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AN INVARIANCE PRINCIPLE FOR STATIONARY ρ-MIXING SEQUENCES WITH INFINITE VARIANCE**

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Abstract

This paper establishes an invariance principle for stationary ρ-mixing sequences under the assumption $EX_0^2g(X_0) = \infty$ for some continuous nondecreasing function $g(x)$. In particular, under the infinite variance assumption, the result improves the theorems of Bradley (1988) and Shao (1989).

§1. Introduction

Suppose $(X_k, k \in Z)$ is a strictly stationary sequence of real-valued random variables on a probability space $(\Omega, \mathcal{F}, p)$. For $-\infty \le m \le n \le \infty$ let $\mathcal{F}_m^n$ denote the σ-field of events generated by the random variables $(X_k, m \le k \le n)$. For each natural $n \ge 1$ define the dependence coefficient

$$\rho(n) := \sup |\text{Corr}(f, g)|$$
$$\text{real } f \in L_2(\mathcal{F}_{-\infty}^0), \text{ real } g \in L_2(\mathcal{F}_n^\infty).$$

The stationary sequence $(X_k, k \in Z)$ is said to be ρ-mixing if $\rho(n) \to 0$ as $n \to \infty$.

Ibragimov (1975) showed that for some stationary ρ-mixing sequence of random variables with finite variance, the partial sums are attracted to a normal law under the assumption $\sum_n \rho(2^n) < \infty$. Peligrad (1987) considered the more general case and obtained the central limit theorem under assumptions $EX_0^2g(X_0) < \infty$ and

$$g(n^{\frac{1}{2}}) >> \exp((2+\varepsilon)\sum_{k=1}^{[\log n]} \rho(2^k))$$

for some increasing function $g(x)$ and $\varepsilon > 0$. Shao (1989) proved that the weak invariance principle also holds under the same hypothesis. Recently, Bradley(1988) established the central limit theorem for some strictly stationary ρ-mixing sequences under infinite variance assumption, which extended the classic result for i.i.d.r.v.'s. The purpose of this paper is to establish the invariance principle under infinite variance, even more general case.

In the statement of our main result we shall use the following notations: log denotes the logarithm with base 2. The notation $a_n \sim b_n$ will mean $\lim_{n\to\infty} a_n/b_n = 1$, and the notation $a_n << b_n$ will mean $a_n = O(b_n)$. The greatest integer $\le x$ will be denoted by $[x]$. $(W(t), 0 \le t \le 1)$ will denote the standard Wiener process. The partial sums of our given

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sequence (X_k) will be denoted by $S_n := X_1 + \cdots + X_n$. Given a function $f : [0, \infty) \to [0, \infty)$ such that $f(x) > 0$ for all x sufficiently large, we shall say that $f(x)$ is slowly varying as $x \to \infty$ if $\forall t > 0$, $\lim_{x \to \infty} f(tx)/f(x) = 1$. Throughout this paper we shall assume that

$$g : (-\infty, \infty) \to [0, \infty) \text{ is an increasing continuous even function and for every } \delta > 0,\ x^\delta / g(x) \text{ is increasing for } x \text{ sufficiently large.} \tag{1.0}$$

Let

$$e(x, \varepsilon) := \exp(\varepsilon \sum_{k=0}^{[\log x]} \rho(2^k)) \text{ and } x_\delta := \exp(\sum_{k=0}^{[\log x]} \rho^{1-\delta}(2^k))$$

for every $x > 0$. Here, and in sequel, $\log x$ means $\log(\max(x, e))$.

Our main result is as follows:

Theorem 1.1. *Suppose $(X_k, k \in Z)$ is a strictly stationary sequence of non-degenerate real-valued random variables. Suppose that*

$$H(x) := EX_0^2 I(|X_0| \le x) \text{ and } G(x) := EX_0^2 g(X_0) I(|X_0| \le x) \tag{1.1}$$

are slowly varying as $x \to \infty$,

$$EX_0 = 0, \tag{1.2}$$

$$\rho(1) < 1, \tag{1.3}$$

$$H(x)g(x) >> G(x)e(x^2, 2 + \varepsilon^*) \text{ for some } 0 < \varepsilon^* < 1, \text{ and} \tag{1.4}$$

$$g(x) << g(x/x_\delta), \text{ or} \tag{1.5a}$$

$$G(x) << G(x/x_\delta), \text{ for some } 0 < \delta < 1. \tag{1.5b}$$

Then there exists a sequence $(A_n, n \in N)$ of positive numbers with $A_n \to \infty$ as $n \to \infty$, such that

$$W_n(t) \Rightarrow W(t)$$

as $n \to \infty$, where $W_n(t) := S_{[nt]}/A_n (0 \le t \le 1)$.

Corollary 1.1. *Suppose $(X_k, k \in Z)$ is a strictly stationary sequence of non-degenerate real-valued random variables. Suppose that (1.2) and (1.3) are satisfied and that*

$$H(x) \text{ is slowly varying as } x \to \infty. \tag{1.1)*}$$

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty. \tag{1.4)*}$$

Then there exists a sequence $(A_n, n \in N)$ of positive numbers with $A_n \to \infty$ as $n \to \infty$, such that $W_n(t) \Rightarrow W(t)$ as $n \to \infty$.

It is well-known that the mixing rate (1.4)* is essentially sharp, even in the case of finite second moments. However, the following result is very interesting: suppose X_0 in Theorem 1.1 has density function

$$p(x) = c(1 + |x|^3)^{-1}$$

for $x \in R^1$, where $c^{-1} = \int_{-\infty}^{\infty} (1 + |x|^3)^{-1} dx$. Let $g(x) = \exp(\log(1 + |x|^3)^\alpha)$ for some $0 < \alpha < 1$. It is easy to see that as $x \to \infty$

$$H(x) \sim 2c \log(1 + |x|^3)/3,$$

$$G(x) \sim 2c(\log(1+|x|^3))^{1-\alpha}g(x)/(3\alpha).$$

Suppose that for some $a < \frac{1}{2}\alpha$ and every n sufficiently large

$$\rho(n) \le a/(2\log n). \tag{1.4}^{**}$$

Then we can easily verify that the conditions in Theorem 1.1 are satisfied but the condition (1.4)* in Corollary 1.1 fails. Hence, we say that the condition (1.4)* may be not essentially sharp in some particular case of infinite variance, even of finite variance.

Theorem 1.1 is an extension of Theorem from Shao (1989) except for (1.3) and will be proved in Section 3.

The following notations will be used: Terms like a_b will be written as $a(b)$ when that is needed for typogrophical convenience. Notations $a \wedge b$ and $a \vee b$ will mean $\min(a,b)$ and $\max(a,b)$, respectively. The norm in L_p will be denoted by $\|\cdot\|_p (p \ge 1)$. The capital letter K will denote a constant that may be different even in the same equation.

