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SOME PROPERTIES OF σ-PRODUCTS**

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Abstract

Three theorems concerning the almost θ-expandability, normality, and $[\theta, k]$-compactness $(\theta > \omega)$ of σ-products are proved.

§1. Introduction and Preliminaries

The notion of σ-products was introduced by H.H.Corson[4] and some interesting results was given in [1-4, 8, 9, 13, 14, 16]. This notion plays an important role in the study of a number of covering and separation properties. In this paper we give some other results concerning it.

All spaces in this paper are Hausdorff. The letter ω denotes the first infinite cardinal; k, λ, θ denote cardinal numbers and other Greek letters will denote ordinal numbers. For a set A, we denote the cardinality of A by $|A|$; $A^{<\omega} = \cup\{A^n : n < \omega\}$, where A^n is the set of functions from n to A. For $t \in A^n$ and $a \in A$, we define $t \oplus a$ is the function from $n+1$ into A such that $t \oplus a|n = t$ and $t(n) = a$. Let

$$[A]^{<\omega} = \cup\{[A]^n : n < \omega\},$$

where

$$[A]^n = \{B \subset A : |B| = n\}.$$

Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces and s be a given point of the product space $P = \prod\{X_\alpha : \alpha \in A\}$. For each $x \in P$, let

$$Q(x) = \{\alpha \in A : x_\alpha \neq s_\alpha\}.$$

The subspace $\{x \in P : |Q(x)| < \omega\}$ of P is called the σ-product of $\{X_\alpha : \alpha \in A\}$ with base point s and is denoted by $\sigma\{X_\alpha : \alpha \in A, s\}$.

Let $X = \sigma\{X_\alpha : \alpha \in A, s\}$. For each finite subset c of A, the product $\prod\{X_\alpha : \alpha \in c\}$ is called a finite subproduct of X. For each $B \subset A$, let

$$X|B = \sigma\{X_\alpha : \alpha \in B, s|B\} \times \{s|(A\backslash B)\}.$$

Define a map $p_B : X \to X|B$ by

$$\begin{aligned}(p_B(x))_\alpha &= x_\alpha, \quad \text{if } \alpha \in B,\\ &= s_\alpha, \quad \text{if } \alpha \in A\backslash B\end{aligned}$$

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for each $x \in X$, where $(p_B(x))_\alpha$ denotes the α-coordinate of $p_B(x)$. For $n < \omega$, let

$$\widetilde{X}_n = \{x \in X : |Q(x)| \le n\}.$$

It is easy to verify the following facts.

Fact 1. $X = \cup\{\widetilde{X}_n : n < \omega\}$, where each $\widetilde{X}_n$ is closed in X;

$$\widetilde{X}_n = \cup\{X|c : c \in [A]^n\}, \quad \widetilde{X}_0 = \{s\}.$$

Fact 2. For each $B \subset A$, $X|B$ is closed in X and p_B is a continuous open map from X onto $X|B$ such that $p_B|(X|B) = \mathrm{id}_{(X|B)}$.

Fact 3. If $n < \omega$ and H is an open set in X such that $\widetilde{X}_n \subset H$, then

$$\{p_c^{-1}(X|c\backslash H) : c \in [A]^{n+1}\}$$

is a locally finite family of closed subsets of X.

Proof. See Remark 1 in [3] or the proof of Theorem 3 in [8].

Fact 4[14]. For each $n < \omega$,

$$\{p_c^{-1}(X|c\backslash \widetilde{X}_n) : c \in [A]^{n+1}\}$$

is a point finite collection of open sets in X.

Proof. Let $x \in X$ and $S = \{c \in [A]^{n+1} : c \subset Q(x)\}$, S is finite. For each $c \in [A]^{n+1}\backslash S$, there is $\alpha \in c\backslash Q(x)$. We have

$$|Q(p_c(x))| = |c \cap Q(x)| < |c| = n + 1.$$

It follows that $p_c(x) \in \widetilde{X}_n$; therefore $x \notin p_c^{-1}(X|c\backslash \widetilde{X}_n)$.

