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ON THE ESSENTIAL SPECTRUM OF A COMPLETE RIEMANNIAN MANIFOLD WITH A POLE**

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Abstract

It is shown that the essential spectrum of a complete Riemannian manifold with a pole and infinite volume can be computed via the growth rate of the volume of the geodesic ball under some technical condition. The growth-rate assumption is somewhat weaker than the curvature assumption used by many authors.

§1. Introduction

Let M^n be a complete Riemannian manifold of dimension n. The Laplacian Δ of M on $C_0^\infty(M)$ has a unique extension Δ to an unbounded self-adjoint operator on $L^2(M)$. One defines the essential spectrum of M, denoted by Ess Spec (M), to be those real numbers which are either cluster points of the spectrum of Δ or eigenvalues of infinite multiplicity for Δ.

In [1–3], the Ess Spec M has been computed where M is assumed to be some Riemannian manifold which can be "compared" with the space form. In [1] M is assumed to be simply connected and negatively curved and the sectional curvature tends to $-c$ at infinity and then the essential spectrum of M is $[\frac{1}{4}a^2, +\infty)$; furthermore, in [2], the authors pointed out that if c is infinite, then the Ess Spec $M = \emptyset$. In [3], it is proved that if M has a rotational invariant metric outside a compact set and the radical sectional curvature is nonnegative, then M has no eigenvalues and Ess Spec $M = [0, +\infty)$. In order to obtain these results, a strong geometric and topological assumption must be given, which implies that the manifold possess a pole. On the other hand, we do not know much about the relation between the essential spectrum and the growth of the volume of a Riemannian manifold. In [4–6], some corse inequalities are obtained (we will refine one of them in §3) under the assumption of the growth of the volume of the manifold. In this paper, we compute the Ess Spec M under the assumption that M has a rotational invariant metric and the growth of the volume of the geodesic ball satisfies $\log(V(r))–ar–b \to 0$ as $r \to \infty$. We also give an example to show that the condition of rotational invariant metric is necessary. In §4, we introduce the concept of "comparison" manifold to generalize the result to some non-rotational invariant metric case.

We will need the following theorem concerning the abstract spectrum theory:

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Theorem A. *The necessary and sufficient condition for $\sigma \in$ Ess Spec M is that for every $\epsilon > 0$ there exists an infinite dimensional subspace G_ϵ of $\mathcal{D}(\Delta)$, the domain of Δ, such that for each $f \in G_\epsilon$ we have $\|\Delta f + \sigma f\| < \epsilon\|f\|$, where $\|\cdot\|$ is the L^2-norm.*

§2. The Case of Rotational Invariant Metric

The main purpose of this section is to prove the following

Theorem 2.1. *Suppose M is an n-dimensional noncompact complete Riemannian manifold with rotational invariant metric (thus must have a pole), i.e., the Riemannian metric can be written as $ds^2 = dr^2 + \omega^2 d\theta^2$ where r is the distance to the pole and $d\theta^2$ is the metric on S^{n-1} with constant sectional curvature 1. And ω is a smooth function only relies on r. If the radical sectional curvature of M is bounded, and let the volume $V(r)$ of the geodesic ball of center o, radius r satisfy*

$$\lim_{r\to+\infty}(\log V(r) - ar - b) = 0, \tag{2.1}$$

where a, b are constant, then Ess Spec $M = [\frac{1}{4}a^2, +\infty)$.

We will need the following lemma.

Lemma 2.1. *Under the hypothesis of Theorem 2.1, in addition if C_1 is the bound of the radical sectional curvature, then there exists a constant C_2 such that for $r > 1$ we have*

$$\left|\frac{\omega''}{\omega}\right| \le C_2, \quad \left|\frac{\omega'}{\omega}\right| \le C_2 \tag{2.2}$$

and a constant C_3 such that, for $r_1, r_2 > 1$, $|r_2 - r_1| < 1$,

$$\omega(r_2)/\omega(r_1) \le C_3, \tag{2.3}$$

where $ds^2 = dr^2 + \omega^2 d\theta^2$ is the Riemannian metric of M and $\omega(r)$ is the function of r and $d\theta^2$ is the canonical metric on S^{n-1}.