§2. Preliminaries

In this section we shall give some lemmas that will be used in Section 3 in the proof of Theorem 1. Lemmas 2.1 and 2.2 below are the general cases of Peligrad [(1987), Lemma 1] and Bradley [(1988), Lemma 2.3].

Lemma 2.1. *Suppose $(r(n), n \in N)$ is a non-increasing sequence of non-negative numbers such that $\lim_{n\to\infty} r(n) = 0$. Then, for every $\varepsilon > 0$, there exists a positive constant $D = D(\varepsilon, r(\cdot))$ such that the following holds:*

For every sequence $(Y_k, k \in Z)$ of square-integrable random variables such that the condition

$$\forall n \ge 1, \ \rho(n) \le r(n)$$

holds, one has that

$$\forall n \ge 1, \mathrm{Var}(Y_1 + \cdots + Y_n) \le Dne(n, 1+\varepsilon) \max_{1\le k\le n} \mathrm{Var}(Y_k). \tag{2.1}$$

Lemma 2.2. *Suppose $(r(n), n \in N)$ is a non-increasing sequence of non-negative numbers such that $r(1) < 1$ and $\lim_{n\to\infty} r(n) = 0$. Then, for every $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon, r(\cdot))$ such that the following holds:*

For every strictly stationary sequence $(X_k, k \in Z)$ of square-integrable random variables such that the condition

$$\forall n \ge 1, \ \rho(n) \le r(n)$$

holds, one has that

$$\forall n \ge 1, \ \mathrm{Var}(S_n) \ge Cne(n, -1-\varepsilon)\mathrm{Var}(X_0). \tag{2.2}$$

Lemma 2.3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two σ-fields, V and W are real-valued random variables such that $V \in L_p(\mathcal{A})$ and $W \in L_q(\mathcal{B})$ for some $1/p + 1/q = 1$, $p > 1$ and $q > 1$. Then*

$$|EVW - EV\ EW| \le 14\rho(\mathcal{A}, \mathcal{B})^{2/p\wedge 2/q}\|V\|_p\|W\|_q. \tag{2.3}$$

Proof. Without loss of generality, assume $p < q$. Let

$$V_1 = VI(|V| \le c) - EVI(|V| \le c) \text{ and } V_2 = VI(|V| > c) - EVI(|V| > c),$$

where $c = \|V\|_p \rho^{-2/p}$ and $\rho := \rho(\mathcal{A}, \mathcal{B})$. Then

$$|EVW - EV\ EW| \le |EV_1W - EV_1\ EW| + |EV_2W - EV_2\ EW|.$$

By the definition of $\rho(\mathcal{A}, \mathcal{B})$

$$|EV_1W - EV_1\ EW| \le \rho\|V_1\|_2\|W\|_2 \le 2\rho c^{(2-p)/2}\|V\|_p^{p/2} \cdot \|W\|_q,$$

$$\begin{aligned}|EV_2W| \le &(E|V_2|^p)^{1-2/q}(E|V_2|^{p/2} \cdot |W|^{q/2})^{2/q} \\ \le &(E|V_2|^p)^{1-2/q}(E|V_2|^{p/2}E|W|^{q/2} + \rho \cdot (E|V_2|^pE|W|^q)^{1/2})^{2/q} \\ \le &8c^{-p/q}\|V\|_p^p\|W\|_q + 2\rho^{2/q}\|V\|_p\|W\|_q \\ \le &10\rho^{2/q}\|V\|_p\|W\|_q.\end{aligned}$$

And

$$|EV_2\ EW| \le 2c^{1-p}\|V\|_p^p\|W\|_q = 2\rho^{2/q}\|V\|_p\|W\|_q.$$

This proves that (2.3) holds.

Lemma 2.4. *Suppose $(X_k, k \in Z)$ is a strictly stationary ρ-mixing sequence of random variables with $EX_0 = 0$ and $E|X_0|^{2+\delta} < \infty$ for some $0 < \delta < 1$. If there exists a constant θ with $0 < \theta < 2 - 2^{2/(2+\delta)}$ such that for every $n \ge n_0$*

$$\max(\sigma_{n_1}^2, \sigma_{n_2}^2) \le (2-\theta)^{-1}\sigma_n^2, \tag{2.4}$$

where $n_1 = [\frac{1}{2}n]$, $n_2 = n - n_1$, $\sigma_n^2 := ES_n^2$, then there exists a constant $K = K(n_0, \theta, \rho(\cdot))$ such that for every $n \ge 1$

$$E|S_n|^{2+\delta} \le K(n \exp(560 \sum_{k=0}^{[\log n]} \rho(2^k)^{2/(2+\delta)})E|X_0|^{2+\delta} + \sigma_n^{2+\delta}). \tag{2.5}$$

Proof. Let $c = \frac{1}{2} + (2-\theta)^{-(2+\delta)/2}$. Fix a natural m_0 such that

$$2(1 + 252\rho(m_0)^{2/(2+\delta)} + 488 \log^{-2} m_0)(2-\theta)^{-(2+\delta)/2} < c.$$

For every $n \le 2n_0 \vee m_0^2 \vee 2^{17}$, (2.5) obviously holds for $K = K_0 := (2n_0 \vee m_0^2 \vee 2^{17})^{2+\delta}$. For $n \ge 2n_0 \vee m_0^2 \vee 2^{17}$, let $n_1 = [\frac{1}{2}n]$, $n_2 = n - n_1$, $n_3 = [n^{\frac{1}{2}}]$ and $S_k(n) = \sum_{i=1}^{n} X_{i+k}$. By a trivial inequality

$$(1+x)^{2+\delta} \le 1 + 9x + 9x^{1+\delta} + x^{2+\delta}$$

for every $x \ge 0$ and $0 < \delta < 1$, we have that

$$\begin{aligned}E|S_n|^{2+\delta} \le &E|S_{n_1}|^{2+\delta} + E|S_{n_1}(n_2)|^{2+\delta} + \\ &+ 9E|S_{n_1}||S_{n_1}(n_2)|^{1+\delta} + 9E|S_{n_1}(n_2)||S_{n_1}|^{1+\delta}.\end{aligned}$$

By (2.3) and (2.4)

$$\begin{aligned}E|S_{n_1}||S_{n_1}(n_2)|^{1+\delta} &\le E|S_{n_1-n_3}||S_{n_1}(n_2)|^{1+\delta} + E|S_{n_1-n_3}(n_3)||S_{n_1}(n_2)|^{1+\delta} \\ &\le \sigma_{n_1-n_3}\sigma_{n_2}^{1+\delta} + 14\rho(n_3)^{2/(2+\delta)}\|S_{n_1-n_3}\|_{2+\delta}\|S_{n_2}\|_{2+\delta}^{1+\delta} \\ &\quad + \|S_{n_3}\|_{2+\delta}\|S_{n_2}\|_{2+\delta}^{1+\delta} \\ &\le \sigma_n^{2+\delta} + 14\rho(n_3)^{2/(2+\delta)}(\|S_{n_1}\|_{2+\delta}^{2+\delta} + \|S_{n_2}\|_{2+\delta}^{2+\delta}) \\ &\quad + 16\log^{-2} n\|S_{n_2}\|_{2+\delta}^{2+\delta} + 16\log^4 n\|S_{n_3}\|_{2+\delta}^{2+\delta}.\end{aligned}$$