A space X is almost θ-expandable if for every locally finite family $\{F_\alpha : \alpha \in A\}$ of closed subsets of X there exists a sequence

$$\langle \mathcal{G}_n = \{G_{n\alpha} : \alpha \in A\}\rangle_{n<\omega}$$

of collections of open subsets of X satisfying the following:

(1) $F_\alpha \subset G_{n\alpha}$ for each $\alpha \in A$ and $n < \omega$.

(2) For each $x \in X$ there is some $n < \omega$ such that $\mathcal{G}_n$ is point finite at x. These spaces were introduced in [6] under the name of "θ-expandable". The name "almost θ-expandable" was suggested by J.C.Smith in [11]. J.Chaber proved that a space is submetacompact if and only if it is almost θ-expandable and has property b_1 (see [12]). An open cover of a space is an A-cover if it has a locally finite refinement. A cover $\mathcal{U}$ of a space X is semi-open if $x \in \mathrm{IntSt}(x,\mathcal{U})$ for each $x \in X$. Let $\mathcal{U}$, $\mathcal{V}_n$, $n < \omega$, be covers of X. The sequence $\langle \mathcal{V}_n\rangle$ is open (semi-open) if $\mathcal{V}_n$ is open (semi-open) for each $n < \omega$; $\langle \mathcal{V}_n\rangle$ is a pointwise W-refining sequence of $\mathcal{U}$ if for each $x \in X$ there is $n < \omega$ and finite $\mathcal{U}' \subset \mathcal{U}$ such that $\{v \in \mathcal{V}_n : x \in V\}$ is a partial refinement of $\mathcal{U}'$ ([15]); $\langle \mathcal{V}_n\rangle$ is a point-star $\dot{F}$-refining sequence of $\mathcal{U}$ if for each $x \in X$ there is some $n < \omega$ and finite $\mathcal{U}' \subset \mathcal{U}$ such that $x \in \cap\mathcal{U}'$ and $\mathrm{St}(x, \mathcal{V}_n) \subset \cup\mathcal{U}'$ (see [5]); $\langle \mathcal{V}_n\rangle$ is a point-star refining sequence of $\mathcal{U}$ if for each $x \in X$ there is some $n < \omega$ and $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}_n) \subset U$.

Fact 5[7]. The following are equivalent for a space X:

(1) X is almost θ-expandable.

(2) For every A-cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n\rangle$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there is $n < \omega$ such that $\mathcal{V}_n$ is point finite at x.

(3) For every directed A-cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n \rangle$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there is some $n < \omega$ and $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}_n) \subset U$.

(4) Every directed A-cover of X has a σ-cushioned refinement.

Let θ, $k \geq \omega$. A space X is called $[\theta, k]$-compact if every open cover of X of cardinality $\leq k$ has a subcover of cardinality $< \theta$. It is easy to verify the following

Fact 6. Suppose that $\mathrm{cf}(\theta) > \gamma$ and $\{F_\alpha : \alpha < \gamma\}$ is a closed cover of a space Y such that F_α is $[\theta, k]$-compact for each $\alpha < \gamma$. Then Y is $[\theta, k]$-compact.

Fact 7[10]. Let $\mathcal{B}$ be a base of a compact space C closed with respect to finite unions and finite intersections. The product space $X \times C$ is normal if and only if X is normal and every $\mathcal{B}$-cover of X has a locally finite open refinement.

§2. Main Results

Theorem 2.1. *Let $X = \sigma\{X_\alpha : \alpha \in A, s\}$. If every finite subproduct of X is almost θ-expandable, then X is almost θ-expandable.*

Proof. Let $\mathcal{U} = \{U_\xi : \xi \in D\}$ be a directed A-cover of X.

Claim. *For each $n < \omega$ and $t \in \omega^n$ there exists a family $\mathcal{W}(t)$ of open subsets of X satisfying the following*