Proof. Let $K(r)$ be radical sectional curvature of M. Then by the Jacobi equation[7], we know

$$\omega'' + K\omega = 0; \tag{2.4}$$

so $|\omega''/\omega| \le |K| \le C_1$. Let $h = \omega'/\omega$. Then

$$h' + h^2 = -K. \tag{2.5}$$

Let F be the zero set of h' on $[1, +\infty)$. Then if $\sup F = +\infty$, i.e., there exists zero points series $r_k \to +\infty$, $k \to +\infty$, then we have

$$\sup_{r\ge1}|h| \le \max_{1\le r\le r_1}\left(|h|, \sqrt{|K|}\right) \le \sqrt{C_1} + |h(1)|; \tag{2.6}$$

otherwise there exists a real number r_0 such that if $r > r_0$ then $h' \ne 0$. So h is monotone. We can conclude that $\lim\limits_{r\to+\infty} h(r) = c$ where $c \ne \pm\infty$. Otherwise by (2.5) we know that $\lim\limits_{r\to+\infty} h(r) = -\infty$, so for arbitrary $A > a > 0$ there exists r_0 such that if $r > r_0$ we have $h(r) < -A$. Then for $r > r_0$ we have

$$\log\frac{\omega(r)}{\omega(r_0)} = \int_{r_0}^{r} h(s)ds < -A(r - r_0). \tag{2.7}$$

So $\omega(r) < \omega(r_0)e^{-A(r-r_0)}$ and

$$V(r) \le \int_0^{r_0} \omega^{n-1}dr + \int_{r_0}^{r} \omega(r_0)^{n-1}e^{-A(n-1)(r-r_0)}dr$$

is finite, which contradicts (2.1). Thus h has to be bounded, i.e., there exists a constant C_2 such that (2.2) holds. By the mean value theorem we have

$$\begin{aligned}\log(\omega(r_2)/\omega(r_1)) &= \log\omega(r_2) - \log\omega(r_1)\\ &= h'(\xi)(r_2 - r_1) \leq C_2.\end{aligned}$$

Then we complete the proof of the lemma by letting $C_3 = e^{C_2}$.

Lemma 2.2. *Under the hypothesis of Theorem 2.1, there exists a constant C_7 such that*

$$\left|\frac{B_\sigma(r+\sigma) - B_\sigma(r)}{\sigma B_\sigma(r)} - \frac{\omega'(r)}{\omega(r)}\right| \leq C_7\sigma, \tag{2.8}$$

where

$$B_\sigma(r) = \sqrt[n-1]{\frac{A_\sigma(r)}{a\sigma}}, \quad A_\sigma(r) = V(r+\sigma) - V(r).$$

Proof. By Taylor formula

$$\begin{aligned}&A_\sigma(r) - \sigma\omega^{n-1}(r) - \frac{1}{2}(n-1)\omega^{n-2}(r)\omega'(r)\sigma^2\\ =&\int_r^{r+\sigma}(\omega^{n-1}(\tau) - \omega^{n-1}(r) - (n-1)\omega^{n-2}(r)\omega'(r)(\tau - r))d\tau\\ =&\frac{1}{6}((n-1)(n-2)\omega^{n-3}(\xi)(\omega'(\xi))^2 + (n-1)\omega^{n-2}(\xi)\omega''(\xi))\sigma^3.\end{aligned} \tag{2.9}$$

Without lossing generality, we assume $\sigma < 1$. By using Lemma 2.1, we have

$$\begin{aligned}&\left|A_\sigma(r) - \sigma\omega^{n-1}(r) - \frac{n-1}{2}\omega^{n-2}(r)\omega'(r)\sigma^2\right|\\ \leq&\frac{1}{6}((n-1)(n-2)C_3^{n-1}C_2^2 + (n-1)C_2C_3^{n-1})\omega^{n-1}(r)\sigma^3.\end{aligned}$$