The above similar inequality also holds for $E|S_{n_1}(n_2)||S_{n_1}|^{1+\delta}$. Hence

$$E|S_n|^{2+\delta} \le (E|S_{n_1}|^{2+\delta} + E|S_{n_2}|^{2+\delta})(1 + 288\log^{-2} n + 252\rho(n_3)^{2/(2+\delta)}) + 18\sigma_n^{2+\delta} + 288E|S_{n_3}|^{2+\delta}\log^4 n. \tag{2.6}$$

Let $(K_n, n \in N)$ and $(J_n, n \in N)$ be two increasing sequences of positive numbers with $J_n \wedge K_n \ge K_0$ such that for every $n \ge 1$

$$E|S_n|^{2+\delta} \le K_n nE|X_0|^{2+\delta} + J_n\sigma_n^{2+\delta}. \tag{2.7}$$

For $n > 2n_0 \vee m_0^2 \vee 2^{17}$, by (2.6), (2.7) and (2.4)

$$\begin{aligned} E|S_n|^{2+\delta} \le{}& K_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 288(\log^{-2} n + n^{-1/2}\log^4 n))nE|X_0|^2 \\ &+ J_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 288(\log^{-2} n)(\sigma_{n_1}^{2+\delta} + \sigma_{n_2}^{2+\delta}) \\ &+ 288J_{n_2}\log^4 n\sigma_{n_3}^{2+\delta} + 18\sigma_n^{2+\delta}. \end{aligned}$$

By (2.4), $\forall n \ge n_0$

$$\sigma_{[n/2]}^2 \le (2-\theta)^{-1}\sigma_n^2 \le 2^{-2/5}\sigma_n^2.$$

Hence, by recurrence, we can get that

$$\sigma_{n_3} \le n^{-1/7}\sigma_n.$$

Taking into account that $n \ge 2^{17}$ we finally obtain that

$$\begin{aligned} E|S_n|^{2+\delta} \le{}& K_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 488\log^{-2} n)nE|X_0|^{2+\delta} \\ &+ (18 + 2J_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 488\log^{-2} n)(2-\theta)^{-(2+\delta)/2})\sigma_n^{2+\delta}. \end{aligned}$$

Hence, we can define

$$K_n = K_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 488\log^{-2} n) \tag{2.8}$$

and

$$J_n = 2J_{n_2}(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 488\log^{-2} n)(2-\theta)^{-(2+\delta)/2} + 18. \tag{2.9}$$

Noting that

$$2(1 + 252\rho(n^{1/2})^{2/(2+\delta)} + 488\log^{-2} n)(2-\theta)^{-(2+\delta)/2} < c < 1,$$

we get that for every n, by (2.9)

$$J_n \le \max(K_0, 18/(1-c)). \tag{2.10}$$

By (2.8), it is easy to see that

$$K_n \le K_0 \exp(560 \sum_{k=0}^{[\log n]} \rho(2^k)^{2/(2+\delta)}) \exp(488), \tag{2.11}$$

(2.5) now follows from (2.10) and (2.11), as desired.

Lemma 2.5. *Suppose that $H(x)$ is slowly varying as $x \to \infty$ and that C_n and D_n are two sequences of positive numbers with $C_n \to \infty$ and $D_n \to \infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} C_n H(D_n)/H(C_n D_n) = \infty. \tag{2.12}$$

Proof. By the well-known property on slowly varying function (e.g., see [5]), $\forall 0 < \varepsilon < \frac{1}{4}$, there is a positive x_0 such that for every $x \ge x_0$

$$\sup_{x\le y\le 2x} H(y)/H(x) \le 1 + \varepsilon.$$

So there is an integer n_0 such that for every $n \ge n_0$ and every $k \ge 1$

$$\sup_{2^{k-1}\le x\le 2^k} H(xD_n)/H(2^{k-1}D_n) \le 1+\varepsilon.$$

Note that

$$H(C_nD_n)/H(D_n) = H(C_nD_n)H(2^{[\log C_n]}D_n)^{-1}\prod_{i=1}^{[\log C_n]} H(2^iD_n)/H(2^{i-1}D_n).$$

It follows that for $n \ge n_0$

$$H(C_nD_n)/H(D_n) \le (1+\varepsilon)^{1+[\log C_n]} \le 4C_n^{2\varepsilon}.$$

Now the lemma follows from the fact that $C_n \to \infty$ as $n \to \infty$.

In the proof of Theorem 1.1 the following result proved by Bradley ((1988), Theorem 1) plays an important role.

Lemma 2.6. *Suppose $(X_k, k \in Z)$ is a strictly stationary sequence of non-degenerate real-valued random variables satisfying the conditions of Corollary 1. Then $S_n/a_n \to N(0,1)$ in distribution as $n \to \infty$, where $a_n^2 = \mathrm{Var}(\sum_{i=1}^n X_iI(|X_i| \le s_n))$ and $s_n^2 = nH(s_n)$.*

§3. Proof of Theorem 1.1

Throughout this section suppose (X_k) is a strictly stationary sequence satisfying the hypotheses of Theorem 1.1.

It follows from (1.1) that $\lim_{x\to\infty} x^{-2}g^{-1}(x)G(x) = 0$. Let M^* be a positive integer such that

$$\sup_{x>0} x^{-2}g^{-1}(x)G(x) > 1/M^*.$$

For each $n \ge M^*$ define the positive number

$$t_n := \sup(x>0 : x^{-2}g^{-1}(x)G(x) \ge 1/n). \tag{3.1}$$

It is clear that

$$t_n \to \infty \text{ monotonically as } n \to \infty. \tag{3.2}$$

Note by a trivial argument that

$$\forall n \ge M^*, \quad t_n^2 g(t_n) = nG(t_n). \tag{3.3}$$

Remark. If $\sum_{i=1}^\infty \rho(2^i) < \infty$, then, w.l.o.g., $g(x)$ in (3.1) is assumed to be equal to 1. By (1.0), (3.3) and the fact that $G(x)$ is slowly varying

$$\forall 0<\varepsilon<\frac{1}{2}, \quad n^{1-\varepsilon} << t_n^2 << n^{1+\varepsilon}. \tag{3.4}$$

Hence, we have

$$\forall \varepsilon > 0, \quad e(n,1) << e(t_n^2, 1+\varepsilon). \tag{3.5}$$

Lemma 3.1. *As $n \to \infty$*

$$G(t_n)(g(t_n/t_n(\delta))H(t_n))^{-1} << e(n, -2-\frac{1}{2}\varepsilon^*). \tag{3.6}$$

Proof. If (1.5a) is satisfied, then (3.6) follows from (1.5a), (1.4) and (3.5). If (1.5b) is satisfied, then by (1.4)

$$G(t_n)/(H(t_n)g(t_n/t_n(\delta))) << G(t_n/t_n(\delta))/(H(t_n/t_n(\delta))g(t_n/t_n(\delta)))$$
$$<< e((t_n/t_n(\delta))^2, -2-\varepsilon^*).$$

It is easy to see that

$$\forall \varepsilon > 0, \quad t_n(\delta) << n^\varepsilon.$$

Hence

$$\forall \varepsilon > 0, \quad n^{1-\varepsilon} << (t_n/t_n(\delta))^2 << n^{1+\varepsilon}$$

and

$$\forall \varepsilon > 0, \quad e(n,1) << e((t_n/t_n(\delta))^2, 1+\varepsilon).$$

It follows that

$$G(t_n)/(H(t_n)g(t_n/t_n(\delta))) << e(n, -2-\frac{1}{2}\varepsilon^*)$$

as desired.

