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ON BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL FUNCTION OPERATOR ON RIEMANNIAN MANIFOLDS

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Abstract

The authors construct a complete Riemannian manifold such that its Hardy-Littlewood maximal function operator is unbounded in L^p for some $p > 1$.

§1. Introduction

For a complete Riemannian manifold N, its Hardy-Littlewood maximal function operator is defined by

$$M(f)(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the geodesic ball with center x and radius r. As known, Hardy-Littlewood maximal operator is very important in Harmonic Analysis. For $N = \mathbb{R}^n$, a classical result shows its L^p and weak type $(1,1)$ boundedness, $1 < p \le \infty$ (see [6]). For positively curved manifolds, Varapolous proved its L^p and weak type $(1,1)$ boundedness[9,1], $1 < p \le \infty$; and the first named author[1] proved its BMO-boundedness. For non-compact symmertic spaces, Clerc and Stein[4] proved its L^p-boundedness for $1 < p \le \infty$; and Strömberg proved its weak type $(1,1)$ boundedness. For general negatively curved manifolds, Lohoué proved its L^p-boundedness for $p > p_0$ where $p_0(> 1)$ depends on the bounds of the sectional curvature of N. A basic problem naturally arise: Is M L^p-bounded for all $p > 1$? In this paper, we shall construct a simply connected complete Riemannian manifold (based on [2]) with sectional curvature $K_N \le 0$, for which M is not L^p-bounded for all $1 < p < 2$. At the same time, the example also shows that the main results of "On the sectional curvature of a Riemannian manifold" (Chinese Annals of Mathematics (Ser. B), Vol. 11, No. 1, 1990) are wrong. For simplicity, we only consider 2-dimensional case, i.e., $\dim(N) = 2$.

In the whole paper, C denotes an absolute positive number and $C_{a,b,\cdots}$ a positive number depending only on $a, b, \cdots$, $f(r) = \overline{O}(g(r))$ means that $C^{-1} \le |f(r)/g(r)| \le C$.

§2. Some Notes on Poincare Plane

Let $\mathbb{M}_{-1} = (D, (1-r^2)^{-2}(dr^2 + r^2 d\theta^2))$ denote the Poincare plane, where $D = \{z : |z| < 1\} \subset \mathbb{R}^2$, $B_{-1}(z,r)$ denote the geodesic ball in $\mathbb{M}_{-1}$ with center z and radius r, $B_0(z,r)$ denote the Euclidean ball in $\mathbb{R}^2$ with center z and radius r, $\rho(.,.)$ denote the geodesic distance function on $\mathbb{M}_{-1}$, $R_0 = \operatorname{th} 1$, $0 < R < 1$. We have (see the figure)

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Lemma 2.1. *$B_{-1}(R,\rho(O,R)) = B_0(O_R,O_R)$ where $O_R = R/(1+R^2)$,*

$$B_{-1}(R,\rho(O,R)+1) = B_0(O'_R, O'_R + R_0)$$

where $O'_R = 4e^2(e^2+1)^{-2}R/(1+2R\,\mathrm{th}\,1+R^2)$.

Proof. It is easy to see that the Riemannian structure of $\mathbb{M}_{-1}$ is preserved by the maps

$$z \longrightarrow (az+b)/(\bar{b}z+\bar{a}) \qquad (\forall z \in D, |a|^2-|b|^2=1) \tag{2.1}$$

and all straight lines through the origin are geodesics. So, all geodesic balls with center O are Euclidean balls and the maps (2.1) preserve both the Euclidean and the $\mathbb{M}_{-1}$-geodesic balls. It is also easy to see that O_R, the center of $B_{-1}(R,\rho(O,R))$ as a Euclidean ball, must be in x-axis because $B_{-1}(R,\rho(O,R))$ is tangent to y-axis at O and the angle between two vectors in the Riemannian structure coincides with the Euclidean angle. Similarly, O'_R, the center of $B_{-1}(R,\rho(O,R)+1)$ as a Euclidean ball, must be in x-axis. Now, let R_* and R'_* be shown in the figure. We shall compute R'_* and O'_R only, here. We have

