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THE COHOMOLOGY OF MODULAR LIE ALGEBRAS WITH COEFFICIENTS IN A RESTRICTED VERMA MODULE**

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Abstract

This paper determines the structure of the cohomology of a modular semisimple Lie algebra with coefficients in an arbitrary restricted Verma module.

§1. Introduction

In [12], Williams gave the structure of the cohomology of a complex semisimple Lie algebra with coefficients in an arbitrary Verma module. Let $(\boldsymbol{g}, [p])$ be a classical semisimple Lie algebra over an algebraically closed field F, char $F = p > 0$, $\boldsymbol{h}$ a Cartan subalgebra of $\boldsymbol{g}$, $\boldsymbol{b}$ a Borel subalgebra such that $\boldsymbol{h} \subset \boldsymbol{b}$, and $u(\boldsymbol{g})$ and $u(\boldsymbol{b})$ the restricted universal enveloping algebras of $\boldsymbol{g}$ and $\boldsymbol{b}$, respectively. For any restricted homomorphism $\lambda : \boldsymbol{b} \to F$ we let $Z(\lambda) := u(\boldsymbol{g}) \otimes_{u(b)} F_\lambda$ denote the restricted Verma module of $\boldsymbol{g}$ with the highest weight λ, where F_λ is a canonical one-dimesional $\boldsymbol{b}$-module. In [5], R. Farnsteiner and H. Strade obtained a modular Lie algebraic version of Shapiro's lemma and showed that if $\lambda - \sigma|_h$ is not a sum of positive roots, then $H^*(\boldsymbol{g}, Z(\lambda)) = 0$, where $\sigma : \boldsymbol{b} \to F$ is the Lie algebra homomorphism given by $\sigma(x) := \mathrm{tr}(ad_{g/b}\, x)$, $\forall x \in \boldsymbol{b}$. In this paper we give, in generalization of the results of [5], the structure of the cohomology $H^*(\boldsymbol{g}, Z(\lambda))$ of $\boldsymbol{g}$ with coefficients in $Z(\lambda)$ for any restricted homomorphism $\lambda : \boldsymbol{b} \to F$. The main result is Theorem 6.1. By Shapiro's lemma and the Hochschild-Serre spectral sequence, we reduce the computation of $H^*(\boldsymbol{g}, Z(\lambda))$ to the computation of the cohomology of nilradicals $\boldsymbol{n}$ of $\boldsymbol{b}$ in certain modules. For $H^*(\boldsymbol{n}, F_\lambda)$, we generalize B. Kostant's fundamental result [10, Theorem 5.14] on the homology (or cohomology) of nilradicals of a complex semisimple Lie algebra to the modular case (see Theorem 5.1).

Most of the results presented here can easily be modified to yield statements regarding homology groups.

§2. The Restricted Verma Modules

In this section, we shall review the notions and the results of [5]. Let V be a $\boldsymbol{b}$-module. We introduce a twisted action on V by setting $x \cdot v := xv + \sigma(x)v$, where $x \in \boldsymbol{b}$, $v \in V$ and $\sigma(x) = \mathrm{tr}(ad_{g/b}\, x)$. The new $\boldsymbol{b}$-module will be called V_σ. Similarly, we can define the $\boldsymbol{b}$-module $V_{-\sigma}$ such that $(V_{-\sigma})_\sigma = V$. Let $\boldsymbol{n}$ (or $\boldsymbol{n}^-$) be the sum of positive (or negative) root spaces of

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$\boldsymbol{g}$. Let $\{h_1, \cdots, h_l\}$ be a canonical basis of $\boldsymbol{h}$ such that $\alpha_i(h_j) \in \mathbb{Z}/p\mathbb{Z}$, $i, j = 1, \cdots, l$, where $\{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots. Let Λ denote the collection of p^l restricted weights μ characterized by the conditions $0 \le \mu(h_i) < p$, $1 \le i \le l$. Then $\sigma|_h \in \Lambda$ and $\lambda|_h \in \Lambda$ for any restricted homomorphism $\lambda : \boldsymbol{b} \to F$. Obviously, $\sigma|_n = 0$ and $\lambda|_n = 0$. Hence λ and σ are determined by $\lambda|_h$ and $\sigma|_h$. For convenience, we still denote $\lambda|_h$ and $\sigma|_h$ by λ and σ, respectively. For each $\mu \in \Lambda$, we can also canonically obtain the one-dimensional $\boldsymbol{b}$-module which is still denoted by F_μ. For any $\lambda \in \Lambda$, by [5, Corollary 1.6], we have

$$Z(\lambda) = u(\boldsymbol{g}) \otimes_{u(b)} F_\lambda \simeq \mathrm{Hom}_{u(b)}(u(\boldsymbol{g}), F_{\lambda-\sigma})(\text{as } \boldsymbol{g}-\text{modules}).$$