(1) *$\mathcal{W}(t)$ is a partial refinement of $\mathcal{U}$,*

(2) *$\widetilde{X}_n \subset \bigcup_{j=0}^{n} (\cup \mathcal{W}(t|j))$, and*

(3) *If $n > 0$, then for each $x \in X$ there is some $i(x) < \omega$ and $\xi \in D$ such that*

$$St(x, \mathcal{W}(t(n-1) \oplus i(x))) \subset U_\xi.$$

Proof of Claim. For $t = \phi \in \omega^0$, pick $\xi_0 \in D$ such that $s \in U_{\xi_0}$. Let $\mathcal{W}(\phi) = \{U_{\xi_0}\}$. Assume that $\mathcal{W}(t)$ has been constructed for each $t \in \bigcup_{i=0}^{n} \omega^i$. For each $t \in \omega^{n+1}$ there is some $r \in \omega^n$ with $t = r \oplus t(n)$. Let $W = \bigcup_{j=0}^{n} (\cup \mathcal{W}(r|j))$. By (2), $\widetilde{X}_n \subset W$. For each $c \in [A]^{n+1}$,

$$\mathcal{U}_c = \{U_\xi \cap (X|c) \backslash \widetilde{X}_n : \xi \in D\} \cup \{W \cap (X|c)\}$$

is an A-cover of $X|c$. Since $X|c$ is almost θ-expandable, by Fact 5 there exists a sequence $< \mathcal{H}_{ci} >$ of open covers of $X|c$ such that each $\mathcal{H}_{ci}$ is a refinement of $\mathcal{U}_c$ and for each $x \in X|c$ there is some $i < \omega$ such that $\mathcal{H}_{ci}$ is point finite at x. We may assume that for each $i < \omega$

$$\mathcal{H}_{ci} = \{H(c, i, \xi) : \xi \in D\} \cup \{W_{ci}\},$$

$W_{ci} \subset W \cap X|c$ and

$$H(c, i, \xi) \subset U_\xi \cap X|c \backslash \widetilde{X}_n \text{ for each } \xi \in D.$$

For each $k < \omega$, let

$$\mathcal{V}_{ck} = \{p_c^{-1}(\bigcap_{j=0}^{k} H(c, j, \xi_j)) \cap U_{\xi_k} : \xi_0, \cdots, \xi_k \in D\}.$$

Then $\mathcal{V}_{ck}$ are collections of open sets of X such that $\mathcal{V}_{ci}$ is a partial refinement of $\mathcal{V}_{cj}$ for each $j \leq i$. Since $\mathcal{U}$ is directed, we have

(4) For each $c \in [A]^{n+1}$ and $x \in X$ there is some $i < \omega$ and $\xi \in D$ such that $\mathrm{St}(x, \mathcal{V}_{ci}) \subset U_\xi$. Define

$$\mathcal{W}(t) = \mathcal{W}(r \oplus t(n)) = \cup\{\mathcal{V}_{ct(n)} : c \in [A]^{n+1}\}.$$

Then $\mathcal{W}(t)$ is an open partial refinement of $\mathcal{U}$. To see (2), let $x \in \widetilde{X}_{n+1}\backslash W$. There is some $c \in [A]^{n+1}$ such that $x \in X|c\backslash W$. Let $k = t(n)$ and pick $\xi_j \in D$ such that $x \in H(c, j, \xi_j)$ for each $j = 0, \cdots, k$. Since, by Fact 2, $p_c(x) = \mathrm{id}_{X|c}(x) = x$,

$$x \in p_c^{-1}(\bigcap_{j=0}^{k} H(c, j, \xi_j)) \cap U_{\xi_k} \in \mathcal{W}(t).$$

It follows that $\widetilde{X}_{n+1}\backslash W \subset \cup\mathcal{W}(t)$, so $\widetilde{X}_{n+1} \subset \bigcup_{j=0}^{n+1}(\cup\mathcal{W}(t|j))$. Thus (2) is satisfied. To see (3), let $x \in X$. By Fact 4, we may assume that

$$\{c \in [A]^{n+1} : x \in p_c^{-1}(X|c\backslash\widetilde{X}_n)\} \subset \{c_0, \cdots, c_m\}.$$

For each $j = 0, \cdots, m$, by (4) there is some $i_j < \omega$ and $\xi_j \in D$ such that $\mathrm{St}(x, \mathcal{V}_{c_j i_j}) \subset U_{\xi_j}$. Let $\xi \in D$ such that $\bigcup_{j=0}^{m} U_{\xi_j} \subset U_\xi$ and let $i(x) = \max\{i_0, \cdots, i_m\}$. We have

$$\mathrm{St}(x, \mathcal{W}(r \oplus i(x))) = \bigcup_{j=0}^{m} \mathrm{St}(x, \mathcal{V}_{c_j i(x)}) \subset \bigcup_{j=0}^{m} \mathrm{St}(x, \mathcal{V}_{c_j i_j}) \subset U_\xi.$$

The condition (3) is satisfied and the proof of Claim is thus complete.

