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THE IDEAL INTERSECTION PROPERTY OF SKEW GROUP RING

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Abstract

Let A be a prime ring, and let G be a group (possibly infinite) as automorphisms acting on A. When the X-inner subgroup G_{inn} of G is finite, a criterion is obtained for every non-zero ideal of the skew group ring AG to have non-zero intersection with A.

Throughout this paper, A will denote a prime associative ring which does not necessarily have a unity. Let G be a group (possibly infinite) as automorphisms acting on A. For $a \in A$ and $g \in G$ we will let a^g denote the image of a under g. The skew group ring AG is definted to be $AG = \oplus_{g\in G} Ag$ with addition given component-wise and multiplication given as follows: if $a, b \in A$ and $g, h \in G$, then $(ag)(bh) = ab^{g^{-1}}gh \in Agh$.

Let F denote the set of all two-sided ideals of A, and let A_F denote the ring of left Martindale quotients of A with respect to F, i.e. $A_F = \varinjlim_{I\in F} \text{Hom}_A({}_AI, A)$. An alternate description of A_F is as follows. Consider the set of all left A-module homomorphisms $f \in \text{Hom}_A({}_AI, A)$ where I ranges over F. Two such functions are said to be equivalent if they agree on their common domain, which is a non-zero ideal since A is prime. It is easy to see that this is an equivalence relation. We let $\hat{f}$ denote the equivalence class of f and let A_F be the set of all such equivalence classes. The arithmetic in A_F is defined in a fairly obvious manner. Suppose $f \in \text{Hom}_A({}_AI, A)$ and $g \in \text{Hom}_A({}_AJ, A)$. Then $\hat{f} + \hat{g}$ is the class of $f + g \in \text{Hom}_A(I \cap J, A)$ and $\hat{f}\hat{g}$ is the class of the composite function $fg \in \text{Hom}_A(JI, A)$. It is easy to see that these definitions make sense and that they respect the equivalence relation. Furthermore A_F is a ring with 1, and A is a subring of A_F regarded as ring multiplications. It is well-known that A_F is also prime, the center C of A_F is a field[1], and every automorphism of A may be uniquely extended to an automorphism of A_F as follows: if $\hat{f} \in A_F$, $f \in \text{Hom}_A(I, A)$, then for $a \in I^g$, let $af' = (a^{g^{-1}}f)^g$. Then $f' \in \text{Hom}_A(I^g, A)$ and then $\hat{f'} = \hat{f}^g$. Thus we are able to form the skew group ring A_FG.

An automorphism g of A is said to be X-inner if it is induced by conjugation by a unit of A_F. Let $G_{\text{inn}} = \{g \in G/g \text{ is } X\text{-inner}\}$. Then we can see that G_{inn} is a normal subgroup of G.

For each $g \in G_{\text{inn}}$ choose a unit $u_g \in A_F$ inducing the automorphism on A and let $\bar{g} = u_g^{-1}g$. Then $E = \sum_{g\in G_{\text{inn}}} C\bar{g}$ is a subring of A_FG. With respect to the basis $\{\bar{g}/g \in G_{\text{inn}}\}$, E has the structure of $C^t[G_{\text{inn}}]$, a twisted group algebra of G_{inn} over the center C.

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We say that a ring extension $A \subseteq B$ has the ideal intersection property if every non-zero ideal of B has a non-zero intersection with A. The importance of the ideal intersection property derives from the fact that if a ring extension $A \subseteq B$ has the ideal intersection property then many semigroup properties (for instance, prime, semiprime, simple) of all ideals of A can be lift onto B. In [2], we obtained a criterion for the smash product of a group graded ring to have the ideal intersection property, and then obtained criteria for the smash product to be prime or simple, which were proven by S.Montgomery and D.Passman with different method.

A subset K of A is said to be G-stable if $K^g \subseteq K$ for all $g \in G$.

In [9, Proposition 2.11], it was shown that $AG \supseteq A$ has the ideal intersection property if and only if $C^t[G_{\text{inn}}]$ is G-simple (i.e. every G-stable ideal is trivial).

