \\
E_2 =& 6[\int \partial_x^5 u \partial_x^4 (uHu_x) dx + \int \partial_x^5 u \partial_x^4 (H(uu_x)) dx] \\
=& 6[\int \partial_x^5 u u H \partial_x^5 u dx + \int \partial_x^5 u H(u \partial_x^5 u) dx + \int \partial_x^5 u \partial_x^4 u H u_x dx \\
&+ 4\int \partial_x^5 u u_x H \partial_x^4 u dx + 5\int \partial_x^5 u H(u_x \partial_x^4 u) dx] + \int P_4(u) dx \\
=& 54 \int \partial_x^5 u u_x U \partial_x^4 u dx + \int P_4'(u) dx \\
\leq& 54 \int \partial_x^5 u u_x H \partial_x^4 u dx + C_4 |u|_4^2 + C_5.
\end{aligned}
$$

Therefore,

$$
\frac{d}{dt}(|u|_4^2) \leq -2\mu |u|_6^2 + 54 \int \partial_x^5 u u_x H \partial_x^4 u dx + C_6 |u|_4^2 + C_7.
$$

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{9}{2}\int u \partial_x^3 u H \partial_x^4 dx\Big) =& \frac{9}{2}\mu[\int H(u\partial_x^3)_{xx} \partial_x^6 u dx \\
&+ \int (uH\partial_x^4 u)_x \partial_x^6 u dx - \int (\partial_x^3 u H \partial_x^4 u)_{xx} \partial_x^2 u dx] \\
&- \frac{9}{2}[\int (uH\partial_x^4 u) \partial_x^4 (u^3) dx + \int H(u\partial_x^3 u)_x \partial_x^4 (u^3) dx] \\
&- \int \partial_x^3 (H\partial_x^4 u) \partial_x (u^3) dx] - \frac{27}{2}[\int H(u\partial_x^3 u)_x \partial_x^4 (uHu_x) dx \\
&+ \int uH\partial_x^4 u \partial_x^4 (uHu_x) dx - \int \partial_x^3 u H \partial_x^4 u (uHu_x)_x dx] \\
&- \frac{27}{2}[\int H(u\partial_x^3 u)_x H(\partial_x^4 (uu_x)) dx + \int uH\partial_x^4 u H \partial_x^4 (uu_x) dx \\
&- \int \partial_x^3 u H \partial_x^4 u H(uu_x)_x dx] - 18[\int H(u\partial_x^3 u)_{xx} \partial_x^5 u dx \\
&+ \int (uH\partial_x^4 u)_x \partial_x^5 u dx + \int \partial_x^3 u H \partial_x^4 u \partial_x^3 u dx] \\
=& \mu F_1 + F_2 + F_3 + F_4 + F_5' + F_5''.
\end{aligned}
$$

$$
\begin{aligned}
F_5' =& -18[3\int \partial_x^5 u u_x H \partial_x^4 u dx + \int \partial_x^4 u u_{xx} H \partial_x^4 u dx \\
&- \int H(u_{xx}\partial_x^4 u) \partial_x^4 u dx - \int H(\partial_x^3 u)^2 \partial_x^4 u dx] \\
\leq& -54 \int \partial_x^5 u u_x H \partial_x^4 u dx + C_8 |u|_4^2 + C_9. \\
F_5'' =& 18 \int \partial_x^3 u H \partial_x^4 u \partial_x^3 u dx \leq C_{10} |u|_4^2 + C_{11}.
\end{aligned}
$$

Therefore,

$$
F_5' + F_5'' \leq -54 \int \partial_x^5 u u_x H \partial_x^4 u dx + C_{12} |u|_4^2 + C_{13}.
$$

In F_4, we have

$$-\int H(u\partial_x^3 u)_x H\partial_x^4(uu_x)dx = \int (uu_x)(\partial_x^4 u)^2 dx + \int P_4(u)dx,$$

$$-\int uH\partial_x^4 uH\partial_x^4(uu_x)dx = -\int uH\partial_x^4 uH(u\partial_x^5 u)dx + \int P_4(u)dx,$$

$$= \int uH\partial_x^5 uH(u\partial_x^4 u)dx + \int P_4'(u)dx$$

and

$$\int \partial_x^3 uH\partial_x^4 uH(uu_x)_x dx \le C_{14}|u|_4^2 + C_{15}.$$

Therefore,

$$F_4 \le \frac{27}{2}\int uH\partial_x^5 uH(u\partial_x^4 u)dx + C_{16}|u|_4^2 + C_{17}.$$

In F_3, we have

$$-\int H(u\partial_x^3 u)_x \partial_x^4(uHu_x)dx = -\int uH\partial_x^5 uH(u\partial_x^4 u)dx + \int P_4(u)dx,$$

and

$$-\int uH\partial_x^4 u\partial_x^4(uHu_x)dx = -\int u^2(H\partial_x^4 u)H\partial_x^5 udx + \int P_4(u)dx.$$

