Chin. Ann. of Math.
14B: 1(1993),55-58.

CLASSIFYING INVOLUTIONS ON $PR(2k)$ UP TO EQUIVARIANT COBORDISM***

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Abstract

It is proved that there are exactly $k+1$ involutions on $RP(2k)$ up to equivariant cobordism.

§1. Introduction

Let (M^n, τ) be a smooth involution on the smooth closed manifold M^n. Let A denote the antipodal involution on the sphere S^m. Then $A \times \tau$ is a free smooth involution on $S^m \times M^n$. By identifying (x, y) with $(Ax, \tau y)$ in $S^m \times M^n$, we obtain a smooth closed manifold $R^m(\tau)$ of dimension $m+n$. In ([1], p.165), we have proved

Theorem 1.1. *(M_1^n, τ_1) is equivariant cobordant to (M_2^n, τ_2) if and only if $R^m(\tau_1)$ is cobordant to $R^m(\tau_2)$ for all $m \geq 0$.*

In [2], we have determined the equivariant cobordism classes of smooth involutions on $RP(2k+1)$ by applying Theorem 1.1. As a further application, we prove in this paper the following theorem.

Theorem 1.2. *There are exactly $k+1$ smooth involutions on $RP(2k)$ up to equivariant cobordism, which are $1, \tau_0, \tau_1, \cdots, \tau_{k-1}$. Here $1 = \tau_{-1}$ is the identical involution on $RP(2k)$ and τ_i is such a smooth involution on $RP(2k)$ that*

$$\tau_i[x_0, x_1, \cdots, x_{2k}] = [-x_0, -x_1, \cdots, -x_i, x_{i+1}, \cdots, x_{2k}].$$

We assume $k > 0$ throughout this paper.

§2. Proof of Theorem 1.2

Firstly, we need the following lemma.

Lemma 2.1. *$R^m(\tau_i)$ is diffeomorphic to the projective space bundle $RP((i+1)\lambda_m \oplus (2k-i)R)$, where λ_m and R are, respectively, the canonical line bundle and the trivial line bundle over $RP(m)$.*

Proof. The vector bundle $(i+1)\lambda_m \oplus (2k-i)R$ over $RP(m)$ may be formed from $S^m \times R^{2k+1}$ by identifying $(x, t_0, t_1, \cdots, t_i, t_{i+1}, \cdots, t_{2k})$ with

$$(Ax, -t_0, -t_1, \cdots, -t_i, t_{i+1}, \cdots, t_{2k}).$$

Manuscript received August 13, 1990. Revised August 5, 1991.
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***Project supported by the National Natural Science Foundation of China .

Let $f: S^m \times R^{2k+1} \to (i+1)\lambda_m \oplus (k-i)R$ be the identification. Then f is a morphism between vector bundles. Thus f induces a morphism $\widetilde{f} :: S^m \times RP(2k) \to RP((i+1)\lambda_m \oplus (2k-i)R)$ between projective space bundles. But $\widetilde{f}$ is still an identification identifying

$$(x, [t_0, t_1, \cdots, t_i, t_{i+1}, \cdots, t_{2k}]) \text{ with } (Ax, [-t_0, -t_1, \cdots, -t_i, t_{i+1}, \cdots, t_{2k}])$$

in $S^m \times RP(2k)$. Thus $RP((i+1)\lambda_m \oplus (2k-i)R)$ is the identification space $R^m(\tau_i) = S^m \times RP(2k)/A \times \tau_i$. Since we put such a differentiable structure on the identification space which makes the identification map smooth, the lemma follows.

Next, we consider the cohomology ring $H^*(R^m(\tau); Z_2)$ for an arbitrary smooth involution τ on $RP(2k)$. Note that $(R^m(\tau), p, RP(m))$ is a differentiable fibre bundle with fibre $RP(2k)$ and with the structure group Z_2. From the Euler characteristic relation $\chi(RP(2k)) = \chi(F)$ (mod 2) (see [3]), we know that the fixed point set F of τ is nonempty. Let y be a fixed point. Then we have a cross-section $\rho_y : RP(m) \to RP(m) \times y \subseteq S^m \times RP(2k)/A \times \tau = R^m(\tau)$ for the fibre projection p. Let $i_m : R^m(\tau) \to R^{m_1}(\tau)$ be the natural inclusion, $m \le m_1$; and let $i : RP(2k) \to R^m(\tau)$ be the fibre inclusion. We have the following lemma.