In this paper, we will give an interior characteristic for $AG \supseteq A$ to have the ideal intersection property when G_{inn} is finite. Incompletely, we are not able to remove the condition $|G_{\text{inn}}| < \infty$.

In [5], using Kharchenko's crucial proposition[6], J.Fisher and S.Montgomery showed that $AG \supseteq AG_{\text{inn}}$ has the ideal intersection property. If G is X-outer (i.e. $G_{\text{inn}} = \{e\}$), then $AG \supseteq A$ has the ideal intersection property. The following theorem improves this result. Before the statement of our theorem, we give two definitions.

Define A to be an AG-module via $(\sum_{g\in G} a_g g)\cdot a = \sum_{g\in G} a_g a^g$ for $a, a_g \in A$, and say a subring B of A is hereditary if $B = RL$ for some non-zero G-stable right ideal R and some non-zero G-stable left ideal L of A.

Theorem. *Let A be a prime ring and let G be a group acting as automorphisms on A. If G_{inn} is finite, then the ring extension $AG \supseteq A$ has the ideal intersection property if and only if*

i) *A is a faithful AG_{inn}-module and*

ii) *$t \cdot B \neq 0$ for all hereditary subrings B, where $t = \sum_{g\in G_{\text{inn}}} g$.*

Proof. We define G acting on $A_F G$ by

$$(\sum_{g\in G} s_g g)^h = h^{-1}(\sum_{g\in G} s_g g)h = \sum_{g\in G} s_g^h g^h$$

for all $h \in G$ and $s_g \in A_F$. For every G-stable ideal I of AG_{inn}, IG is an ideal of AG. Since every non-zero ideal of AG has a non-zero intersection with A and since A is prime, AG is also prime and hence AG_{inn} is G-prime (i.e. two non-zero G-stable ideals have non-zero intersection).

Since G_{inn} is a normal subgroup of G, At is a G-stable left ideal of AG_{inn}. Therefore, $\text{ann}_{AG_{\text{inn}}}(A) = 0$ and hence A is a faithful AG_{inn}-module.

Let $B = RL$ for some non-zero G-stable right ideal R and some non-zero G-stable left ideal L of A. Since Lt is a non-zero G-stable left ideal of AG_{inn} and tR is a non-zero G-stable right ideal of AG_{inn}, we have $(t \cdot B)t = (tR)(Lt) \neq 0$, and then $t \cdot B \neq 0$. This proves the necessity.

We now prove the sufficiency. Let J be a non-zero G-stable ideal of $C^t[G_{\text{inn}}]$. For any

$s \in A_F$,

$$\bar{g}s = u_g^{-1}gs = u_g^{-1}s^{g^{-1}}g = u_g^{-1}u_g s u_g^{-1}g = su_g^{-1}g = s\bar{g}$$

and $cs = sc$ for all $c \in C$. Thus $J_1 = A_F J = JA_F$ is a non-zero G-stable ideal of $A_F G_{\text{inn}}$. By the property of the Martindale quotient ring (i.e. if $0 \neq s_i \in A_F$ for $i = 1, \cdots, n$, then there exists $a \in A$ with $0 \neq as_i \in A$ for all i),

$$J_2 = J_1 \cap AG_{\text{inn}}$$

is a non-zero G-stable ideal of AG_{inn}.

Let $0 \neq \sum_{g \in G_{\text{inn}}} a_g g \in J_2$. Then $0 \neq \sum_{g \in G_{\text{inn}}} a_g g^{-1} \in AG_{\text{inn}}$. By the condition i), there exists $a \in A$ with

$$\sum_{g \in G_{\text{inn}}} a_g a^{g^{-1}} \neq 0.$$

Thus

$$0 \neq \sum_{g \in G_{\text{inn}}} a_g a^{g^{-1}} t = (\sum_{g \in G_{\text{inn}}} a_g g)at \in J_2 t \subseteq J_2.$$

Set

$$L = \{a \in A / at \in J_2\}.$$

Since $a^g t = a^g t^g = (at)^g \in J_2$ for all $a \in L$ and $g \in G$, we can see that L is a non-zero G-stable left ideal of A such that $Lt \subseteq J_2$.