If we let

$$C_4 = \frac{1}{6}(n-1)(n-2)C_3^{n-1}C_2^2 + (n-1)C_2C_3^{n-1},$$

$$S(r,\sigma) = (A_\sigma(r) - \sigma\omega^{n-1}(r) - \frac{n-1}{2}\omega^{n-2}(r)\omega'(r)\sigma^2)/\omega^{n-1}(r)\sigma^3,$$

we have $|S(r,\sigma)| \leq C_4$. Thus

$$\begin{aligned}B_\sigma(r) &= \sqrt[n-1]{\frac{A_\sigma(r)}{a\sigma}}\\ &= \frac{\omega(r)}{\sqrt[n-1]{a}} \cdot \sqrt[n-1]{1 + \frac{n-1}{2}\frac{\omega'(r)}{\omega(r)}\sigma + S(r,\sigma)\sigma^2}.\end{aligned} \tag{2.10}$$

Since

$$\lim_{x\to 0}(\sqrt[n-1]{1+x} - 1 - \frac{1}{n-1}x)/x^2 = \frac{2-n}{2(n-1)^2}$$

is bounded, we have a constant C_5 such that

$$\left|B_\sigma(r) - \frac{\omega(r)}{\sqrt[n-1]{a}} \cdot \left(1 + \frac{1}{2}\frac{\omega'(r)}{\omega(r)}\sigma\right)\right| \leq \frac{\omega(r)}{\sqrt[n-1]{a}}C_5\sigma^2. \tag{2.11}$$

If we let

$$T(r,\sigma) = \left(B_\sigma(r) - \frac{\omega(r)}{\sqrt[n-1]{a}} \cdot \left(1 + \frac{1}{2}\frac{\omega'(r)}{\omega(r)}\sigma\right)\right) \Big/ \left(\frac{\omega(r)}{\sqrt[n-1]{a}}\sigma^2\right), \tag{2.12}$$

we know

$$|T(r,\sigma)| \leq C_5.$$

Thus

$$\begin{aligned}
&\left|B_\sigma(r+\sigma)-B_\sigma(r)-\frac{\omega'(r)}{\sqrt[n-1]{a}}\sigma\right| \\
\leq&\left|\frac{\omega(r+\sigma)-\omega(r)-\omega'(r)\sigma}{\sqrt[n-1]{a}}\right|+\frac{1}{2}\left|\frac{\omega'(r+\sigma)-\omega'(r)}{\sqrt[n-1]{a}}\cdot\sigma\right| \\
&+\left|B_\sigma(r+\sigma)-\frac{\omega(r+\sigma)}{\sqrt[n-1]{a}}\Big(1+\frac{1}{2}\frac{\omega'(r+\sigma)}{\omega(r+\sigma)}\sigma\Big)\right| \\
&+\left|B_\sigma(r)-\frac{\omega(r)}{\sqrt[n-1]{a}}\Big(1+\frac{1}{2}\frac{\omega'(r)}{\omega(r)}\sigma\Big)\right| \\
\leq& C_5\left|\frac{\omega(r+\sigma)}{\sqrt[n-1]{a}}\cdot\sigma^2\right|+C_5\left|\frac{\omega(r)}{\sqrt[n-1]{a}}\cdot\sigma^2\right|+\frac{1}{2}\left|\frac{\omega''(\xi)}{\sqrt[n-1]{a}}\sigma^2\right|+\frac{1}{2}\left|\frac{\omega''(\eta)}{\sqrt[n-1]{a}}\sigma^2\right| \\
=&\left[\frac{C_5}{\sqrt[n-1]{a}}(C_3+1)+\frac{C_3C_1}{\sqrt[n-1]{a}}\right]\omega(r)\sigma^2.
\end{aligned}$$

Let

$$U(r,\sigma)=(B_\sigma(r+\sigma)-B_\sigma(r)-\frac{\omega'(r)}{\sqrt[n-1]{a}}\sigma)/(\omega(r)\sigma^2),$$

$$C_6=\left[\frac{C_5}{\sqrt[n-1]{a}}(C_3+1)+\frac{C_3C_1}{\sqrt[n-1]{a}}\right].$$