$$\rho(O,R_0)+\rho(O,R) = \rho(R,R'_*).$$

An easy computation by maps (1) shows

$$\rho(z',z'') = \frac{1}{2}\ln\frac{|1-\bar{z}'z''|+|z'-z''|}{|1-\bar{z}'z''|-|z'-z''|}$$

and thus $\rho(O,R_0)=1$. Therefore

$$(1-R)(1-RR'_*+R'_*-R) = (1+R)e^2(1-RR'_*-R'_*+R),$$
$$R'_* = (\mathrm{th}\,1+2R+R^2\,\mathrm{th}\,1)/(1+2R\,\mathrm{th}\,1+R^2),$$

and

$$O'_R = \frac{1}{2}(R_0+R'_*) = 4e^2(e^2+1)^{-2}R/(1+2R\,\mathrm{th}\,1+R^2).$$

Similarly, we have

$$R_* = 2R/(1+R^2) \qquad \text{and} \qquad O_R = R/(1+R^2).$$

Let α_R and α'_R be shown in the figure. Then

Lemma 2.2. $\alpha_R = \arccos\frac{1}{2}(1+R^2) = \overline{O}((1-R)^{\frac{1}{2}}) \quad (R\to 1^-)$,

$$\alpha'_R = \arccos(2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1-R^2)/(-2RO'_R) = \overline{O}((1-R)^{\frac{1}{2}}) \quad (R\to 1^-).$$

Proof. Here, we only compute α'_R. We have

$$|R\exp(i\alpha'_R)-O'_R| = O'_R+R_0,$$

so

$$\alpha'_R = \arccos(2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1-R^2)/(-2RO'_R).$$

Now

$$\arccos x = \int_x^1\frac{dx}{\sqrt{1-x^2}} = \overline{O}(1)\int_x^1\frac{dx}{\sqrt{1-x}}$$
$$= \overline{O}(1)(1-x)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}\alpha'_R =&\overline{O}(1)(2RO'_R+2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1-R^2)^{\frac12}\\ =&\overline{O}(1)(((8e^2R^2+8e^2R^2\,\mathrm{th}\,1)/(1+2R\,\mathrm{th}\,1+R^2)+(e^2-1)^2-R^2(e^2+1)^2)^{\frac12})\\ =&\overline{O}(1)(8e^4R(1+R)-8e^2R(1-R)+((e^4+1)(1-R^2)-2e^2(1+R^2))\\ &\cdot(e^2(1+2R+R^2)+(1-2R+R^2)))^{\frac12}\\ =&\overline{O}(1)(2e^4(4R(1+R)-(1+R^2)(1+2R+R^2))-\\ &-2e^2(1+R^2)(1-R)^2+\overline{O}_+(1-R))^{\frac12}\\ =&\overline{O}(1)(O((1-R)^2)+2e^4(R^2+2R+1)(1-R)^2+\overline{O}_+(1-R))^{\frac12}\\ =&\overline{O}((1-R)^{\frac12})\qquad (R\to 1^-),\end{aligned}$$

where $h(R)=\overline{O}_+(1-R)$ $(R\to 1^-)$ means that when $R\to 1^-$,

$$h(R)>0 \text{ and } C^{-1}\le |h(R)/(1-R)|\le C.$$

Lemma 2.2 is proved.

Let r_θ and r'_θ be shown in the figure. We have

Lemma 2.3. $r_\theta=(2r/(1+r^2))\cos\theta$,

$$r'_\theta=O'_R\cos\theta+((O'_R\cos\theta)^2+(2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1))^{\frac12}.$$

Proof. We only compute r'_θ. We have

$$|r'_\theta e^{i\theta}-O'_R|=O'_R+R_0,$$
$${r'_\theta}^2-2O'_R\cos\theta\cdot r'_\theta-(2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1)=0.$$

Its positive solution is

$$r'_\theta=O'_R\cos\theta+((O'_R\cos\theta)^2+(2O'_R\,\mathrm{th}\,1+\mathrm{th}^2\,1))^{\frac12}.$$

Now, we have

Lemma 2.4. $|B_{-1}(O,\rho(O,R))|=\pi(1/(1-R^2)-1)=\overline{O}(1/(1-R))$,

$$|B_{-1}(O,\rho(O,R))\cap B_{-1}(R,\rho(O,R)+1)|=\overline{O}(1/(1-R)^{\frac12})$$

when $R\to 1^-$.