Lemma 3.2. *As* $n \to \infty$

$$\exp(600\delta^{-1}\sum_{i=0}^{[\log n]}\rho(2^i)^{2/(2+\delta)}) << t_n(\delta). \tag{3.7}$$

Proof. It suffices to prove that if $t_n(\delta) \to \infty$, then

$$\sum_{i=0}^{[\log n]}\rho(2^i)^{2/(2+\delta)} = o(\sum_{i=0}^{[\log t_n]}\rho(2^i)^{1-\delta}). \tag{3.8}$$

Note that for every natural k

$$\sum_{i=0}^{[\log n]}\rho(2^i)^{2/(2+\delta)} \le \sum_{i=0}^{[\log t(n)]}\rho(2^i)^{2/(2+\delta)} + \rho(t_n)^{2/(2+\delta)}\log n$$
$$\le k + \rho(2^k)^{\delta/2}\sum_{i=0}^{[\log t(n)]}\rho(2^i)^{1-\delta} + 3\rho(t_n)^{2/(2+\delta)}\log t_n$$
$$\le k + (\rho(2^k)^{\delta/2} + 3\rho(t_n)^{\delta/2})(\sum_{i=0}^{[\log t_n]}\rho(2^i)^{1-\delta}).$$

(3.8) now follows from the fact that $t_n(\delta) \to \infty$ and $\rho(2^n) \to 0$ as $n \to \infty$.

Lemma 3.3. *If* $\sum_{i=1}^{\infty}\rho(2^i) < \infty$, *then for every* $0 < t \le 1$

$$t_{[nt]}^2/t_n^2 \to t, \text{ as } n \to \infty. \tag{3.9}$$

Proof. In this case, we have for every $n \ge M^*$ that by (3.3)

$$t_n^2 = nH(t_n). \tag{3.10}$$

It follows that

$$t_n^2 H(t_{[nt]})/(t_{[nt]}^2 H(t_n)) \to 1/t, \text{ as } n \to \infty. \tag{3.11}$$

We now show that

$$\limsup_{n\to\infty} t_n^2/t_{[nt]}^2 < \infty. \tag{3.12}$$

In fact, if $\limsup_{n\to\infty} t_n^2/t_{[nt]}^2 = \infty$, i.e., there is a subsequence n_k such that $\lim_{k\to\infty} t_{n_k}^2/t_{[n_k t]}^2 = \infty$, then by Lemma 2.5

$$t_{n_k}^2 H(t_{[n_k t]})/(t_{[n_k t]}^2 H(t_{[n_k t]} t_{n_k}/t_{[n_k t]})) \to \infty,$$

which is contrary to (3.11). By (3.12), there is a positive constant M such that

$$t_n/t_{[nt]} \le M, \text{ for every } n \ge 1.$$

By (3.10), it follows that

$$n/[nt] \le t_n^2/t_{[nt]}^2 \le nH(Mt_{[nt]})/([nt]H(t_{[nt]})),$$

which implies (3.9) by the fact $\lim_{n\to\infty} H(Mt_{[nt]})/H(t_{[nt]}) = 1$ and $\lim_{n\to\infty} n/[nt] = 1/t$.

The following lemma is similar to that of Bradley (1988).

Lemma 3.4.

$$\lim_{n\to\infty} nP(|X_0| \ge t_n) = 0, \text{ and} \tag{a}$$

$$\lim_{n\to\infty} (ng(t_n)G^{-1}(t_n))^{\frac{1}{2}} E|X_0|I(|X_0| \ge t_n) = 0. \tag{b}$$

Next by the trivial fact that $E|X_0| < \infty$ and $EX_0 = 0$, we have

$$\text{Var}X_0 I(|X_0| \le t_n) \sim H(t_n) \text{ as } n \to \infty. \tag{3.13}$$

Define the positive constants

$$C := C(r(\cdot), \varepsilon_1 = \varepsilon^*/6) \text{ and } D := D(r(\cdot), \varepsilon_1)$$
$$\text{from Lemmas 2.1 and 2.2 with } r(n) := \rho(n),\ \forall n \in N. \tag{3.14}$$

Define the following random variables:

$$\forall n \ge M^*,\ k \in Z,\ X_k^{(n)} := X_k I(|X_k| \le t_n) - EX_k I(|X_k| \le t_n); \text{ and}$$

$$\forall n \ge M^*,\ m \in N,\ S_m^{(n)} := X_1^{(n)} + \cdots + X_m^{(n)}.$$

Note that by the definition of C and D in (3.14)

$$\forall n \ge M^*,\ \forall m \in N,\ Cme(m, -1-\varepsilon_1)\|X_0^{(n)}\|_2^2 \le \|S_m^{(n)}\|_2^2 \le Dme(m, 1+\varepsilon_1)\|X_0^{(n)}\|_2^2 \tag{3.15}$$

and

$\forall n \ge M^*$, finite non-empty sets $S \subset N$

$$\|\sum_{k\in S} X_k^{(n)}\|_2^2 \le Dse(s, 1+\varepsilon_1)\|X_0^{(n)}\|_2^2, \tag{3.16}$$

where $s =$ Card S and $\varepsilon_1 := \varepsilon^*/6$.