Let J' be any non-zero ideal of $C^t[G_{\text{inn}}]$. Then, as shown above, $J_1' = J'A_F = A_F J'$ is a non-zero G-stable left ideal of A_F. By the condition ii), there exists $0 \neq a \in t \cdot L' \subseteq L'$ which is fixed by G_{inn} (i.e. $a^g = a$ for all $g \in G_{\text{inn}}$).

Since G_{inn} is a normal subgroup of G, for all $g \in G$ and $h \in G_{\text{inn}}$

$$(a^g)^h = (a^{ghg^{-1}})^g = a^g.$$

Hence $ta^g = a^g t \in J_2'$.

Since $0 \neq \sum_{g \in G} a^g A$ is a G-stable right ideal of A, by the condition ii) we have

$$\begin{aligned} J_2' J_2 \supseteq & (\sum_{g \in G} a^g t A)(Lt) = (t(\sum_{g \in G} a^g A))(Lt) \\ = & (t \cdot (\sum_{g \in G} a^g AL))t \neq 0 \end{aligned}$$

since $\sum_{g \in G} a^g AL$ is a hereditary subring of A. It follows that

$$A_F(J'J)A_F = J_1' J_1 \supseteq J_2' J_2 \neq 0.$$

This proves that $J'J \neq 0$, that is, $C^t[G_{\text{inn}}]$ is G-prime.

$C^t[G_{\text{inn}}]$ is a finite dimensional algebra over C; furthermore $C^t[G_{\text{inn}}]$ is semisimple since the nilpotent radical of $C^t[G_{\text{inn}}]$ is G-stable and since $C^t[G_{\text{inn}}]$ is G-prime. Since the images of simple components under automorphisms are still simple components, it is clear from the Wedderburn-Artin theorem that $C^t[G_{\text{inn}}]$ is G-simple. Our result now follows from Montgomery-Passman's result ([9, Proposition 2.11]).

As an illustration, we consider the following G.Bergman's example.

Example 1. Let F be a field of characteristic $p \in 0$, with an element $w \neq 0, 1$ of finite multiplicative order, say $w^n = 1$. Let $A' = M_2(F\{x, y\})$. Let $G \subseteq Aut(A')$ be generated by the inner automorphisms induced by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$, and let A be the subring of A' of those matrices whose entries have zero constant term. A is certainly prime, $G = G_{\text{inn}}$, $|G| < \infty$, and A^G, the fixed subring of G, is zero (see [8, Example 1.1] for detail). Since A is a hereditary subring and since $t \cdot A \subseteq A^G = 0$, by our theorem, the extension $AG \supseteq A$ has no the ideal intersection property.

The next two examples show that the conditions i) and ii) are independent. The example 2 appears in [3], which gives the negative answer to S. Montgomery's conjecture.

Example 2. Let $A = M_2(\mathbf{C})$, where $\mathbf{C}$ is the complex number field, let w be a cube root, $u, v \in Aut(A)$ be the inner automorphisms induced by $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively, and let G be generated by u, v. A is the unique hereditary subring and $t \cdot A = A^G \neq 0$. But A is not faithful AG-module, for instance,

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u^2 \right) \cdot A = 0.$$

Example 3. Let $A = M_2(\mathbf{C})$, and let $g \in Aut(A)$ be the inner automorphism induced by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $G = \langle g \rangle$. It is straightforward that A is a faithful AG-module. However, $(e+g) \cdot \begin{pmatrix} 0 & 0 \\ \mathbf{C} & 0 \end{pmatrix} = 0$, where $\begin{pmatrix} 0 & 0 \\ \mathbf{C} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{C} & 0 \\ \mathbf{C} & 0 \end{pmatrix}$ is a hereditary subring; thus the condition ii) is not satisfied.

In the case of G to be a finite group, the following result was shown in [4], which gives a correct statement of Montgomery's conjecture. Let A be a prime k-algebra with a finite group G of automorphisms. Then the skew group ring AG is prime (or equivalently the ring extension $AG \supseteq A$ has the ideal intersection property) if and only if A is a faithful AG-module and $h \cdot B \neq 0$ for all $0 \neq h \in k[G]$ and all hereditary subalgebras B.

Here, our theorem impoves that result.

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