For each $n < \omega$ and $t \in \omega^n$, let

$$\mathcal{G}(t) = \bigcup_{j=0}^{n} \mathcal{W}(t|j) \cup \mathcal{U}|(X\backslash\widetilde{X}_n).$$

Then $\langle\mathcal{G}(t) : t \in \omega^{<\omega}\rangle$ is a sequence of open refinements of $\mathcal{U}$. For each $x \in X$, there is some $n < \omega$ such that $x \in \widetilde{X}_n$. If $n = 0$, then $x = s \in U_{\xi_0} \in \mathcal{W}(\phi)$, so $\mathrm{St}(x, \mathcal{G}(\phi)) \subset U_{\xi_0}$. Now assume $n > 0$. By (3), there is some sequence $\langle i_0(x), \cdots, i_{n-1}(x)\rangle$ of natural numbers and sequence $\langle\eta_1, \cdots, \eta_n\rangle$ such that $\mathrm{St}(x, \mathcal{W}(t_j)) \subset U_{\eta_j}$ for each $j = 1, \cdots, n$, where $t_j = t_{j-1} \oplus i_{j-1}(x)$, $t_0 = \phi$. Define $t(x) = t_n$, then $t(x) \in \omega^n$ and $t(x)|j = t_j$ for each $j = 0, \cdots, n$. Let $\xi \in D$ such that

$$U_{\xi_0} \cup U_{\eta_1} \cup \cdots \cup U_{\eta_n} \subset U_\xi.$$

Then

$$\mathrm{St}(x, \mathcal{G}(t(x))) = \bigcup_{j=0}^{n} \mathrm{St}(x, \mathcal{W}(t(x)|j)) \subset U_\xi.$$

By Fact 5, X is almost θ-expandable.

Theorem 2.2. *For any space X the following are equivalent.*

(1) *X is almost θ-expandable.*

(2) *Every A-cover of X has an open pointwise W-refining sequence.*

(3) *Every A-cover of X has a semi-open point-star $\dot{F}$-refining sequence.*

(4) *Every directed A-cover of X has a semi-open point-star refining sequence.*

(5) *Every directed A-cover of X has a σ-closure-preserving closed refinement.*

Proof. (1) $\rightarrow$ (2) By Fact 5, every A-cover $\mathcal{U}$ of an almost θ-expandable space has a sequence $\langle \mathcal{V}_n \rangle$ of open refinements such that for each $x \in X$ there is some $n < \omega$ such that $\mathcal{V}_n$ is point finite at x. Then $\langle \mathcal{V}_n \rangle$ is a pointwise W-refining sequence of $\mathcal{U}$.

(2) $\rightarrow$ (3) Clear.

(3) $\rightarrow$ (4) Let $\mathcal{U}$ be a directed A-cover of X. By (3), $\mathcal{U}$ has a semi-open point-star $\dot{F}$-refining sequence $\langle \mathcal{V}_n \rangle$. Since $\mathcal{U}$ is directed, $\langle \mathcal{V}_n \rangle$ is a point-star refining sequence of $\mathcal{U}$.

(4) $\rightarrow$ (1) Let $\mathcal{U}$ be a directed A-cover of X. By (4), $\mathcal{U}$ has a semi-open point-star refining sequence $\langle \mathcal{V}_n \rangle$. For each $A \subset X$ and $n < \omega$, let

$$W(n, A) = \{x \in X : \mathrm{St}(x, \mathcal{V}_n) \subset A\}.$$

Then $\mathrm{Cl}(W(n, A)) \subset A$. For each $n < \omega$, let

$$\mathcal{W}_n = \{W(n, U) : U \in \mathcal{U}\}.$$

It is easy to prove that $\bigcup_{n=0}^{\infty} \mathcal{W}_n$ is a σ-cushioned refinement of $\mathcal{U}$. By Fact 5, X is almost θ-expandable.