Then

$$|U(r,\sigma)| \leq C_6,$$

so there exists a constant C_7 such that

$$\begin{aligned}
&\left|\frac{B_\sigma(r+\sigma)-B_\sigma(r)}{\sigma B_\sigma(r)}-\frac{\omega'(r)}{\omega(r)}\right| \\
=&\left|\frac{\frac{\omega'(r)}{\sqrt[n-1]{a}}\sigma+U(r,\sigma)\omega(r)\sigma^2}{\sigma(\omega(r)+\frac{1}{2}\omega'(r)\sigma+T(r,\sigma)\omega(r)\sigma^2)/\sqrt[n-1]{a}}-\frac{\omega'(r)}{\omega(r)}\right| \\
=&\left|\frac{\frac{\omega'(r)}{\omega(r)}+U(r,\sigma)(\sqrt[n-1]{a})\sigma}{(1+\frac{1}{2}\frac{\omega'(r)}{\omega(r)}\sigma+T(r,\sigma)\sigma^2)}-\frac{\omega'(r)}{\omega(r)}\right| \leq C_7\sigma.
\end{aligned}$$

Lemma 2.3. *Under the assimption of Theorem 2.1, there exists a constant C_{10} such that for every $\epsilon>0$ and $\sigma=\epsilon^{\frac{1}{4}}/a$ there exists an r_0 such that for $r\geq r_0$ we have*

$$\left|\frac{B_\sigma(r+\sigma)-B_\sigma(r)}{\sigma B_\sigma(r)}-\frac{a}{n-1}\right| \leq C_{10}\epsilon^{\frac{1}{4}}. \tag{2.13}$$

Proof. Under the assumption of the lemma, we know for every $\epsilon>0$ there exists an r_0 such that if $r\geq r_0$ we have

$$e^{ar+b-\epsilon} \leq V(r) \leq e^{ar+b+\epsilon}. \tag{2.14}$$

Thus

$$e^{ar+a\sigma+b-\epsilon}-e^{ar+b+\epsilon} \leq A_\sigma(r) \leq e^{ar+a\sigma+b+\epsilon}-e^{ar+b-\epsilon} \tag{2.15}$$

and

$$e^{\frac{ar+b}{n-1}} \cdot \sqrt[n-1]{\frac{e^{a\sigma-2\epsilon}-1}{a\sigma}} \le B_\sigma(r) \le e^{\frac{ar+b}{n-1}} \cdot \sqrt[n-1]{\frac{e^{a\sigma+2\epsilon}-1}{a\sigma}}. \tag{2.16}$$

So there exists a constant C_8 such that

$$\left|B_\sigma(r) - e^{\frac{ar+b}{n-1}}\left(1+\frac{1}{2(n-1)}\epsilon^{\frac{1}{4}}\right)\right| \le C_8 e^{\frac{ar+b}{n-1}}\epsilon^{\frac{1}{2}}.$$

From this inequality we have

$$\left|B_\sigma(r+\sigma) - B_\sigma(r) - \frac{1}{n-1}e^{\frac{ar+b}{n-1}}\epsilon^{\frac{1}{4}}\right| \le C_9 e^{\frac{ar+b}{n-1}}\epsilon^{\frac{1}{2}},$$

where C_9 is a constant. By using the same method used in Lemma 2.2 we have the inequality

$$\left|\frac{B_\sigma(r+\sigma)-B_\sigma(r)}{\sigma B_\sigma(r)} - \frac{a}{n-1}\right| \le C_{10}\epsilon^{\frac{1}{4}},$$

which completes the proof of the lemma.

Lemma 2.4. *Under the assumption of Theorem 2.1, $\lambda_0^{\mathrm{ess}} \ge \frac{1}{4}a^2$, where λ_0^{ess} is the infimum of the essential spectrum of M.*

Proof. Let $\omega^{n-1} = \theta(r)r^{n-1}$. Then by the theorem in [5, p.506] we know

$$\lambda_o^{ess} > \left(\frac{1}{4}\lim_{r\to+\infty}\inf_{d(x,x_0)\ge r}\frac{1}{\theta}\frac{\partial\theta}{\partial r}\right)^2 = \frac{1}{4}a^2, \tag{2.17}$$

and the lemma is proved.