For each $n \ge M^*$ define the numbers

$$A_n^2(m) := \|S_m^{(n)}\|_2^2 \text{ and } A_n^2 := A_n^2(n). \tag{3.17}$$

By (3.15) and (3.13)

$$ne(n, -1-\varepsilon_1)H(t_n) << A_n^2 << ne(n, 1+\varepsilon_1)H(t_n). \tag{3.18}$$

In order to prove Theorem 1.1 it suffices to show that

$$W_n(t) \Rightarrow W(t),$$

where $W_n(t) := S_{[nt]}/A_n$. By Lemma 3.4, (1.4) and (3.18) we can easily verify that

$$\forall \varepsilon > 0,\ \lim_{n\to\infty} P(\sup_{0\le t\le 1} |S_{[nt]} - S_{[nt]}^{(n)}| > \varepsilon A_n) = 0.$$

Hence, by [Billingsley (1968), Theorem 4.1, p.25], to prove Theorem 1 it suffices to prove that

$$W_n^*(t) \Rightarrow W(t) \text{ as } n \to \infty, \text{ where } W_n^*(t) := S_{[nt]}^{(n)}/A_n. \tag{3.19}$$

By [Billingsley (1968, Theorem 19.2, p.157], it is enough to show that

$$\forall 0 < t \le 1,\ A_n^2([nt])/A_n^2 \to t \text{ as } n \to \infty, \tag{3.20}$$

$$\forall 0 < t \le 1,\ \{W_n^*(t)^2 : n \ge 1\} \text{ is uniformly integrable}, \tag{3.21}$$

and $\forall \varepsilon > 0$, there exists a constant $\lambda > 1$ such that

$$P(\max_{0 \le m \le n} |S_m^{(n)}| \ge \lambda A_n) < \varepsilon/\lambda^2 \tag{3.22}$$

for all sufficiently large n.

We first prove (3.20).

Lemma 3.5. *For every $0 < t < 1$, as $m \to \infty$ in $n \ge M^*$ uniformly*

$$A_n^2([mt])A_n^{-2}(m) \to t, \text{ and} \tag{3.23}$$

$$A_n^{-2}(m)\max(A_n^2([\tfrac{1}{2}m]),\ A_n^2(m - [\tfrac{1}{2}m])) \to \frac{1}{2}. \tag{3.24}$$

In particular,

$$A_n^2([nt])A_n^{-2}(n) \to t, \text{ as } n \to \infty. \tag{3.25}$$

Proof. We first prove that for every integer $p \ge 2$

$$A_n^2([m/p])A_n^{-2}(m) \to 1/p \text{ as } m \to \infty \text{ in } n \ge M^* \text{ uniformly}. \tag{3.26}$$

Let $q := [m/p]$, $Y_i := \sum_{j=1+iq}^{(i+1)q} X_j^{(n)}$ for $i = 0, 1, \cdots, p-1$ and $Y_p := \sum_{j=1+pq}^{m} X_j^{(n)}$. Note that

$$A_n^2(m) = pA_n^2(q) + \sum_{i \ne j} EY_iY_j + EY_p^2.$$

For $i \ne j$ and every natural k, by the definition of ρ-mixing and Minkowski inequality

$$|EY_iY_j| \le 2\|Y_i\|_2\|S_k^{(n)}\|_2 + \rho(k)\|Y_i\|_2\|Y_j\|_2.$$

Hence, by (3.15)

$$\begin{aligned} |A_n^{-2}(q)A_n^2(m) - p| &\le (p+1)^2(A_n^{-1}(q)\|S_k^{(n)}\|_2 + \rho(k) + A_n^{-2}(q)\|Y_p\|_2^2) \\ &\le (p+1)^2(C^{-1}(k^2+p^2)q^{-1/2}e(q, 1+\varepsilon_1) + \rho(k)) \\ &\le (p+1)^2((k^2+p^2)m^{-1/3} + \rho(k)) \end{aligned} \tag{3.27}$$

for every m sufficiently large. For arbitrary $\varepsilon > 0$, but fixed, by $\rho(k) \to 0$ as $k \to \infty$, choose an integer k such that $(p+1)^3\rho(k) < \varepsilon/2$. Hence for every m sufficiently large in $n \ge M^*$ uniformly

$$|A_n^2(m)A_n^{-2}([m/p]) - p| < \varepsilon$$

as desired.

Case I. If t is a rational number, that is, $t = p/q$ for some integers p and q with $p < q$, by (3.26)

$$\begin{aligned} A_n^2([mp/q])A_n^{-2}(m) &= A_n^2([mp/q])A_n^{-2}(mp)A_n^2(mp)A_n^{-2}(m) \\ &\to p/q = t \end{aligned}$$

as $m \to \infty$ in $n \geq M^*$ uniformly.

Case II. If t is an irrational number, then for arbitrary $0 < \varepsilon < \frac{1}{2}$, but fixed, take a rational number $t_1 > 0$ such that

$$\frac{1}{4}\varepsilon < t - t_1 < \frac{1}{2}\varepsilon. \tag{3.28}$$

By the Minkowski inequality

$$|A_n([mt]) - A_n([mt_1])| \leq A_n([mt] - [mt_1]). \tag{3.29}$$

Let $p = [m/([mt] - [mt_1])]$, then

$$m/(mt - mt_1 + 1) - 1 \leq p \leq m/(mt - mt_1 - 1).$$

By (3.28), for every $m > 20/\varepsilon$

$$\frac{1}{2}\varepsilon^{-1} < p < 5\varepsilon^{-1}.$$

Similar to the proof of (3.27), we can get that for every natural k

$$(p-(p+1)^2\rho(k))A_n^2([mt]-[mt_1]) \leq A_n^2(m)+(p+1)^2A_n^2([mt]-[mt_1])A_n^2(k)+(p+1)^2\|X_0^{(n)}\|_2^2.$$

Take k such that $\rho(k) < \varepsilon/24$, then

$$\begin{aligned} A_n^2([mt] - [mt_1])A_n^{-2}(m) &\leq 6p^{-1}(1 + 3(p+1)^4(A_n^2(k) + \|X_0^{(n)}\|_2^2)A_n^{-2}(m)) \\ &\leq 12\varepsilon + 36 \cdot 5^4 \cdot k^2\varepsilon^{-4} \cdot C^{-1} \cdot m^{-1} \cdot e(m, 1+\varepsilon_1) \\ &\leq 13\varepsilon \end{aligned} \tag{3.30}$$

for every $m \geq 30^8 \cdot k^4 \cdot C^{-2} \cdot \varepsilon^{-10}$.

By (3.29), (3.30) and the result of Case I, we can get that

$$|A_n^2([mt])A_n^{-2}(m) - t| \leq 16\varepsilon$$

for all sufficiently large m in $n \geq M^*$ uniformly. This completes the proof of (3.23). The proof of (3.24) is similar. We omit it.

Next we prove (3.21).

Lemma 3.6. *For every* $0 < t \leq 1$

$$\lim_{A\to\infty} \sup_{n\geq 1} A_n^{-2}E|S_{[nt]}^{(n)}|^2 I(|S_{[nt]}^{(n)}| \geq AA_n) = 0. \tag{3.31}$$

Proof. The proof will be divided into two cases.

Case I. $\sum\limits_{k=1}^{\infty} \rho(2^k) < \infty$.