Lemma 2.2. *For every integer $m \ge 1$, there exists $c_m \in H^1(R^m(\tau); Z_2)$ such that*

(1) $i_m^* c_{m_1} = c_m$ for $m \le m_1$;

(2) $i^* c_m$ is a generator of $H^1(RP(2k); Z_2)$; thus $H^*(R^m(\tau); Z_2)$ is a free $H^*(RP(m); Z_2)$ module with basis $\{1, c_m, c_m^2, \cdots, c_m^{2k}\}$, the module action is given by $b.e = p^*(b) \cup e$, where $b \in H^*(RP(m); Z_2)$ and $e \in H^*(R^m(\tau); Z_2)$.

Proof. Firstly, we prove $H^1(R^m(\tau); Z_2) = Z_2 \oplus Z_2$. In fact, by the homotopy exact sequence associated with a fibration, we have the following short exact sequence

$$0 \to \Pi_1(RP(2k), y) \xrightarrow{i_*} \Pi_1(R^m(\tau), y) \xrightarrow{p_*} \Pi_1(RP(m), x) \to 0,$$

where $x \in RP(m)$ and $y \in p^{-1}(x) = RP(2k)$. Thus

$$\hat{\Pi}_1(R^m(\tau), y)) = \Pi_1(R^m(\tau), y)/\mathrm{Comm}\Pi_1(R^m(\tau), y) \approx A_1 \oplus A_2,$$

where $A_i = Z_2$ or Z, $i = 1, 2$. By Hurewicz theorem, we get $H_1(R^m(\tau); Z) \approx A_1 \oplus A_2$. Thus the universal coefficient theorem implies $H^1(R^m(\tau); Z_2) = Z_2 \oplus Z_2$.

Secondly, $H^*(R^m(\tau), R^{m-1}(\tau); Z_2) \approx H^*(RP(2k); Z_2) \otimes H^*(D^m, S^{m-1}; Z_2)$, where $D^m = \{x |\ \|x\| \le 1, x \in R^m\}$ and S^{m-1} is the boundary of D^m.

The normal bundle of the natural embedding of $RP(m-1)$ in $RP(m)$ is precisely the canonical bundle λ over $RP(m-1)$. Let $D(\lambda)$ denote disk bundle associated with λ over $RP(m-1)$. By the tubular neighborhood theorem ([4], p.115), we may regard $D(\lambda)$ as a tube neighborhood around $RP(m-1)$ in $RP(m)$ such that $RP(m) - \mathring{D}(\lambda) = D^m$. Since $D(\lambda)$ is homotopic to $RP(m-1)$ and since D^m is contractible, the inclusion $R^{m-1}(\tau) \to R^m(\tau)$ is a homotopy equivalence and $p^{-1}(D^m)$ may be regarded as $RP(2k) \times D^m$. Thus

$$\begin{aligned} H^*(R^m(\tau), R^{m-1}(\tau); Z_2) &\approx H^*(R^m(\tau), p^{-1}(D(\lambda)); Z_2) \\ &\approx H^*(p^{-1}(D^m), p^{-1}(S^{m-1}); Z_2) \quad \text{(excision)} \\ &\approx H^*(RP(2k) \times D^m, RP(2k) \times S^{m-1}; Z_2) \\ &\approx H^*(RP(2k); Z_2) \otimes H^*(D^m, S^{m-1}; Z_2) \quad \text{(Kunneth Theorem).} \end{aligned}$$

Finally, $i_{m-1}^* : H^1(R^m(\tau); Z_2) \to H^1(R^{m-1}(\tau); Z_2)$ is an isomorphism for $m \ge 2$ and a surjection for $m = 1$. Thus we may choose $c_m \in H^1(R^m(\tau); Z_2)$ for every $m \ge 1$ such that $i_m^* c_{m_1} = c_m$ and $i^* c_m$ is a generator of $H^1(RP(2k); Z_2)$. By Leray-Hirsch theorem ([5], p.365), we see that $H^*(R^m(\tau); Z_2)$ is a free $H^*(RP(m); Z_2)$ module with basis $\{1, c_m, c_m^2, \cdots, c_m^{2k}\}$, and the module action is just as the form in the lemma.