Lemma 2.5. *Let $\lambda > \frac{1}{4}a^2$. For every $\epsilon > 0$, there exist infinite positive integer pairs (k,l) and smooth functions $\eta_{k,l}$ such that*

$$\|\Delta\eta_{k,l} + \lambda\eta_{k,l}\| \le \epsilon\|\eta_{k,l}\|, \tag{2.18}$$

where $\|\cdot\|$ is L^2–norm.

Proof. The Laplacian can be written as

$$\Delta = \frac{d^2}{dr^2} + (n-1)\frac{\omega'(r)}{\omega(r)}\cdot\frac{d}{dr} + \frac{1}{\omega^2}\Delta_{S^{n-1}} \tag{2.19}$$

under the metric $ds^2 = dr^2 + \omega^2 d\theta^2$ where $\Delta_{S^{n-1}}$ is the Laplacian on S^{n-1}. Let $H(x)$ be a C^∞ smooth function such that

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & \pi \le x \le 2\pi, \\ 0, & x \ge 3\pi. \end{cases}$$

Then we know that H, H', H'' are bounded, i.e., there exists a constant C_{11} such that $|H| + |H'| + |H''| \le C_{11}$. Let

$$\beta = \sqrt{\lambda - \frac{1}{4}a^2}, \qquad f(r) = e^{-\frac{a}{2}r}\cdot\sin\beta r. \tag{2.20}$$

Then $f(r)$ satisfies the equation

$$f''(r) + af'(r) + \lambda f(r) = 0 \tag{2.21}$$

or

$$\Delta f = \left((n-1)\frac{\omega'(r)}{\omega(r)} - a\right)f'(r) - \lambda f(r). \tag{2.22}$$

For arbitrary integers k, l, define

$$\eta_{k,l}(x) = H\Big(\frac{r - k\pi/\beta}{l/\beta}\Big) f(r). \tag{2.23}$$

For the sake of simplicity, we use η instead of $\eta_{k,l}$. Then

$$\begin{aligned}\Delta\eta =& H\Delta f + 2\nabla f \nabla H + f\Delta H - H\cdot\Big(\Big((n-1)\frac{\omega'(r)}{\omega(r)} - a\Big) f'(r)\\ &- \lambda H f + \frac{2\beta}{l} f'(r)H' + f\Big(H''\frac{\beta^2}{l^2} + (n-1)\frac{\omega'(r)}{\omega(r)}H'\frac{\beta}{l}\Big).\end{aligned}$$

So

$$|\Delta\eta + \lambda\eta| \le \Big(C_{11}\Big|(n-1)\frac{\omega'}{\omega} - a\Big| + \frac{C_{12}}{l}\Big)e^{-\frac{a}{2}r}. \tag{2.24}$$

Because

$$\lim_{r\to+\infty}\frac{\omega'(r)}{\omega(r)} = \frac{a}{n-1},$$

for arbitrary $\widetilde{\epsilon} > 0$ there exists an r_0 such that for $r > r_0$ we have

$$\Big|(n-1)\frac{\omega'(r)}{\omega(r)} - a\Big| < \widetilde{\epsilon}.$$

So we let $k > \frac{\beta r_0}{\pi}$. Then

$$|\Delta\eta + \lambda\eta| \le \Big(C_{11}\widetilde{\epsilon} + \frac{C_{12}}{l}\Big)e^{-\frac{1}{2}ar},$$

because η is zero on $[0, \frac{k\pi}{\beta}]$ and $[\frac{k\pi+3l\pi}{\beta}, +\infty)$. So

$$\|\Delta\eta + \lambda\eta\|^2 \le \Big(C_{11}\widetilde{\epsilon} + \frac{C_{12}}{l}\Big)^2 C_{n-1}\int_{\frac{k\pi}{\beta}}^{\frac{k\pi}{\beta}+\frac{3l\pi}{\beta}} e^{-ar}\omega^{n-1}(r)dr, \tag{2.25}$$