Thus we can quote the modular Lie algebraic veraion of Shapiro's lemma due to Farnsteiner and Strade.

Lemma 2.1[5, Theorem 3.2]. *For any* $\lambda \in \Lambda$,

$$H^k(\boldsymbol{g}, Z(\lambda)) \simeq H^k(\boldsymbol{g}, \mathrm{Hom}_{u(b)}(u(\boldsymbol{g}), F_{\lambda-\sigma})$$
$$\simeq \bigoplus_{i+j=k} \Lambda^i(\boldsymbol{g}/\boldsymbol{b}) \otimes_F H^j(\boldsymbol{b}, F_{\lambda-\sigma}).$$

Thus the computation of $H^*(\boldsymbol{g}, Z(\lambda))$ *can be reduced to the computation of* $H^*(\boldsymbol{b}, F_{\lambda-\sigma})$. *By means of the Hochschild-Serre spectral sequence, we can easily show that*

$$E_2^{i,j} \simeq \Lambda^i(\boldsymbol{h}) \otimes_F H^j(\boldsymbol{n}, F)_{-\mu} \underset{i}{\Rightarrow} H^k(\boldsymbol{b}, F_\mu), \tag{2.1}$$

where $\mu \in \Lambda$ *and* $k = i + j$.

In the next section we shall discuss $H_*(\boldsymbol{n}^-, F)$ for computing $H^*(\boldsymbol{n}, F)_{-\mu}$.

§3. The $\boldsymbol{h}$-Module Complex C

Let $\mathcal{U}(\boldsymbol{g})$, $\mathcal{U}(\boldsymbol{b})$ and $\mathcal{U}(\boldsymbol{n}^-)$ be the universal enveloping algebras of $\boldsymbol{g}$, $\boldsymbol{b}$ and $\boldsymbol{n}^-$, respectively. For each $j \in \mathbb{N}$, let D_j be the $\boldsymbol{g}$-module $\mathcal{U}(\boldsymbol{g}) \otimes_{\mathcal{U}(b)} \Lambda^j(\boldsymbol{g}/\boldsymbol{b})$. Then there is an exact sequence of $\boldsymbol{g}$-modules (cf. [6, Proposition 1.1])

$$\boldsymbol{B} : \cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \xrightarrow{\mathcal{E}_0} F \to 0, \tag{3.1}$$

where $\mathcal{E}_0 : D_0 \to F$ is defined by the condition that $\mathcal{E}_0(x \otimes 1)$ $(x \in \mathcal{U}(\boldsymbol{g}))$ is the constant term of x, and the $\boldsymbol{g}$-module map $d_j : D_j \to D_{j-1}$ is defined as follows:

Let $x_1, \cdots, x_j \in \boldsymbol{g}/\boldsymbol{b}$, and choose representations $y_1, \cdots, y_j \in \boldsymbol{g}$, then for all $x \in \mathcal{U}(\boldsymbol{g})$,

$$d_j(x \otimes x_1 \wedge \cdots \wedge x_j) = \sum_{i=1}^{j} (-1)^{i+1}(xy_i) \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_j$$
$$+ \sum_{1 \le r < s \le j} (-1)^{r+s} x \otimes \pi[y_r, y_s] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_j,$$

where $\pi : \boldsymbol{g} \to \boldsymbol{g}/\boldsymbol{b}$ denotes the canonical map, and $\hat{}$ signifies the omission of a symbol. Since D_j is isomorphic as $\mathcal{U}(\boldsymbol{n}^-)$-module and as $\boldsymbol{h}$-module to $\mathcal{U}(\boldsymbol{n}^-) \otimes_F \Lambda^j(\boldsymbol{n}^-)$ with $\mathcal{U}(\boldsymbol{n}^-)$ acting by left multiplication on the first factor, and $\boldsymbol{h}$ acting on $\mathcal{U}(\boldsymbol{n}^-) \otimes_F \Lambda^j(\boldsymbol{n}^-)$ by the tensor product action. The above free resolution of F gives rise to an $\boldsymbol{h}$-module complex