Proof. We have

$$\begin{aligned}|B_{-1}(O,\rho(O,R))|=|B_0(O,R)|=&\int_{|z|<R}\frac{rdrd\theta}{(1-r^2)^2}\\ =&2\pi\int_0^{R^2}\frac{\frac12 dt}{(1-t)^2}=\pi((1-R^2)^{-1}-1).\end{aligned}$$

And, by Lemma 2.1, we have

$$\begin{aligned}&|B_{-1}(O,\rho(O,R))\cap B_{-1}(R,\rho(O,R)+1)|\\ =&|B_0(O,R)\cap B_0(O'_R,O'_R+R_0)|\\ =&\overline{O}(1)\Big(\int_0^{\alpha'_R}\int_0^{R_\epsilon}+\int_{\alpha'_R}^{\frac{\pi}{2}}\int_0^{r'_\theta}+\int_{\frac{\pi}{2}}^{\pi}\int_0^{r_\theta}\Big)\frac{rdrd\theta}{(1-r^2)^2}\\ =&\overline{O}(1)(\mathrm{I}+\mathrm{II}+\mathrm{III}).\end{aligned}$$

It is easy to see that

$$\mathrm{III} = \overline{O}(1) = o(1/(1-R)^{\frac{1}{2}}) \qquad (R \to 1^-),$$
$$\mathrm{I} = \overline{O}(1)\alpha'_R/(1-R^2) = \overline{O}(1/(1-R)^{\frac{1}{2}}) \qquad (R \to 1^-)$$

by Lemma 2.2. To estimate II, we consider $1-r'_\theta$ first. We have

$$\begin{aligned}
1-r'_\theta &= 1 - O'_R\cos\theta - ((O'_R\cos\theta)^2 + (2O'_R\,\mathrm{th}\,1 + \mathrm{th}^2\,1))^{\frac{1}{2}} \\
&= \frac{(1-O'_R\cos\theta)^2 - (O'_R\cos\theta)^2 - (2O'_R\,\mathrm{th}\,1 + \mathrm{th}^2\,1)}{1 - O'_R\cos\theta + ((O'_R\cos\theta)^2 + (2O'_R\,\mathrm{th}\,1 + \mathrm{th}^2\,1))^{\frac{1}{2}}} \\
&= \overline{O}(1)(1 - 2O'_R\cos\theta - (2O'_R\,\mathrm{th}\,1 + \mathrm{th}^2\,1)) \\
&= \overline{O}(1)((1-\mathrm{th}^2\,1)/2O'_R - \mathrm{th}\,1 - \cos\theta) \\
&= \overline{O}(1)((1-\mathrm{th}^2\,1)(1+2R\,\mathrm{th}\,1+R^2)(e^2+1)^2/4e^2R - \mathrm{th}\,1 - \cos\theta) \\
&= \overline{O}(1)((1+R^2)/2R - \cos\theta) \\
&= \overline{O}(1)(1 - (2R\cos\theta/(1+R^2))^2).
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{II} &= \int_{\alpha'_R}^{\frac{\pi}{2}}\int_0^{r'_\theta}\frac{r\,dr\,d\theta}{(1-r^2)^2} = \frac{1}{2}\int_{\alpha'_R}^{\frac{\pi}{2}}\frac{d\theta}{1-r'^2_\theta} \\
&= \overline{O}(1)\int_{\alpha'_R}^{\frac{\pi}{2}}\frac{d\theta}{1-r'_\theta} = \overline{O}(1)\int_{\alpha'_R}^{\frac{\pi}{2}}\frac{d\theta}{1-(2O_R\cos\theta)^2} \\
&= \overline{O}(1)(1-4O_R^2)^{-\frac{1}{2}}\Big(\frac{\pi}{2} - \mathrm{arc\,tg}((\mathrm{tg}\,\alpha'_R)/(1-4O_R^2)^{\frac{1}{2}})\Big).
\end{aligned}$$