By Theorem 1 of Bradley (1988), i.e., Lemma 2.6, we have

$$S_{[nt]}/A_{[nt]} \to N(0,1) \text{ in distribution as } n \to \infty.$$

In terms of Lemma 3.4, (1.4), (3.5) and (3.18) we can get that

$$(S_{[nt]} - S_{[nt]}^{(n)})/A_{[nt]} \to 0 \text{ in distribution as } n \to \infty.$$

Thus we have

$$S_{[nt]}^{(n)}/A_{[nt]} \to N(0,1) \text{ in distribution as } n \to \infty.$$

By Lemmas 2.1, 2.2 and 3.3, we have

$$\begin{aligned}E(S_{[nt]}^{(n)} - S_{[nt]}^{([nt])})^2 A_{[nt]}^{-2} &= A_{[nt]}^{-2}\mathrm{Var}(\sum_{i=1}^{[nt]} X_i I(t_{[nt]} < |X_i| \le t_n)) \\ &\le DC^{-1}EX_0^2 I(t_{[nt]} < |X_0| \le t_n)/EX_0^2 I(|X_0| \le t_{[nt]}) \\ &= DC^{-1}(H(t_n) - H(t_{[nt]}))/H(t_{[nt]}) \\ &\to 0, \text{ as } n \to \infty.\end{aligned}$$

It follows that

$$A_n([nt])/A_{[nt]} \to 1 \text{ as } n \to \infty.$$

Therefore, $S_{[nt]}^{(n)}/A_n([nt]) \to N(0,1)$ in distribution as $n \to \infty$. By a well-known result on uniformly integrablel (e.g. see [1], Theorem 5.4, p.32), $(S_{[nt]}^{(n)}/A_n([nt]))^2$ is uniformly integrable and so is $(S_{[nt]}^{(n)}/A_n)^2$ by (3.25).

Case II. $\sum_{k=1}^{\infty} \rho(2^k) = \infty$.

For every n fixed, let $l := l_n := t_n(\delta)$ and $p := p_n := [nt]$.

$\forall i \ge 1$, define the random variables:

$$X_{i1}^{(n)} := X_i I(|X_i| \le t_n/l) - EX_i I(|X_i| \le t_n/l),$$

$$X_{i2}^{(n)} := X_i I(t_n/l < |X_i| \le t_n) - EX_i I(t_n/l < |X_i| \le t_n),$$

$$S_{k1}^{(n)} := \sum_{i=1}^{k} X_{i1}^{(n)} \text{ and } S_{k2}^{(n)} := \sum_{i=1}^{k} X_{i2}^{(n)}. \tag{3.32}$$

Obviously, $S_p^{(n)} = S_{p1}^{(n)} + S_{p2}^{(n)}$ and

$$E|S_p^{(n)}|^2 I(|S_p^{(n)}| \ge AA_n) \le 4E|S_{p1}^{(n)}|^2 I(|S_{p1}^{(n)}| \ge \frac{1}{2}AA_n) + 4E|S_{p2}^{(n)}|^2 I(|S_{p2}^{(n)}| \ge \frac{1}{2}AA_n). \tag{3.33}$$

By Lemmas 2.1, 2.2 and 3.1

$$\begin{aligned}E|S_{p2}^{(n)}|^2 &<< pe(p, 1+\varepsilon_1)EX_0^2 I(t_n/l < |X_0| \le t_n) \\ &<< pe(p, 1+\varepsilon_1)G(t_n)g^{-1}(t_n/t_n(\delta)) \\ &<< A_n^2 e(p, -\frac{1}{4}\varepsilon^*).\end{aligned} \tag{3.34}$$

By Lemmas 3.5, 2.4, 3.2 and 3.1 and (3.3)

$$\begin{aligned}E|S_{p1}^{(n)}|^{2+\delta} &<< p\exp(560 \sum_{i=0}^{[\log n]} \rho(2^i)^{2/(2+\delta)})E|X_0|^{2+\delta} I(|X_0| \le t_n/l) + A_n^{2+\delta}(p) \\ &<< p\exp(560 \sum_{i=0}^{[\log n]} \rho(2^i)^{2/(2+\delta)})G(t_n/l)g^{-1}(t_n/l)(t_n/l)^{\delta} + A_n^{2+\delta}(p) \\ &<< A_n^{2+\delta}.\end{aligned} \tag{3.35}$$

Hence, by (3.33)–(3.35)

$$A_n^{-2}E|S_p^{(n)}|^2 I(|S_p^{(n)}| \ge AA_n) << e(p, -\frac{1}{4}\varepsilon^*) + A_n^{-\delta}. \tag{3.36}$$

(3.31) now follows from (3.36), the hypothesis $\sum_{i=0}^{\infty} \rho(2^i) = \infty$ and the fact that

$$\lim_{A\to\infty} \sup_{n\le M} A_n^{-2} E|S_{nt}^{(n)}|^2 I(|S_{nt}^{(n)}| > AA_n) = 0$$

for every M fixed.

We finally prove (3.22).

Lemma 3.7. *For every $\varepsilon > 0$, there exists a constant $\lambda > 1$ such that*

$$P(\max_{1\le i\le n} |S_i^{(n)}| \ge 6\lambda A_n) \le 8\varepsilon/\lambda^2 \tag{3.37}$$

for every n sufficiently large.

Proof. The proof of (3.37) is somehow similar to that of (2.5) from Shao (1989). For every $i \ge 1$, $X_{i1}^{(n)}$, $X_{i2}^{(n)}$, $S_{i1}^{(n)}$ and $S_{i2}^{(n)}$ are defined as in (3.32). Obviously

$$P(\max_{i\le n} |S_i^{(n)}| \ge 6\lambda A_n) \le P(\max_{i\le n} |S_{i1}^{(n)}| \ge 5\lambda A_n) + P(\max_{i\le n} |S_{i2}^{(n)}| \ge \lambda A_n).$$

By (3.24) and Lemma 2.4

$$E|S_{i1}^{(n)}|^{2+\delta} << i \exp(560 \sum_{i=0}^{[\log n]} \rho(2^i)^{2/(2+\delta)}) E|X_0|^{2+\delta} I(|X_0| \le t_n/l) + A_n^{2+\delta}(i),$$

where the constant implied by "<<" does not depend on i and n. By Corollary 3 from Moricz (1982), Lemmas 3.2 and 3.5, analogy to (3.35)

$$\begin{aligned} E\max_{i\le n} |S_{i1}^{(n)}|^{2+\delta} &<< n\log^3 n \exp(560 \sum_{i=0}^{[\log n]} \rho(2^i)^{2/(2+\delta)}) E|X_0|^{2+\delta} I(|X_0| \le t_n/l) + A_n^{2+\delta} \\ &<< n \exp(600 \sum_{i=0}^{[\log n]} \rho(2^i)^{2/(2+\delta)}) G(t_n/l) g^{-1}(t_n/l)(t_n/l)^{\delta} + A_n^{2+\delta} \\ &<< A_n^{2+\delta}, \end{aligned}$$

here, and in sequel, we can assume, w.l.o.g., that $\rho(2^i) >> 1/(i\log^2 i)$. Whence there exists a constant $\lambda > 1$ such that for every n sufficiently large

$$P(\max_{i\le n} |S_{i1}^{(n)}| \ge \lambda A_n) < \varepsilon/\lambda^2. \tag{3.38}$$