Now we turn to the Stiefel-Whitney total classes $W(R^m(\tau))$ of $R^m(\tau)$. We need the following lemma.

Lemma 2.3([6], p.442], Proposition 8.4 and its proof). *Let $c \in H^1(R^m(\tau); Z_2)$ be such an element that $\{1, c, c^2, \cdots, c^{2k}\}$ is a basis of $H^*(R^m(\tau); Z_2)$ as a free $H^*(RP(m); Z_2)$ module. Then there is a unique element $1 + a_1 + a_2 + \cdots + a_{2k+1} \in H^*(RP(m); Z_2)$ such that $a_i \in H^i(RP(m); Z_2)$, the elements $1, a_1, \cdots, a_{2k+1}$ follow the Wu formula ([4], p.94), and*

$$c^{2k+1} = p^*(a_1)c^{2k} + p^*(a_2)c^{2k-1} + \cdots + p^*(a_{2k})c + p^*(a_{2k+1}),$$

$$W(R^m(\tau)) = p^*(W(RP(m)))((1+c)^{2k+1} + p^*(a_1)(1+c)^{2k} + \cdots + p^*(a_{2k})(1+c) + p^*(a_{2k+1})).$$

Lemma 2.3 allows us to prove the following proposition.

Proposition 2.1. *Let $\{c_1, \cdots, c_m, \cdots\}$ be an element sequence as in Lemma 2.2. Then there is a unique integer d, $0 \le d \le 2k+1$, such that*

$$W(R^m(\tau)) = p^*(W(RP(m)))(1 + p^*(a) + c_m)^d (1 + c_m)^{2k+1-d},$$

$$c_m^{2k+1} = \sum_{i=1}^{d} \binom{d}{i} p^*(a^i) c_m^{2k+1-i}$$

for all $m \ge 1$, where $a \in H^1(RP(m); Z_2)$ is the generator.

Proof. Choose a sufficient large integer $m_0 \ge 2k+1$. Then there is a unique integer d, $0 \le d \le 2k+1$, such that for this m_0 the formula $1 + a_1 + \cdots + a_{2k+1} = (1+a)^d$ holds.

Let $j = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_t}$, $j_1 > j_2 > \cdots > j_t \ge 0$. Suppose $2^i < 2^{j_t}$ for some $i \ge 0$. Then the Wu formula and the fact that $\binom{n_1}{n_2} \equiv 1 \pmod 2$ if and only if the powers which occur in the binary expression of n_2 occur in the binary expression of n_1 imply

$$0 = S_q^{2^i}(a_j) = a_{2^i} a_j + a_{j+2^i}$$

(Since $a_j = \varepsilon a^j$, $\varepsilon = 0$ or 1) and $a_{j+2^i} = a_{2^i} a_j$. Thus $a_j = a_{2^{j_1}} a_{2^{j_2}} \cdots a_{2^{j_t}}$.

Let $i_1 > i_2 > \cdots > i_r \ge 0$ be all the numbers such that $a_{2^{i_1}}, a_{2^{i_2}}, \cdots, a_{2^{i_r}}$ are nonzero. Define $d = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}$. We claim that for this fixed m_0, there must hold $(1+a)^d = 1 + a_1 + \cdots + a_{2k+1}$. In fact, let $0 \le j \le m_0$ and $a_j \ne 0$. From $a_j = a_{2^{j_1}} a_{2^{j_2}} \cdots a_{2^{j_t}}$ where $j = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_t}$ and $j_1 > j_2 > \cdots > j_t \ge 0$, we have $\{j_1, j_2, \cdots, j_t\} \subseteq \{i_1, i_2, \cdots, i_r\}$. Thus $\binom{d}{j} \equiv 1 \pmod 2$ and the jth homogeneous element in $(1+a)^d$ is exactly the a_j. If $a_j = 0$, then there must exist some j_g such that $a_{2^{j_g}} = 0$. Thus $\{j_1, j_2, \cdots, j_t\} \not\subseteq \{i_1, i_2, \cdots, i_r\}$ and the jth homogeneous element in $(1+a)^d$ is zero. In the case $j > m_0$, both the jth elements $\binom{d}{j} a^j$ and a_j are zero. Put the discussions above together, we see $1 + a_1 + a_2 + \cdots + a_{2k+1} = (1+a)^d$ for this m_0.