Now

$$(1-4O_R^2)^{\frac{1}{2}} = (1-4(R/(1+R^2))^2)^{\frac{1}{2}} = \overline{O}(1-R),$$

$$\begin{aligned}
\mathrm{tg}\,\alpha'_R &= (\cos^{-2}\alpha'_R - 1)^{\frac{1}{2}} = \overline{O}(1)(1-\cos\alpha'_R)^{\frac{1}{2}} \\
&= \overline{O}(1)\alpha'_R = \overline{O}((1-R)^{\frac{1}{2}}),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\pi - \mathrm{arc\,tg}((\mathrm{tg}\,\alpha'_R)/(1-4O_R^2)^{\frac{1}{2}}) \\
&= \int_0^{\mathrm{tg}\,\alpha'_R/\sqrt{1-4O_R^2}}\frac{dx}{1+x^2} = \overline{O}\Big(\Big(\frac{\mathrm{tg}\,\alpha'_R}{\sqrt{1-4O_R^2}}\Big)^{-1}\Big) \\
&= \overline{O}(1)((1-R)^{\frac{1}{2}}).
\end{aligned}$$

Therefore

$$\mathrm{II} = \overline{O}(1)((1-R)^{-\frac{1}{2}}) \qquad (R \to 1^-).$$

§3. Construction of the Counterexample

Let

$$
\begin{aligned}
&D_n = \{z : |z - z_n| < 1\}, \quad z_n = (n+2)^2, \\
&R_n = \operatorname{th} n, \\
&ds_n^2(z) = f_{R_n}^2(r)(dr^2 + r^2 d\theta) \quad \text{where} \quad z = z_n + re^{i\theta}, \\
&f_R(r) = (1 - (r\chi_R(r))^2)^{-1}, \\
&\chi_R(t) = \widetilde{\chi}_R * \varphi_{\epsilon/3}(t) = \int_{-\infty}^{+\infty} \widetilde{\chi}_R(u)\varphi_{\epsilon/3}(t-u)du \qquad (\epsilon = (1-R)^2), \\
&\varphi_r(t) = r^{-1}\varphi(t/r), \\
&\varphi(t) = \begin{cases} Ce^{-(1-t^2)^{-1}}, & |t| < 1, \\ 0, & |t| \geq 1, \end{cases} \qquad C^{-1} = \int_{R^1} e^{-(1-t^2)^{-1}} dt, \\
&\widetilde{\chi}_R(t) = \begin{cases} 1, & t \leq R + \frac{1}{3}\epsilon, \\ 0, & t \geq R + \frac{2}{3}\epsilon, \\ \text{linear}, & R + \frac{1}{3}\epsilon \leq t \leq R + \frac{2}{3}\epsilon. \end{cases}
\end{aligned}
$$

Then, take (compare with [2])

$$
N = (R^2, ds^2),
$$

$$
ds^2(z) = \begin{cases} ds_n^2(z), & z \in D_n, \quad n = 1, 2, \cdots, \\ dzd\bar{z}, & \text{otherwise}. \end{cases}
$$

We have

Lemma 3.1. *N is a smooth complete Riemannian manifold.*

$$
f_{R(r)} \begin{cases} = (1-r^2)^{-1} & \text{for } r \leq R, \\ = 1 & \text{for } r \geq R + \epsilon, \\ \leq (1-r^2)^{-1} & \text{for } R \leq r \leq R + \epsilon. \end{cases}
$$

Proof. Obviously, $f_{R_n} \in C^\infty(D_n)$ and $f_{R_n} = 1$ for

$$
z \in D_n - \{z : 1 > |z - z_n| \geq 1 - R_n - \epsilon_n\}.
$$

So, ds^2 is a smooth Riemannian metric. Thus, it is easy to see that N is a smooth complete Riemannian manifold. Now, noticing that

$$
0 \leq \chi_R(t) \leq 1, \qquad \chi_R \in C^\infty,
$$

$$
\chi_R(t) = \begin{cases} 1 & \text{for } t \leq R, \\ 0 & \text{for } t \geq R + \epsilon, \end{cases}
$$

we can easily get the estimates of f_R.