We now estimate $P(\max_{i\le n} |S_{i2}^{(n)}| \ge 5\lambda A_n)$. Let

$$r_1 = [n/l],\ r_2 = [n/l^2],\ l_1 = [(n-r_1)/(r_1+r_2)],\ l_2 = [n/(r_1+r_2)],$$

$$Y_i = \sum_{j=1+i(r_1+r_2)}^{i(r_1+r_2)+r_1} X_{j2}^{(n)}, \quad i = 0, 1, \cdots, l_1,$$

$$Z_i = \sum_{j=1+r_1+i(r_1+r_2)}^{(i+1)(r_1+r_2)} X_{j2}^{(n)}, \quad i = 0, 1, \cdots, l_2,$$

$$T_i(1) = \sum_{j=0}^{i} Y_j \text{ and } T_i(2) = \sum_{j=0}^{i} Z_j,$$

$$Y_i^* = \sum_{j=1+i(r_1+r_2)}^{i(r_1+r_2)+r_1} |X_j|I(t_n/l < |X_j| \le t_n) - E|X_j|I(t_n/l < |X_j| \le t_n),$$

$$T_i^* = \sum_{j=0}^{i} Y_j^*, \quad i = 0, 1, \cdots, l_1.$$

It is easy to see that

$$\begin{aligned} P(\max_{i\le n} |S_{i2}^{(n)}| \ge 5\lambda A_n) &\le P(\max_{i\le l_1} |T_i(1)| \ge 2\lambda A_n) + P(\max_{i\le l_2} |T_i(2)| \ge 2\lambda A_n) \\ &\quad + P(\max_{i\le l_1} \sum_{k=1+i(r_1+r_2)}^{i(r_1+r_2)+r_1} |X_{j2}^{(n)}| \ge \frac{1}{2}\lambda A_n) + 2lP(\max_{i\le r_2} |S_{i2}^{(n)}| \ge \frac{\lambda}{2} A_n) \\ &= I_1 + I_2 + I_3 + I_4 \text{ (say)}. \end{aligned} \tag{3.39}$$

By (3.3), (1.4), (3.5) and (2.2)

$$\begin{aligned} r_1 E|X_0|I(t_n/l < |X_0| \le t_n) &\le r_1 l t_n^{-1} H(t_n) \\ &= r_1 l H(t_n)(nG(t_n)/g(t_n))^{-1/2} \le \lambda_0 A_n, \end{aligned}$$

here λ_0 is a finite constant. Hence for $\lambda > 8\lambda_0$

$$I_3 \le P(\max_{i\le l_1} |Y_i^*| \ge \frac{1}{4}\lambda A_n) \tag{3.40}$$

and

$$I_4 \le 2lP(\sum_{i=1}^{r_2} |X_i|I(t_n/l < |X_i| \le t_n) - E|X_i|I(t_n/l < |X_i| \le t_n) \ge \frac{1}{4}\lambda A_n). \tag{3.41}$$

In order to establish the estimation of I_1, let

$$G_{-1} = (\Omega, \Phi), \; G_k = \sigma(X_i, 1 \le i \le r_1 + k(r_1 + r_2)),$$

$$u_k = E(Y_k | G_{k-1}), \; k = 0, 1, \cdots, l_1,$$

$$U_0(0) = 0, \; U_i(k) = \sum_{j=1}^{k} u_{j+1} \text{ and } T^*(k) = T_k(1) - U_0(k).$$

Obviously

$$\begin{aligned} I_1 &\le P(\max_{i\le l_1} |T^*(i)| \ge \lambda A_n) + P(\max_{i\le l_1} |U_0(i)| \ge \lambda A_n) \\ &= I_1^{(1)} + I_1^{(2)} \text{ (say)}. \end{aligned} \tag{3.42}$$

Noting that $\{T^*(i), G_i, i = 0, 1, \cdots, l_1\}$ is a martingale sequence and using a maximal inequality due to Brown (1971), we have

$$I_1^{(1)} \le 4A_n^{-2}\lambda^{-2} ET^*(l_1)^2 I(|T^*(l_1)| \le \lambda A_n). \tag{3.43}$$

We below prove that for every i, k and n, by induction on k

$$EU_i^2(k) \le Dkr_1\rho(r_2)^2 \log^2(2k) EX_0^2 I(t_n/l < |X_0| \le t_n) e(n, 1 + \frac{1}{4}\varepsilon^*). \tag{3.44}$$

If $k = 1$, by the defintion of ρ-mixing

$$EU_i^2(1) = EY_{i+1}E(Y_{i+1}|G_i)) \le \rho(r_2)\|Y_{i+1}\|_2 \|E(Y_{i+1}|G_i)\|_2.$$

Thus (3.44) is true for $k = 1$ by (2.1).

If $k \geq 2$, assume (3.44) holds for every integer less than k. Put $k_1 = [k/2]$, $k_2 = k - k_1$. Then

$$\begin{aligned} EU_i^2(k) &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2) \\ &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1) \sum_{j=1+k_1}^{k} Y_{j+1} \\ &\leq EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2\rho(r_2)\|U_i(k_1)\|_2 \| \sum_{j=1+k_1}^{k} Y_{j+1}\|_2 \\ &\leq D(k_1 \log^2(2k_1) + k_2 \log^2(2k_2) + 2k_1^{1/2}k_2^{1/2} \log(2k_1)) \cdot \\ &\quad \cdot r_1\rho(r_2)^2 e(n, 1 + \frac{1}{4}\varepsilon^*) EX_0^2 I(t_n/l < |X_0| \leq t_n) \\ &\leq Dkr_1\rho(r_2)^2 \log^2(2k) e(n, 1 + \frac{1}{4}\varepsilon^*) EX_0^2 I(t_n/l < |X_0| \leq t_n), \end{aligned}$$

by induction hypotheses and (2.1). This proves (3.44).

From (3.44) and Corollary 4 of Moricz (1982) we obtain that

$$\begin{aligned} E \max_{i \leq l_1} U_0^2(i) &\leq 3Dl_1 r_1 \rho(r_2)^2 \log^4(2l_1) e(n, 1 + \frac{1}{4}\varepsilon^*) EX_0^2 I(t_n/l < |X_0| \leq t_n) \\ &<< A_n^2 \rho(r_2)^2 \log^4(2l) e(n, -\frac{1}{4}\varepsilon^*) \\ &<< A_n^2 \rho(r_2)^2 e(n, -\frac{1}{4}\varepsilon^*) (\sum_{i=1}^{[\log n]} \rho(2^i)^{1-\delta})^4 \end{aligned}$$

by (3.6) and (2.2). On the other hand

$$\begin{aligned} \sum_{i=1}^{[\log n]} \rho(2^i)^{1-\delta} &\leq \sum_{i=1}^{[\log r_2]} \rho(2^i)^{1-\delta} + \rho(r_2)^{1-\delta}(2 + \log(n/r_2)) \\ &\leq \rho(r_2)^{-\delta} \sum_{i=1}^{[\log r_2]} \rho(2^i) + \rho(r_2)^{1-\delta} \sum_{i=1}^{[\log n]} \rho(2^i)^{1-\delta}, \end{aligned}$$

from which follows

$$\rho(r_2) \sum_{i=1}^{[\log n]} \rho(2^i)^{1-\delta} << \sum_{i=1}^{[\log n]} \rho(2^i)$$

by the fact that $\rho(r_2) \to 0$ as $n \to \infty$. Therefore we obtain that

$$E \max_{i \leq l_1} U_0^2(i) << \rho(r_2) (\sum_{i=1}^{[\log n]} \rho(2^i))^4 e(n, -\frac{1}{4}\varepsilon^*) A_n^2 << \rho(n^{\frac{1}{2}}) A_n^2 \tag{3.45}$$

and

$$I_1^{(2)} \leq \varepsilon/\lambda^2 \tag{3.46}$$

for every $\lambda > 1$ and every n sufficiently large.