To complete the proof of this proposition, we note that

$$TR^m(\tau) \approx TRP(m) \oplus \bar{T}_m RP(2k)$$

holds for all $m \ge 1$ ([7], p.482), where $TR^m(\tau)$ and $TRP(m)$ are the tangent bundles of $R^m(\tau)$ and $RP(m)$ respectively, and $\bar{T}_m RP(2k)$ is the tangent bundle along the fibres of the differentiable fibre bundle $R^m(\tau)$ over $RP(m)$. Thus

$$W(\bar{T}_m RP(2k)) = (1+c_m)^{2k+1} + p^*(a_1)(1+c_m)^{2k} + \cdots + p^*(a_{2k})(1+c_m) + p^*(a_{2k+1})$$

for all $m \geq 1$. Since $i_m^* \bar{T}_{m_1} RP(2k) = \bar{T}_m RP(2k)$, $i_m^* c_{m_1} = c_m$ for $m \leq m_1$, we have

$$(1+a)^d = 1 + a_1 + \cdots + a_{2k+1}$$

for all $m \geq 1$. This leads to

$$W(R^m(\tau)) = p^*(W(RP(m)))(1+c_m)^{2k+1-d}(1+p^*(a)+c_m)^d$$

for all $m \geq 1$. Consequently, the dimension $2k$ of the vector bundle $\bar{T}_m RP(2k)$ implies

$$c_m^{2k+1} = \sum_{i=1}^{d} \binom{d}{i} p^*(a^i) c_m^{2k+1-i}$$

for all $m \geq 1$. The uniqueness of the integer d is obvious.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Suppose d is the unique integer in Proposition 2.1 for the smooth involution τ on $RP(2k)$. By ([3], p.75), for every $m \geq 1$, there is $c'_m \in H^1(RP(d\lambda_m \oplus (2k+1-d)R); Z_2)$ such that $i_m^* c'_{m_1} = c'_m$ and $H^*(RP(d\lambda_m \oplus (2k+1-d)R); Z_2)$ is a free $H^*(RP(m); Z_2)$ module with basis $\{1, c'_m, (c'_m)^2, \cdots, (c'_m)^{2k}\}$, the module action is just as in Lemma 2.2, and

$$W(RP(d\lambda_m \oplus (2k+1-d)R)) = p^*(W(RP(m)))(1+c'_m)^{2k+1-d}(1+p^*(a)+c'_m)^d,$$

$$(c'_m)^{2k+1} = \sum_{i=1}^{d} \binom{d}{i} p^*(a^i)(c'_m)^{2k+1-i},$$

where i_m is the natural inclusion for $m \leq m_1$, and $p: RP(d\lambda_m \oplus (2k+1-d)R) \to RP(m)$ is the fibre projection. Thus the homomorphism

$$f: H^*(RP(d\lambda_m \oplus (2k+1-d)R); Z_2) \to H^*(R^m(\tau); Z_2)$$

such that $f(c'_m) = c_m$ and $f(p^*(a)) = p^*(a)$ is a ring isomorphism. Let W denote the total Stiefel-Whitney classes of $RP(d\lambda_m \oplus (2k+1-d)R)$. By Proposition 2.1, the total Stiefel-Whitney classes of $R^m(\tau)$ is $f(W)$. Therefore both $R^m(\tau)$ and $RP(d\lambda_m \oplus (2k+1-d)R)$ have the Stiefel-Whitney numbers and $R^m(\tau)$ is cobordant to $RP(d\lambda_m \oplus (2k+1-d)R) = R^m(\tau_{d-1})$ for all $m \geq 1$ (Lemma 2.1). Since $R^0(\tau) = RP(2k) = R^0(\tau_{d-1})$, by Theorem 1.1, τ is equivariant cobordant to τ_{d-1}. Since both τ_{d-1} and τ_{2k-d-1} have the same fixed point sets and the same normal bundles of the fixed point sets in $RP(2k)$, τ_{d-1} is equivariant cobordant τ_{2k-d-1} ([3], 25.2, p.88). The proof is complete.

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