where C_{n-1} is the volume of S^{n-1}. Let ψ denote the characteristic function of the set $\{x \in M|\ \ \frac{k\pi}{\beta} + \frac{l\pi}{\beta} \le \mathrm{dist}(x, o) \le \frac{k\pi}{\beta} + \frac{2l\pi}{\beta}\}$ and δ be $\frac{k\pi+l\pi}{\beta}$. Then

$$\begin{aligned}&\|\eta\|^2 \ge \|f\psi\|^2\\ &= C_{n-1}\int_{\delta}^{\delta+\frac{l\pi}{\beta}} e^{-ar}\omega^{n-1}(r)\sin^2\beta r dr\\ &= \sum_{j=0}^{l-1}\int_{\delta+\frac{j\pi}{\beta}}^{\delta+\frac{(j+1)\pi}{\beta}} e^{-ar}\omega^{n-1}(r)\sin^2\beta r dr\\ &\ge \frac{1}{2}\sum_{j=0}^{l-1}\int_{\delta+\frac{j\pi}{\beta}+\frac{\pi}{4\beta}}^{\delta+\frac{j\pi}{\beta}+\frac{\pi}{2\beta}} e^{-ar}\omega^{n-1}(r)dr.\end{aligned} \tag{2.26}$$

By (2.22) we get

$$\int_{\delta+\frac{j\pi}{\beta}}^{\delta+\frac{(j+1)\pi}{\beta}} e^{-ar}\omega^{n-1}(r)dr \le C_{13}\int_{\delta+\frac{j\pi}{\beta}+\frac{\pi}{4\beta}}^{\delta+\frac{j\pi}{\beta}+\frac{\pi}{2\beta}} e^{-ar}\omega^{n-1}(r)dr. \tag{2.27}$$

Using (2.26) and (2.27), we obtain

$$\int_{\delta}^{\delta+\frac{j\pi}{\beta}} e^{-ar}\omega^{n-1}(r)dr \le 2C_{13}\|\eta\|^2. \tag{2.28}$$

By the mean value theorem,

$$\left|\log\left[\frac{(\omega(r+l\pi/\beta))^{n-1}}{e^{a(r+l\pi/\beta)}}\right]-\log\frac{(\omega(r))^{n-1}}{e^{ar}}\right|\leq\frac{\pi}{\beta}l\tilde{\epsilon}, \tag{2.29}$$

$$e^{-ar}\omega^{n-1}(r)\Big/\big((\omega(r+l\pi/\beta))^{n-1}e^{-a(r+l\pi/\beta)}\big)\leq\exp(\frac{\pi}{\beta}l\tilde{\epsilon}), \tag{2.30}$$

$$e^{-ar}\omega^{n-1}(r)\Big/\big((\omega(r-l\pi/\beta))^{n-1}e^{-a(r-l\pi/\beta)}\big)\leq\exp(\frac{\pi}{\beta}l\tilde{\epsilon}). \tag{2.31}$$

Let $l=[\frac{1}{\epsilon}]$. Then by (2.25), (2.28), (2.30), (2.31), there exists a constant C such that

$$\|\Delta\eta+\lambda\eta\|\leq C\sqrt{\tilde{\epsilon}}\|\eta\|. \tag{2.32}$$

Thus we can prove the lemma if we let $\epsilon=C\sqrt{\tilde{\epsilon}}$.

Proof of Theorem 2.1. For every $\lambda>\frac{1}{4}a^2$, there exist infinite pairs (k,l) such that $\|\Delta\eta_{k,l}+\lambda\eta_{k,l}\|\leq\epsilon\|\eta_{k,l}\|$, and their supports do not intersect each other. Letting $\epsilon\to 0$, by Theorem A we know $\lambda\in$ Ess Spec M. By Lemma 2.4 the essential spectrum of M is $[\frac{1}{4}a^2,+\infty)$.