For I_2, having analogy to (3.43) and (3.46), we can get that for every $\lambda > 1$ and every n sufficiently large

$$I_2 \leq \varepsilon/\lambda^2 + 4A_n^{-2}\lambda^{-2} ET_{l_2}^2(2). \tag{3.47}$$

By (2.1), (3.6), (1.4) and (2.2)

$$\begin{aligned} ET_{l_2}^2(2) &\le Dl_2r_2e(n,1+\varepsilon_1)EX_0^2I(t_n/l<|X_0|\le t_n) \\ &\le Dnl^{-1}e(n,1+\varepsilon_1)G(t_n)/g(t_n/l) \\ &<< A_n^2l^{-1}e(n,-\frac{1}{4}\varepsilon^*). \end{aligned} \tag{3.48}$$

Noting that $1/l=1/l_n\to 0$ as $n\to\infty$, we finally obtain that

$$I_2\le 2\varepsilon/\lambda^2 \tag{3.49}$$

for every $\lambda>1$ and every n sufficiently large.

By (3.40), (2.2), (3.6), (2.1) and (3.5), we have

$$\begin{aligned} I_4 &<< \lambda^{-2}A_n^{-2}\cdot l\cdot r_2e(n,1+\varepsilon_1)EX_0^2I(t_n/l<|X_0|\le t_n) \\ &<< \lambda^{-2}A_n^{-2}\cdot n\cdot l^{-1}e(n,1+\varepsilon_1)G(t_n)/g(t_n/l) << \lambda^{-2}e(n,-\frac{1}{4}\varepsilon^*)\cdot l^{-1}. \end{aligned}$$

Hence

$$I_4\le \varepsilon/\lambda^2 \tag{3.50}$$

for every $\lambda\ge 1$ and every n sufficiently large.

We now estimate $I_1^{(1)}$ and consider two cases.

Case I. $\sum\limits_{i=1}^{\infty}\rho(2^i)=\infty.$ (3.51)

From (3.43), (2.2), (3.6), (2.1) and (3.5)

$$I_1^{(1)} << \lambda^{-2}A_n^{-2}\cdot l\cdot r_1e(n,1+\varepsilon_1)EX_0^2I(t_n/l<|X_0|\le t_n) << \lambda^{-2}e(n,-\frac{1}{4}\varepsilon^*).$$

Hence, by (3.51) and (3.46)

$$I_1<2\varepsilon/\lambda^2. \tag{3.52}$$

Case II. $\sum\limits_{i=1}^{\infty}\rho(2^i)<\infty.$ (3.53)

We have

$$\begin{aligned} ET^{*^2}(l_1)I(|T^*(l_1)|\ge\lambda A_n) &\le 4ET_{l_1}^2(1)I(|T_{l_1}(1)|\ge\frac{1}{2}\lambda A_n)+4EU_0^2(l_1) \\ &\le 36(ES_{n2}^{(n)2}I(|S_{n2}^{(n)}|\ge\lambda\frac{A_n}{6})+E(\sum_{i=l_2(r_1+r_2)}^{n}X_{i2}^{(n)})^2+EU_0^2(l_1)) \\ &\le 144(ES_n^{(n)}I(|S_n^{(n)}|\ge\lambda\frac{A_n}{12})+ES_{n1}^{(n)2}I(|S_{n1}^{(n)}|\ge\lambda\frac{A_n}{12}) \\ &\quad +ET_{l_2}^2(2)+EU_0^2(l_1)+E(\sum_{i=l_2(r_1+r_2)}^{n}X_{i2}^{(n)})^2). \end{aligned} \tag{3.54}$$

By (3.53), (2.1), (2.2) and (3.45)

$$\begin{aligned} A_n^{-2}(ET_{l_2}^2(2)+E(\sum_{i=l_2(r_1+r_2)}^{n}X_{i2}^{(n)})^2+EU_0^2(l_1) &<< n^{-1}(l_2r_1+r_2)+\rho(n^{\frac{1}{2}}) \\ &<< l^{-1}+\rho(n^{\frac{1}{2}}). \end{aligned}$$

It follows that

$$(ET_{l_2}^2(2)+E(\sum_{i=l_2(r_1+r_2)}^{n}X_{i2}^{(n)})^2+EU_0^2(l_1))A_n^{-2}\le C/3000 \tag{3.55}$$

for every n sufficiently large.

From (3.31) we find that there is a constant λ_1 such that for every $\lambda > \lambda_1$ and every $n \geq 1$

$$A_n^{-2} E S_n^{(n)2} I(|S_n^{(n)}| \geq \lambda A_n/12) \leq \varepsilon/3000. \tag{3.56}$$

By (3.35)

$$E S_{n1}^{(n)2} I(|S_{n1}^{(n)}| \geq \lambda A_n/12) A_n^{-2} \leq 12\lambda^{-\delta/2} A_n^{-2-\delta} E|S_{n1}^{(n)}|^{2+\delta} << \lambda^{\delta/2},$$

where the constant implied by "<<" does not depend on n. It follows that there is a constant $\lambda_2 > 0$ such that for every $\lambda > \lambda_2$ and every $n \geq 1$

$$A_n^{-2} E S_{n1}^{(n)2} I(|S_{n1}^{(n)}| \geq \lambda A_n/12) \leq \varepsilon/3000. \tag{3.57}$$

Whence, we find by (3.43) and (3.44)–(3.57) that for every $\lambda > \max(\lambda_1, \lambda_2)$ and every n sufficiently large

$$I_1^{(1)} \leq \varepsilon/\lambda^2, \tag{3.58}$$

which together with (3.46) yields

$$I_1 \leq 2\varepsilon/\lambda^2. \tag{3.59}$$

Proceeding exactly as the proof of (3.59), we have also

$$I_3 \leq 2\varepsilon/\lambda^2 \tag{3.60}$$

for every sufficiently large λ and n.

It follows from (3.38), (3.39), (3.49), (3.50), (3.59) and (3.60) that (3.37) holds, as desired.

Now (3.20)-(3.22) follow from Lemmas 3.5-3.7. This completes the proof of Theorem 1.1.

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