(1) $\rightarrow$ (5) Let $\mathcal{U} = \{U(t) : t \in T\}$ be a directed A-cover of an almost θ-expandable space X. By Fact 5, there exists a sequence $\langle \mathcal{V}_n \rangle$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there is $n < \omega$ such that $\mathcal{V}_n$ is point-finite at x. We may assume that $\mathcal{V}_n = \{V(n, t) : t \in T\}$ and $V(n, t) \subset U(t)$ for each $n < \omega$, $t \in T$. Let $S = \{s \subset T : |s| < \omega\}$. For any n, $m < \omega$ and $s \in S$, let

$$H(n, m) = \{x \in X : |\{V \in \mathcal{V}_n : x \in V\}| \leq m + 1\},$$

$$F(n, m, s) = \{H(n, m) \setminus \bigcup \{V(n, t) : t \in T \setminus s\}.$$

Since $H(n, m)$ is closed in X and $\{H(n, m) \cap V(n, t) : t \in T\}$ is a point-finite family of open subsets of $H(n, m)$, the family

$$\mathcal{F}_{nm} = \{F(n, m, s) : s \in S\}$$

is a closure-preserving family of closed subsets of X. For each $x \in X$, there is some $n(x) < \omega$ and $m(x) \geq 1$ such that

$$\{V \in \mathcal{V}_{n(x)} : x \in V\} = \{V(n(x), t_1), \cdots, V(n(x), t_{m(x)})\}.$$

Then $x \in H(n(x), m(x))$. Let $s(x) = \{t_1, \cdots, t_{m(x)}\}$. Take a $t(x) \in T$ such that

$$U(t_1) \cup \cdots \cup U(t_{m(x)}) \subset U(t(x)).$$

Then

$$x \in F(n(x), m(x), s(x)) \subset U(t(x)).$$

It follows that $\{F(n(x), m(x), s(x)) : x \in X\}$ is a σ-closure-preserving closed refinement of $\mathcal{U}$.

(5) $\rightarrow$ (1) Any σ-closure-preserving closed refinement of a cover of X is clearly a σ-cushioned refinement. By Fact 5, X is almost θ-expandable.

Corollary 2.1. *A continuous image of an almost θ-expandable space under a closed mapping is almost θ-expandable.*

Proof. Let f be a closed and continuous mapping from an almost θ-expandable space X onto a space Y and $\mathcal{U}$ be a directed A-cover of Y. Then

$$\mathcal{V}=\{f^{-1}(U):U\in\mathcal{U}\}$$

is a directed A-cover of X. By Theorem 2.2, the cover $\mathcal{V}$ has a refinement $\bigcup_{n=0}^{\infty}\mathcal{F}_n$, where each $\mathcal{F}_n$ is a closure-preserving family of closed subsets of X. For each $n<\omega$, let

$$\mathcal{K}_n=\{f(F):F\in\mathcal{F}_n\}.$$

Then $\bigcup_{n=0}^{\infty}\mathcal{K}_n$ is a σ-closure-preserving closed refinement of $\mathcal{U}$. By Theorem 2.2, Y is almost θ-expandable.

Theorem 2.3. *Let $X=\sigma\{X_\alpha:\alpha\in A,s\}$ and C be a compact space. Suppose that X is normal and the product $\prod\{X_\alpha:\alpha\in c\}\times C$ is normal for each finite subset c of A. Then $X\times C$ is normal.*

Proof. Let $\mathcal{B}$ be a base of C closed with respect to finite unions and finite intersections and

$$\mathcal{G}=\{G(B,D):(B,D)\in\mathcal{E}\}$$

be a $\mathcal{B}$-cover of X, where

$$\mathcal{E}=\{(B,D):B,D\in\mathcal{B}\text{ and }\mathrm{Cl}(B)\cap\mathrm{Cl}(D)=\emptyset\}.$$

Claim. *For each $n<\omega$, there is an open subset H_n of X and a collection $\mathcal{V}_n$ of open subsets of X satisfying the following*

(1) *$\mathcal{V}_n$ is locally finite and is a partial refinement of $\mathcal{G}$.*

(2) *$\widetilde{X}_n\subset H_n\subset H_{n+1}$ and $\mathrm{Cl}(H_n)\subset\bigcup_{i=0}^{n}(\cup\mathcal{V}_i)$.*