Using the same method we can prove

Theorem 2.2. *Let N be $(n-1)$-dimensional compact Riemannian manifold whose metric is g. Let M be n-dimensional complete Riemannian manifold which is diffeomorphism to $N\times(0,+\infty)$ outside some compact set and whose metric can be written as $ds^2=dr^2+\omega^2 g$ where r is the distance to $N\times\{0\}$. Assume the sectional curvature of M is bounded and $V(r)$ denotes the geodesic ball of radius r with center a fixed point of M which satisfies*

$$\lim_{r\to+\infty}(\log V(r)-ar-b)=0,$$

where a,b are constants. Then Ess Spec $M=[\frac{1}{4}a^2,+\infty)$.

Remark. J. F. Escobar[3] proved that if the metric of a complete noncompact manifold is rotational invariant and the radius curvature is nonnegative outside some compact set, then M has no eigenvalues. But in general case we cannot expect such results (see the counterexample of H. Donnelly[9]).

§3. A Counterexample

In this section, we will show that the assumption in Theorem 2.1 that the metric is rotational invariant is necessary.

Theorem 3.1. *There exists a Riemannian manifold with a pole which has bounded radical curvature and the volume $V(r)$ of whose geodesic ball of radius r and center o satisfies*

$$\lim_{r\to+\infty}(\log V(r)-ar-b)=0 \tag{3.1}$$

but Ess Spec $M\neq[\frac{1}{4}a^2,+\infty)$.

Proof. We consider the metric on $\mathbf{R}^n$

$$ds^2=dr^2+e^{2\eta(\theta)r}d\theta^2, \tag{3.2}$$

where (r,θ) are the radical coordinates on $\mathbf{R}^n$ and smooth function $\eta : S^{n-1} \to \mathbf{R}^n$ satisfies

$$\eta(\theta) = \begin{cases} 1, & \theta \in \text{the neighborhood near the north pole of } S^{n-1} \text{ with the measure } 1, \\ \frac{1}{2}, & \theta \in \text{the neighborhood near the south pole of } S^{n-1} \text{ with the measure } 1, \\ \frac{1}{2} \le \eta \le 1. \end{cases}$$

We estimate $V(r)$

$$\begin{aligned} V(r) &= \int_{S^{n-1}} \Big(\int_0^r e^{(n-1)\eta(\theta)r} dr \Big) d\theta \\ &= \int_{S^{n-1}} \Big(\frac{1}{(n-1)\eta(\theta)} (e^{(n-1)\eta(\theta)r} - 1) \Big) d\theta. \end{aligned} \tag{3.3}$$

Using the Laplace method ([9, p.163]) we know when $r \to +\infty$,

$$\lim_{r\to+\infty} (\log V(r) - (n-1)r + \log((n-1)/C_{n-1})) = 0. \tag{3.4}$$

But we will show that the essential spectrum of $(\mathbf{R}^n, ds^2)$ is not $[\frac{(n-1)^2}{4}, +\infty)$. In fact we will prove that the infimum of essential spectrum $\lambda_0^{\text{ess}} \le \frac{1}{16}(n-1)^2$. In order to prove this we will first establish the following theorem which slightly generalizes the result in [5].

Theorem 3.2. *Suppose M is a complete Riemannian manifold with a pole whose metric can be written as*

$$ds^2 = dr^2 + \omega^2 d\theta^2.$$

For every connected open subset U of S^{n-1}, define

$$M_U = \{(r,\theta) | \theta \in U\}, \quad \mu = \lim_{r\to+\infty} \sup \frac{1}{r} \log V_U(r),$$

where $V_U(r)$ is the volume of the set $\{(\tilde r,\theta)| \ \theta \in U, \tilde r \le r\}$. If M has infinite volume, then $\lambda_0^{ess} \le \frac{1}{4} \inf_U \mu_0^2$.

The proof of the theorem is like that of the proof of Theorem 1 in [5] and we omit it.

Proof of Theorem 3.1. Let U be the open neighborhood near the north pole with measure 1. Then

$$\begin{aligned} V_U(r) &= \int_U \frac{1}{(u-1)\frac{1}{2}} (e^{\frac{1}{2}(n-1)r} - 1) d\theta \\ &= \frac{2}{(n-1)} (e^{\frac{1}{2}(n-1)r} - 1). \end{aligned}$$

Thus

$$\mu_U = \lim_{r\to+\infty} \sup \frac{1}{r} \log V_U(r) = \frac{1}{2}(n-1).$$

So by Theorem 3.2 we complete the proof of Theorem 3.1.