$$\boldsymbol{C} : \cdots \xrightarrow{1 \otimes d_2} F \otimes_{\mathcal{U}(n^-)} D_1 \xrightarrow{1 \otimes d_1} F \otimes_{\mathcal{U}(n^-)} D_0 \to 0 \tag{3.2}$$

and its homology is $H_*(\boldsymbol{n}^-, F)$ and $\boldsymbol{h}$ acts in the obvious way on $H_*(\boldsymbol{n}^-, F)$.

§4. The Symmetric Bilinear Form τ on h^*

Let $A = (A_{ij})_{l\times l}$ be the Cartan matrix of g. Then there are positive integers $q_1, \cdots, q_l$ such that $\mathrm{diag}(q_1, \cdots, q_l)A$ is a symmetric matrix. Let $g_{\mathbb{Q}}$ be the semisimple Lie algebra over $\mathbb{Q}$ with the same Cartan matrix A. Let $h_i, e_i, f_i (i = 1, \cdots, l)$ be the canonical generators with the relations

$$[h_i, h_j] = 0,\ [e_i, f_j] = \delta_{ij}h_i,\ [h_i, e_j] = A_{ij}e_i,\ [h_i, f_j] = -A_{ij}f_i,\ \forall i, j = 1, \cdots, l,$$

and $(ade_i)^{-A_{ij}+1} = 0 = (adf_j)^{-A_{ij}+1}$ whenever $i \neq j$. Let $h_{\mathbb{Q}}$ be a Cartan subalgebra of $g_{\mathbb{Q}}$, $\Phi \subset h_{\mathbb{Q}}^*$ the root system (relative to $h_{\mathbb{Q}}$), $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ a basis of Φ, Φ_+ (or Φ_-) the set of positive (or negative) roots and W its Weyl group. For every subset Ψ of Φ, define $\langle\Psi\rangle = \sum\limits_{\varphi\in\Phi} \varphi \in h^*$. For all $w \in W$, define

$$\Phi_w = \Phi_+ \cap w\Phi_- = \{\varphi \in \Phi_+ | w^{-1}\varphi \in \Phi_-\}.$$

Write $\rho = \frac{1}{2}\langle\Phi_+\rangle \in h^*$. Then $-\langle\Phi_w\rangle = w\rho - \rho$ (cf. [6, Proposition 2.5]) and $\rho(h_i) = 1$ for all $i = 1, \cdots, l$. Write $T = \{-\langle\Psi\rangle | \Psi \subseteq \Phi_+\}$.

Remark 4.1. For g we still adopt the above notations, except that the integeral coefficients A_{ij} in the relations are reduced modulo a prime p (> 2 or 3).

Define a symmetric bilinear form $\tau_{\mathbb{Q}}$ on $h_{\mathbb{Q}}^*$ by the conditions

$$\tau_{\mathbb{Q}}(\alpha_i, \alpha_j) = q_iA_{ij}, \quad \forall i, j = 1, \cdots, l.$$

Then $\tau_{\mathbb{Q}}(\rho, \alpha_i) = q_i$.

Lemma 4.1.[6,2.13] *Let $\mu = -\langle\Psi\rangle \in T$. Then*

(1) *We have*

$$\tau_{\mathbb{Q}}(\rho, \rho) - \tau_{\mathbb{Q}}(\mu + \rho, \mu + \rho) \geq 0. \tag{4.1}$$

(2) *In (4.1), equality holds if and only if there exists $w \in W$ such that $\Psi = \Phi_w$, or equivalently, such that $\mu = -\langle\Phi_w\rangle = w\rho - \rho$. In case of equality, μ determines w.*