(3) *$\mathrm{Cl}(H_n)\cap(\cup\mathcal{V}_{n+1})=\emptyset$.*

Proof of Claim. For $n=0$, let H_0 be an open subset of X and $(B_0,D_0)\in\mathcal{E}$ such that

$$s\in H_0\subset\mathrm{Cl}(H_0)\subset G(B_0,D_0).$$

Let $\mathcal{V}_0=\{G(B_0,D_0)\}$. Let us assume that H_i and $\mathcal{V}_i$ has been constructed for each $i\le n$. Let $V=\bigcup_{i=0}^{n}(\cup\mathcal{V}_i)$. By (2),

$$\widetilde{X}_n\subset H_n\subset\mathrm{Cl}(H_n)\subset V.$$

For each $c\in[A]^{n+1}$,

$$\{G(B,D)\cap X|c:(B,D)\in\mathcal{E}\}$$

is a $\mathcal{B}$-cover of $X|c$. Since $(X|c)\times C$ is normal, there is a locally finite open cover $\{V(c,B,D):(B,D)\in\mathcal{E}\}$ of $X|c$ such that

$$V(c,B,D)\subset G(B,D)\cap X|c$$

for each $(B,D)\in\mathcal{E}$. Let

$$\mathcal{V}_{n+1}=\cup\{\mathcal{V}_c:c\in[A]^{n+1}\},$$

where

$$\mathcal{V}_c=\{p_c^{-1}(V(c,B,D)\setminus\mathrm{Cl}(H_n))\cap G(B,D)\setminus\mathrm{Cl}(H_n):(B,D)\in\mathcal{E}\}.$$

Since $\cup\mathcal{V}_c \subset p_c^{-1}(X|c\backslash H_n)$ and, by Fact 3,

$$\{p_c^{-1}(X|c\backslash H_n) : c \in [A]^{n+1}\}$$

is locally finite, $\mathcal{V}_{n+1}$ is a locally finite collection of open subsets of X. It is also a partial refinement of $\mathcal{G}$. For each $x \in \widetilde{X}_{n+1}\backslash V$, there is some $c \in [A]^{n+1}$ such that $x \in X|c\backslash V$. Let $(B, D) \in \mathcal{E}$ such that $x \in V(c, B, D)$. Since $p_c(x) = \mathrm{id}_{(X|c)}(x) = x$,

$$x \in p_c^{-1}(V(c,B,D)\backslash \operatorname{Cl}(H_n)) \cap G(B,D)\backslash \operatorname{Cl}(H_n) \in \mathcal{V}_{n+1}.$$

Then $\widetilde{X}_{n+1}\backslash V \subset \cup\mathcal{V}_{n+1}$, so $\widetilde{X}_{n+1} \subset \bigcup\limits_{i=0}^{n+1}(\cup\mathcal{V}_i)$. Since X is normal and

$$\widetilde{X}_{n+1} \cup \operatorname{Cl}(H_n) \subset \widetilde{X}_{n+1} \cup V \subset \bigcup_{i=0}^{n+1}(\cup\mathcal{V}_i).$$

there is an open subset H_{n+1} of X such that

$$\widetilde{X}_{n+1} \cup \operatorname{Cl}(H_n) \subset H_{n+1} \subset \operatorname{Cl}(H_{n+1}) \subset \bigcup_{i=0}^{n+1}(\cup\mathcal{V}_i).$$

It follows that $\mathcal{V}_{n+1}$ and H_{n+1} satisfy the conditions (1)-(3); the proof of Claim is complete.

Let $\mathcal{V} = \bigcup\limits_{n=0}^{\infty} \mathcal{V}_n$. By (1) and (2), $\mathcal{V}$ is an open refinement of $\mathcal{G}$. Let $x \in X$. There exists $m < \omega$ such that $x \in \widetilde{X}_m \subset H_m$. For each $i \geq m$, by (3),

$$H_m \cap (\cup\mathcal{V}_{i+1}) \subset \operatorname{Cl}(H_i) \cap (\cup\mathcal{V}_{i+1}) = \phi.$$

For each $i \leq m$, there is a neighbourhood S_i of x which intersects only finitely many members of $\mathcal{V}_i$. Then $H_m \cap S_0 \cap \cdots \cap S_m$ is a neighbourhood of x which intersects only finitely many members of $\mathcal{V}$. $\mathcal{V}$ is a locally finite open refinement of $\mathcal{G}$. By Fact 7, $X \times C$ is normal.