Let $d(.,.)$ denote the geodesic distance function on N, $B(z, r)$ the geodesic ball with center z and radius r. Then, we have

Lemma 3.2. *When $n \to \infty$, we have*

$$
|B(z_n, d(z_n, z_n + R_n))| = \overline{O}(1/(1-R_n)),
$$

$$
|B(z_n + R_n, d(z_n, z_n + R_n) + 1)| = \overline{O}(1/(1-R_n)^{\frac{1}{2}}).
$$

Proof. The first estimate is the same as the first estimate in Lemma 2.4. The second estimate can be obtained by the second estimate in Lemma 2.4. Since

$$B(z_n+R_n,d(z_n,z_n+R_n)+1)$$

is convex in N, we have

$$\begin{aligned}&B(z_n+R_n,d(z_n,R_n+z_n)+1)\cap B_0(z_n,R_n+\epsilon_n)\\ \subset&B_0(z_n,R_n+\epsilon_n)\cap\{z:|\arg(z-z_n)|<\alpha'_{R_n}\}\\ &\cup(B(z_n+R_n,d(z_n,R_n+z_n)+1)\cap\{z:|\arg(z-z_n)|>\alpha'_{R_n}\}).\end{aligned}$$

So, by Lemma 2.4 and Lemma 3.1

$$\begin{aligned}&|B(z_n+R_n,d(z_n,R_n+z_n)+1)\cap B_0(z_n,R_n+\epsilon_n)|\\ =&\overline{O}(1/(1-R_n)^{\frac{1}{2}})\qquad(n\to\infty).\end{aligned}$$

Finally, the Riemannian metric on

$$B(z_n+R_n,1+d(z_n,R_n+z_n))\cap(B_0(z_n,R_n+\epsilon_n))^c$$

is Euclidean and the geodesic distance

$$d(z_n,R_n+z_n)=\frac{1}{2}\ln((1+R_n)/(1-R_n))=n,$$

so we have

$$\begin{aligned}&|B(z_n+R_n,d(z_n,R_n+z_n)+1)\cap(B_0(z_n,R_n+\epsilon_n))^c|\\ =&\overline{O}(1)(d(z_n,R_n+z_n))^2=\overline{O}(1)(\ln((1+R_n)/(1-R_n))^2.\end{aligned}$$

Therefore

$$B(z_n+R_n,d(z_n,z_n+R_n)+1)=\overline{O}((1-R_n)^{-\frac{1}{2}})\qquad(n\to\infty).$$

Now, take

$$h_n(z)=|B(z_n,1)|^{-1}\chi_{B(z_n,1)}(z).$$

Then, for $z\in B_0(z_n,R_n)$

$$\begin{aligned}M(h_n)(z)&\ge|B(z,d(z_n,z)+1)|^{-1}\\ &\ge|B(z_n+R_n,d(z_n,z_n+R_n)+1)|^{-1}\\ &\ge C(1-R_n)^{\frac{1}{2}}\end{aligned}$$

by Lemma 3.2. So

$$\begin{aligned}&|\{z:M(h_n)(z)>C(1-R_n)^{\frac{1}{2}}\}|\ge|B_0(z_n,R_n)|\\ =&|B(z_n,d(z_n,z_n+R_n))|\\ =&\overline{O}(1/(1-R_n)).\end{aligned}$$

But

$$((1-R_n)^{\frac{1}{2}})^{-p}\int_N h_n^p(z)d\sigma(z)=\overline{O}((1-R_n)^{-p/2}).$$

Thus, M is not weak type (p,p) bounded for $1\le p<2$. Of course, it is not L^p-bounded for $1\le p<2$.