For $\mu = -\sum\limits_i n_i\alpha_i \in T$, we have

$$\begin{aligned}\tau_{\mathbb{Q}}(\rho, \rho) - \tau_{\mathbb{Q}}(\mu + \rho, \mu + \rho) =& \tau_{\mathbb{Q}}(\rho, \sum n_i\alpha_i) + \tau_{\mathbb{Q}}(\sum n_i\alpha_i, -\sum n_i\alpha_i + \rho)\\ =& 2\sum n_iq_i - \sum_i\sum_j q_iA_{ij}n_in_j\\ =& 2(\sum_i q_in_i - \sum_i q_in_i^2 - \sum_{i<j} q_iA_{ij}n_in_j).\end{aligned}$$

Let

$$f(x_1, \cdots, x_l) = \sum_i q_ix_i - \sum_i q_ix_i^2 - \sum_{i<j} q_iA_{ij}x_ix_j$$

be a polynomial in variable $x_1, \cdots, x_l$ and $c_1, \cdots, c_l$ is the solution of the following system of linear equations

$$2q_ix_i + \sum_{i<k} q_iA_{ik}x_k + \sum_{k<i} q_kA_{ki}x_k = q_i,\ i = 1, \cdots, l. \tag{4.2}$$

Then we have

$$\frac{1}{2}(\tau_{\mathbb{Q}}(\rho, \rho) - \tau_{\mathbb{Q}}(\mu + \rho, \rho + \rho)) \leq f(c_1, \cdots, c_l), \ \text{for}\ \mu \in T.$$

Write $q' = \max\{2, [f(c_1, \cdots, c_l)]\}$. Define a symmetric bilinear form τ on $\boldsymbol{h}^*$ by the conditions

$$\tau(\alpha_i, \alpha_j) \equiv q_i A_{ij} \mathrm{mod}(p), \ \forall i, j = 1, \cdots, l.$$

Then using Lemma 4.1, we have

Corollary 4.1. *Suppose* char.$F = p > q'$. *Then for* $\mu = -\langle \Psi \rangle \in T$,

$$\tau(\rho, \rho) - \tau(\mu + \rho, \mu + \rho) = 0$$

if and only if there exists (unique) $w \in W$ *such that* $\Psi = \Phi_w$ *and* $\mu = -\langle \Phi_w \rangle = w\rho - \rho$.

Example. If $\boldsymbol{g}$ is of type A_l, then

$$c_i = \frac{il}{2} - \sum_{k=1}^{i-1} k, \quad i = 1, \cdots, l,$$

satisfy (4.2) and

$$q' = \max\{2, \sum_i c_i - \sum_i c_i^2 + \sum_{i=1}^{l-1} c_i c_{i+1}\}.$$

Thus we have

l	2	3	4	5	6	$\cdots$
q'	2	2	5	8	14	$\cdots$

§5. The Casimir Element

Let char $F = p > q'$. Set $t_i = q_i h_i$ and $k_i = \frac{1}{4}\tau(\alpha_i, \alpha_i) h_i$, $i = 1, \cdots, l$. If $\alpha = \sum\limits_{i=1}^{l} n_i \alpha_i$, then set $t_\alpha = \sum\limits_{i=1}^{l} n_i t_i$. For $\alpha \in \Phi$, choose nonzero x_α in $\boldsymbol{g}_\alpha$ and $z_\alpha \in \boldsymbol{g}_{-\alpha}$ such that

$$[x_\alpha, z_\alpha] = t_\alpha,$$

where $\boldsymbol{g}_\alpha$ and $\boldsymbol{g}_{-\alpha}$ are the root spaces of α and $-\alpha$ in $\boldsymbol{g}$, respectively. Define $\Gamma = \sum\limits_{i=1}^{l} h_i k_i + \sum\limits_{\alpha \in \Phi} x_\alpha z_\alpha$ which is called a universal Carimir element of $\boldsymbol{g}$. Similarly as in the case of characteristic 0 (cf. [8, §22.1]), we can show that for any representation φ of $\boldsymbol{g}$, $\varphi(\Gamma)$ commutes with $\varphi(\boldsymbol{g})$.

For $\lambda \in \Lambda$, we let $V(\lambda) := \mathcal{U}(\boldsymbol{g}) \otimes_{\mathcal{U}(b)} F_\lambda$ denote the Verma module of $\boldsymbol{g}$ with the highest weight λ.