Theorem 2.4. *Let $X = \sigma\{X_\alpha : \alpha \in A, s\}$ and θ be a regular cardinal strictly greater than ω. Suppose every finite subproduct of X is $[\theta, k]$-compact. Then X is $[\theta, k]$-compact.*

Proof. Since $X = \cup\{\widetilde{X}_n : n < \omega\}$, by Fact 6 it suffices to show that each $\widetilde{X}_n$ is $[\theta, k]$-compact. Clearly $\widetilde{X}_0 = \{s\}$ is $[\theta, k]$-compact. Let us assume $\widetilde{X}_n$ is $[\theta, k]$-compact. Suppose $\mathcal{U} = \{U_\gamma : \gamma < k\}$ is a collection of basic open sets in X such that $\cup\,\mathcal{U} \supset \widetilde{X}_{n+1} \supset \widetilde{X}_n$. There is a subfamily $\mathcal{U}_0 = \{U_\gamma : \gamma \in S\}$ of $\mathcal{U}$ with $S \subset k$, $|S| < \theta$ and $\cup\,\mathcal{U}_0 \supset \widetilde{X}_n$. For each $\gamma \in S$, there is a finite subset $b(\gamma) \subset A$ and an open set $V_{\gamma\beta}$ of X for each $\beta \in b(\gamma)$ such that

$$U_\gamma = X \cap (\prod\{V_{\gamma\beta} : \beta \in b(\gamma)\} \times \prod\{X_\beta : \beta \in A\backslash b(\gamma)\}).$$

Let $B = \cup\{b(\gamma) : \gamma \in S\}$. It is easy to see that

$$|B| < \theta \text{ and } p_B^{-1}(p_B(U_\gamma)) = U_\gamma \text{ for each } \gamma \in S.$$

Let

$$Y = \cup\{X|c : c \in [A]^{n+1} \cap p(B)\},$$

where $p(B)$ denotes the power set of B. For each $c \in [A]^{n+1}\backslash p(B)$ and $x \in X|c$, $c \not\subset B$, so

$$|Q(p_B(x))| \leq |B \cap c| < |c| = n + 1.$$

Then $p_B(x) \in \widetilde{X}_n$. There is $\gamma \in S$ such that $p_B(x) \in U_\gamma$. Since, by Fact 2,

$$p_B(x) = \mathrm{id}_{(X|B)}(p_B(x)) = p_B(p_B(x)),\ x \in p_B^{-1}(p_B(U_\gamma)) = U_\gamma.$$

It follows that

$$\cup\{X|c : c \in [A]^{n+1}\backslash p(B)\} \subset \cup\,\mathcal{U}_0.$$

Thus

$$\widetilde{X}_{n+1}\backslash \cup\ \mathcal{U}_0 \subset Y \subset \widetilde{X}_{n+1}.$$

Since $|[A]^{n+1} \cap p(B)| < \theta$, by Fact 6, Y is $[\theta, k]$-compact. There is a subfamily $\mathcal{U}_1$ of $\mathcal{U}$ such that $|\mathcal{U}_1| < \theta$ and $Y \subset \cup\,\mathcal{U}_1$. Then $\widetilde{X}_{n+1} \subset \cup(\mathcal{U}_0 \cup \mathcal{U}_1)$. Therefore, $\widetilde{X}_{n+1}$ is $[\theta, k]$-compact.

The following example shows that Theorem 2.4 is false when $\theta = \omega$.

Example 6. For each $n < \omega$, let X_n denotes the two-point discrete space $D(2)$ and

$$X = \sigma\{X_n : n < \omega, 0\},$$

where $0 = (0, 0, \cdots)$. Every finite product of $\{X_n : n < \omega\}$ is compact. Because X is a dense proper subspace of the compact space $D(2)^\omega$, X is not compact. On the other hand, it follows from Theorem 5 that X is Lindelöf.

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