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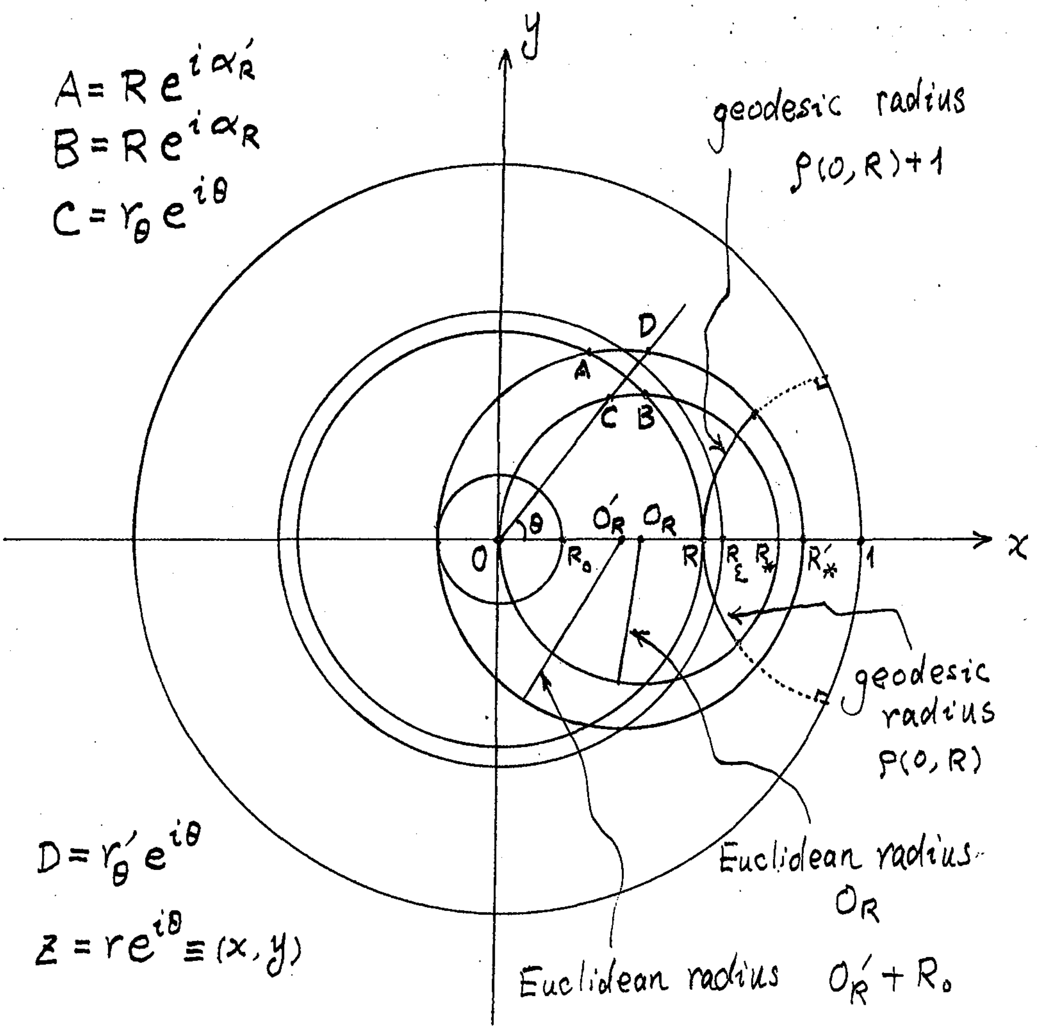
A = R e^{iα'_R}
B = R e^{iα_R}
C = r_θ e^{iθ}
y
geodesic radius
ρ(O, R)+1
D
A
C B
θ
O'_R O_R
O R_0 R R_ε R_* R'_* 1
x
geodesic radius
ρ(O, R)
D = r'_θ e^{iθ}
z = re^{iθ} ≡ (x, y)
Euclidean radius
O_R
Euclidean radius O'_R + R_0