Lemma 5.1. *Let* $\lambda \in \Lambda$. *Then* Γ *acts on* $V(\lambda)$ *as scalar multiplication by* $\tau(\lambda + \rho, \lambda + \rho) - \tau(\rho, \rho)$.

Proof. Since $V(\lambda) = \mathcal{U}(\boldsymbol{g})(1 \otimes 1)$, it is enough to show that $\Gamma(1 \otimes 1) = (\tau(\lambda + \rho, \lambda + \rho) -$

$\tau(\rho,\rho))(1\otimes 1)$. Since $\lambda(t_\alpha)=\tau(\lambda,\alpha)$, we have

$$\begin{aligned}\Gamma(1\otimes 1)&=\sum_{i=1}^{l} h_i k_i(1\otimes 1)+\sum_{\alpha\in\Phi_+} t_\alpha(1\otimes 1)\\&=\sum_{i=1}^{l}\lambda(h_i)\lambda(k_i)+\sum_{\alpha\in\Phi_+}\lambda(t_\alpha)\\&=(\tau(\lambda+\rho,\lambda+\rho)-\tau(\rho,\rho))(1\otimes 1).\end{aligned}$$

Let V be a $\boldsymbol{g}$-module and

$$\Theta(V)=\{c\in F|\Gamma v=cv,\ \text{for some } v\in V,\ v\neq 0\}.$$

For all $c\in F$, let

$$V_{(c)}=\{v\in V|(\Gamma-c)^n v=0,\ \text{for some } n>0\}.$$

Then $\Theta(V)=\{c\in F|V_{(c)}\neq 0\}$. In particular,

$$\Theta(V(\lambda))=\{\tau(\lambda+\rho,\lambda+\rho)-\tau(\rho,\rho)\},\ \text{for } \lambda\in\Lambda.$$

Let $\Psi=\{\lambda_1,\lambda_2,\cdots\}\subset\Lambda$. A $\boldsymbol{g}$-module V is said to be of type Ψ if V has a strictly increasing (finite) $\boldsymbol{g}$-module filtration $0=V_0\subset V_1\subset V_2\subset\cdots$ such that $V=\cup V_i$ and such that the sequence of $\boldsymbol{g}$-modules $V_1/V_0,\ V_2/V_1,\cdots$ coincides up to rearrangement with the sequence of induced modules $V(\lambda_1),V(\lambda_2),\cdots$.

Lemma 5.2. *For each $j\in\mathbb{Z}_+$, let $\Psi^j=\{\lambda_{j_1},\lambda_{j_2},\cdots\}$ be the set of the weights of $\Lambda^j(\boldsymbol{n}^-)$, then D_j is of type Ψ^j. Furthermore, we have*

$$D^j=\bigoplus_{c\in\Theta(D_j)}(D_j)_{(c)},$$

where

$$\Theta(D_j)=\{\tau(\lambda_{j_i}+\rho,\lambda_{j_i}+\rho)-\tau(\rho,\rho)\}_i.$$

Proof. As $\boldsymbol{h}$-modules, $\boldsymbol{g}/\boldsymbol{b}\simeq\boldsymbol{n}^-$ and $\Lambda^j(\boldsymbol{g}/\boldsymbol{b})\simeq\Lambda^j(\boldsymbol{n}^-)$. Clearly $\Lambda^j(\boldsymbol{n}^-)$ is a weight module with weights all of the form

$$-\sum_{k=1}^{l} n_k\alpha_k\in\mathbb{Z}\alpha_1+\cdots+\mathbb{Z}\alpha_l,$$

and the same is true of $\Lambda^j(\boldsymbol{g}/\boldsymbol{b})$. Note that if $m_i=n_i \bmod(p),\ i=1,\cdots,l$, then

$$-\sum_{i=1}^{l} m_i\alpha_i=-\sum_{i=1}^{l} n_i\alpha_i\in\Lambda.$$

We arrange the weights of $\Lambda^j(\boldsymbol{g}/\boldsymbol{b})$ in a certain order

$$\lambda_{j_1}(=-\sum_{k=1}^{l} n_{k1}\alpha_k),\ \lambda_{j_2}(=-\sum_{k=1}^{l} n_{k2}\alpha_k),\cdots,$$