§4. Generalized Case

We introduce the concept of comparison manifolds.

Definition 4.1. *If M is an n-dimensional complete Riemannian manifold with a pole o and whose metric can be written as $ds^2 = dr^2 + \omega^2(r,\theta)d\theta^2$; if there exists a complete Riemannian manifold M' with rotational invariant metric and bounded radical curvature, the volume of whose geodesic ball of radius r and center o satisfies*

$$\lim_{r\to+\infty} (\log V'(r) - ar - b) = 0,$$

where a, b are constants; and if there exists a constant c such that

$$\lim_{r\to+\infty}\left(\log\int_0^r \omega^{n-1}(r,\theta)dr - \log\int_0^r (\omega')^{n-1}(r)dr\right) = C \tag{4.1}$$

uniformly about θ; then we say M' is the comparison manifold of M and M is comparable.

Remark. The concept of the comparison manifold is useful because by investigating some class of regular manifolds we can obtain the spectra information of much more irregular Riemannian manifolds.

Theorem 4.1. *If M' is a comparison manifold of a given Riemannian manifold M', then* Ess Spec $M =$ Ess SpecM'.

Proof. By Theorem 2.1 we can assume Ess Spec $M' = [\frac{1}{4}a^2, +\infty)$ without loss of generality. So by (4.1) we know there exists a constant d such that

$$\lim_{r\to+\infty}\left(\log\int_0^r \omega^{n-1}(r,\theta)dr - ar - d\right) = 0$$

uniformly about θ. Using the same method as in the proof of Lemmas 2.1, 2.2, 2.3 we can prove

$$\lim_{r\to+\infty}\left(\frac{\omega_r'(r,\theta)}{\omega(r,\theta)} - \frac{a}{(n-1)}\right) = 0 \tag{4.2}$$

uniformly about θ. If Δ, Δ' represent the Laplacian of the manifold M and M' respectively, and the metric of M' can be written as $d(s')^2 = dr^2 + (\omega')^2 d\theta^2$, then when $\lambda > \frac{1}{4}a^2$, we have

$$\|\Delta\eta_{k,l} - \Delta'\eta_{k,l}\|^2 \le \max_{\theta, r > k\pi/\beta}\left|\frac{\omega_r'(r,\theta)}{\omega(r,\theta)} - \frac{(\omega')'(r)}{\omega(r)}\right| \int_{\text{supp }(\eta_{k,l})} |\nabla\eta_{k,l}|^2.$$

Let

$$C(r) = \max_\theta\left|\frac{\omega_r'(r,\theta)}{\omega(r,\theta)} - \frac{(\omega')'(r)}{\omega(r)}\right|.$$

Then

$$\begin{aligned}\int_{\text{supp }(\eta_{k,l})} |\nabla\eta_{k,l}|^2 &\le \int_{\text{supp }(\eta_{k,l})} \eta_{k,l}\Delta'\eta_{k,l}\\ &\le \|\eta_{k,l}\|\cdot\|\Delta'\eta_{k,l}\|\\ &= (\lambda+\epsilon)\|\eta_{k,l}\|^2,\end{aligned}$$

where we use the Cauchy Inequality in the second inequality. So

$$\|\Delta\eta_{k,l} + \lambda\eta_{k,l}\| \le \left(\sqrt{\lambda+1}\cdot\sqrt{C(r)} + \epsilon\right)\cdot\|\eta_{k,l}\|.$$

Since $C(r) \to 0 \quad (r \to +\infty)$, we have

$$\sqrt{\lambda+1}\cdot\sqrt{C(r)} + \epsilon \to 0.$$

By the method used in the proof of Lemma 2.5 we know $\lambda \in$ Ess Spec M. On the other hands, we know from Lemma 2.4 that $\lambda_0^{\text{ess}} \ge \frac{1}{4}a^2$. Thus the theorem is proved.

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