We then know that

$$\begin{aligned}F_3 =& -\frac{27}{2}[\int H(u\partial_x^3 u)_x\partial_x^4(uHu_x)dx + \int uH\partial_x^4 u\partial_x^4(uHu_x)dx \\ & -\int \partial_x^3 uH\partial_x^4 u(uHu_x)_x dx] \\ \le & -\frac{27}{2}\int uH\partial_x^5 uH(u\partial_x^4 u)dx + C_{18}|u|_4^2 + C_{19}. \\ F_2 =& -\frac{9}{2}[\int (uH\partial_x^4 u)\partial_x^4(u^3)dx + \int H(u\partial_x^4 u)_x\partial_x^4(u^3)dx \\ & -\int \partial_x^3 u(H\partial_x^4 u)\partial_x(u^3)dx] = \int P_4(u)dx \le C_{20}|u|_4^2 + C_{21}.\end{aligned}$$

In F_1, we have

$$\begin{aligned}\int H(u\partial_x^3 u)_{xx}\partial_x^6 udx =& \int H(u\partial_x^5 u)\partial_x^6 udx + 2\int H(u_x\partial_x^4 u)\partial_x^6 udx \\ & + \int H(u_{xx}\partial_x^3 u)\partial_x^6 udx \le q^{-1}|u|_6^2 + C_{22}|u|_4^2 + C_{23},\end{aligned}$$

$$\begin{aligned}-\int (\partial_x^3 uH\partial_x^4 u)_{xx}\partial_x^2 udx =& \frac{1}{2}\int (\partial_x^3 u)^2 H\partial_x^5 udx \\ \le & q^{-1}|u|_6^2 + C_{24}|u|_4^2 + C_{25},\end{aligned}$$

and

$$\int (uH\partial_x^4 u)_x\partial_x^6 udx \le q^{-1}|u|_6^2 + C_{26}|u|_4^2 + C_{27},$$

where q is a positive integer to be determined later. Therefore

$$\mu F_1 \le \mu\frac{27}{2q}|u|_6^2 + C_{28}|u|_4^2 + C_{29}.$$

Taking $q = 14$ we obtain

$$\frac{d}{dt}(\frac{9}{2}\int u\partial_x^3 uH\partial_x^4 udx) \leq \mu|u|_6^2 - 54\int u_x\partial_x^5 uH\partial_x^4 udx + C_{30}|u|_4^2 + C_{31}.$$

Considering the above inequality and that of (1) we know

$$\frac{d}{dt}(|u|_4^2 + \frac{9}{2}\int u\partial_x^3 H\partial_x^4 udx) \leq C_{32}|u|_4^2 + C_{33},$$

where positive constants C_{32}, C_{33} are obviously independent of μ. Integrating with respect to t we get

$$|u|_4^2 + \frac{9}{2}\int u\partial_x^3 uH\partial_x^4 udx \leq \int_0^t C_{32}|u|_4^2 dt + C_{34},$$

where $C_{34} > 0$ is dependent on T and ϕ. On the other hand, we have

$$\frac{9}{2}|\int u\partial_x^3 uH\partial_x^4 udx| \leq \frac{1}{2}|u|_4^2 + C_{35}.$$

Consequently,

$$|u|_4^2 \leq 2\int_0^t C_{32}|u|_4^2 dt + C_{36}.$$

By using Gronwall's inequality we conclude that

$$|u|_4^2 \leq M_4,$$

where M_4 is independent of μ. This completes the proof of the theorem for $s = 4$. By induction we can prove the theorem for general s. In fact, suppose that we have proved the theorem for i, $4 \leq i \leq s-1$, i.e., there exists a positive constant M_i' independent of $\mu \in (0,1)$ such that

$$|u|_i^2 \leq M_i'. \quad t \in [0,T], \text{ uniformly for } \mu \in (0,1).$$

Considering $\frac{d}{dt}(|u|_{i+1}^2 + \frac{2i+3}{2}\int u\partial_x^i uH\partial_x^{i+1}udx)$, similar to the proof of $|u|_4^2$ we can show there exists $M_{i+1}' > 0$ independent of $\mu \in (0,1)$ such that

$$|u|_{i+1}^2 \leq M_{i+1}', \quad t \in [0,T], \quad \text{uniformly for } \mu \in (0,1).$$

This finishes the proof.

Proof of Theorem. Combining the local existence result with the result of Theorem 3.5 and using the technique introduced in [4] (see also [5]) we can easily establish the existence part of the Theorem. It suffices to show the uniqueness. For this purpose, we write $w = u - v$, where u and v are the solutions to problem (I) with the same initial data φ. Considering the difference equation of equations and the provious bounds about u and v we can obtain

$$\frac{d}{dt}(||w||_1^2 + \frac{3}{2}\int vwH\partial_x wdx) \leq C||w||_1^2.$$

Using the restriction on φ and Lemma 1.1 we have

$$(1-\beta)||w||_1^2 \leq ||w||_1^2 + \frac{3}{2}\int vwH\partial_x wdx \leq (1+\beta)||w||_1^2$$

for some $\beta \in [0,1)$. Therefore

$$\frac{d}{dt}(||w||_1^2 + \frac{3}{2}\int vwH\partial_x wdx) \leq C(||w||_1^2 + \frac{3}{2}\int vwH\partial_x wdx).$$

Gronwall's inequality yields

$$(1-\beta)||w||_1^2 \le ||w||_1^2 + \frac{3}{2}\int vwH\partial_x w dx \le 0,$$

$t \in [0,T]$ for all T, which implies that $u = v$. This ends the proof.

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