such that $\sum_{k=1}^{l} n_{k,i+1}$ is not less than $\sum_{k=1}^{l} n_{ki}$. Let $v_{j_1},v_{j_2},\cdots$ be the corresponding weight vectors in $\Lambda^j(\boldsymbol{g}/\boldsymbol{b})$ and

$$V_{j_1}=\bigoplus_{k\leq i} Fv_{j_k},\ i=1,2,\cdots.$$

Then

$$0 = V_{j_0} \subset V_{j_1} \subset V_{j_2} \subset \cdots$$

is a filtration of $\boldsymbol{b}$-modules such that $\Lambda^j(\boldsymbol{g}/\boldsymbol{b}) = \cup V_{j_i}$ and each quotient $V_{j_{i+1}}/V_{j_i}$ is a trivial $\boldsymbol{n}$-module and is isomorphic as an $\boldsymbol{h}$-module to a module $F_{\lambda_{j_{i+1}}}$, and $\underset{i}{\oplus} V_{j_{i+1}}/V_{j_i}$ is isomorphic as an $\boldsymbol{h}$-module to $\Lambda^j \boldsymbol{n}^-$.

By [6, Proposition 1.10], the $\boldsymbol{g}$-module D_j is of type Ψ^j. Hence D_j is annihilated by a product of powers of operators of the form $\Gamma - (\tau(\lambda_{j_i} + \rho, \lambda_{j_i} + \rho) - \tau(\rho, \rho))$. By standard linear algebra,

$$D_j = \bigoplus_{c \in \Theta(D_j)} (D_j)_{(c)}.$$

Since the action of Γ on D_j commutes with the action of $\boldsymbol{g}$ of D_j, $(D_j)_{(c)}$ is also a $\boldsymbol{g}$-submodule of D_j. Let $f : X \to Y$ be a $\boldsymbol{g}$-module map. Then

$$f \cdot \Gamma(v) = \Gamma \cdot f(v), \quad \forall v \in X. \tag{5.1}$$

Thus we have

$$(X_{i+1}/X_i)_{(c)} = (X_{i+1})_{(c)}/(X_i)_{(c)}, \text{ for } c \in F$$

and the following subcomplex of the exact sequence (3.1)

$$\boldsymbol{B}_{(c)} : \cdots \to (D_1)_{(c)} \to (D_0)_{(c)} \to F_{(c)} \to 0$$

is also exact.

Let $0 = X_{j_0} \subset X_{j_1} \subset X_{j_2} \subset \cdots$ be any filtration of D_j with the properties stated in Lemma 5.2 and $\lambda_{j_i} \in \Lambda$ be the highest weight of X_{i+1}/X_i, that is, $X_{i+1}/X_i \simeq V(\lambda_{j_i})$ for each i. Then $(D_j)_{(c)}$ has a $\boldsymbol{g}$-module filtration $0 = Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ such that $(D_j)_{(c)} = \underset{k}{\cup} Y_k$, and the family of $\boldsymbol{g}$-modules Y_{k+1}/Y_k coincides (up to isomorphism) with the family of $\boldsymbol{g}$-modules X_{i+1}/X_i for which $\tau(\lambda_{j_i}+\rho, \lambda_{j_i}+\rho) - \tau(\rho,\rho) = C$. Hence by Lemma 5.2, we have

Lemma 5.3. *Let $\Psi_0^j = \{\lambda \in \Psi^j | \tau(\lambda+\rho, \lambda+\rho) - \tau(\rho,\rho) = 0\}$. Then the resolution $\boldsymbol{B}$ is a direct sum of an exact $\boldsymbol{g}$-module complex*

$$\boldsymbol{B}_{(0)} : \cdots \to (D_1)_{(0)} \to (D_0)_{(0)} \to F \to 0 \tag{5.2}$$

and a finite number of exact $\boldsymbol{g}$-module complexes

$$\boldsymbol{B}_{(c)} : \cdots \to (D_1)_{(c)} \to (D_0)_{(c)} \to 0, \text{ for } c \neq 0 \tag{5.3}$$

such that for each $j \in \mathbb{Z}_+$, $(D_j)_{(0)}$ is of type Ψ_0^j.

By Lemma 5.3, the complex (3.2) is the direct sum of the $\boldsymbol{h}$-module complex

$$\boldsymbol{C}_{(0)} : \cdots \to F \otimes_{\mathcal{U}(n^-)} (D_1)_{(0)} \to F \otimes_{\mathcal{U}(n^-)} (D_0)_{(0)} \to 0$$

and the $\boldsymbol{h}$-module complexes

$$\boldsymbol{C}_{(c)} : \cdots \to F \otimes_{\mathcal{U}(n^-)} (D_1)_{(c)} \to F \otimes_{\mathcal{U}(n^-)} (D_0)_{(c)} \to 0, \text{for } c \neq 0.$$

Since D_j is free as a $\mathcal{U}(\boldsymbol{n}^-)$-module, (5.3) is a $\mathcal{U}(\boldsymbol{n}^-)$-projective resolution of the zero module 0. Thus $\mathrm{Tor}_*^{n^-}(F, 0) = 0$, that is, the homology of $\boldsymbol{C}_{(c)} (c \neq 0)$ is zero. Hence we have

Lemma 5.4. *$H_*(\boldsymbol{n}^-, F)$ is $\boldsymbol{h}$-module isomorphic to the homology of $\boldsymbol{C}_{(0)}$.*

Note that $F \otimes_{\mathcal{U}(n^-)} (D_j)_{(0)}$ is isomorphic to the sum of all the $\boldsymbol{h}$-weight spaces $\Lambda^j(\boldsymbol{n}^-)_\lambda$ with weights $\lambda \in \Lambda$ satisfying $\tau(\lambda+\rho, \lambda+\rho) = \tau(\rho, \rho)$.

For all $j \in \mathbb{Z}_+$, write $W_j = \{w \in W | l(w) = j\}$ and $F_w = F_{-\langle\Phi_w\rangle}$.

Theorem 5.1. *Suppose that* char$F = p > q'$. *For each* $j \subset \mathbb{Z}_+$, *the* $\boldsymbol{h}$*-module*

$$H_j(\boldsymbol{n}^-, F) \simeq \bigoplus_{w \in W_j} F_{w\rho-\rho}.$$

Proof. First, we shall show that

$$F \oplus_{\mathcal{U}(n^-)} (D_j)_{(0)} = \bigotimes_{w \in W_j} F_{-\langle\Phi_w\rangle}.$$

Let $w \in W_j$. Set $\lambda = -\langle\Phi_w\rangle$, $\Phi_w = \{\beta_1, \cdots, \beta_j\}$ and $x_{\beta_i} \in \boldsymbol{g}_{\beta_i}$. Then we have $x_{\beta_1} \wedge \cdots \wedge x_{\beta_j} \in \Lambda^j(\boldsymbol{n}^-)_\lambda$ (the weight space in $\Lambda^j(\boldsymbol{n}^-)$ with the weight λ). Since $\lambda = -\langle\Phi\rangle = w\rho - \rho$, we have $\tau(\lambda+\rho, \lambda+\rho) = \tau(w\rho, w\rho) = \tau(\rho, \rho)$. Conversely, let $x_{\beta_1} \wedge \cdots \wedge x_{\beta_j} \in \Lambda^j(\boldsymbol{n}^-)_\lambda$ such that $\tau(\lambda+\rho, \lambda+\rho) = \tau(\rho, \rho)$ and $x_{\beta_i} \in \boldsymbol{g}_{\beta_i}$. By Corollary 4.1, there exists (unique) $w \in W_j$ such that $\Phi_w = \{\beta_1, \cdots, \beta_j\}$. Hence $F(x_{\beta_i} \wedge \cdots \wedge x_{\beta_j}) = F_w$ and

$$F \otimes_{\mathcal{U}(n^-)} (D_j)_{(0)} = \bigoplus_{w \in W_j} F_w = \bigoplus_{w \in W_j} F_{-\langle\Phi_w\rangle} = \bigoplus_{w \in W_j} F_{w\rho-\rho}.$$

Next, we claim that all of the maps $d_j : F \otimes_{\mathcal{U}(n^-)} (D_j)_{(0)} \to F \otimes_{\mathcal{U}(n^-)} (D_{j-1})_{(0)}$ are zero. Indeed, if α, $\beta \in \Phi_w$, then $\alpha + \beta \in \Phi_w$ or $\alpha + \beta \notin \Phi$. Hence

$$D_j(x_{\beta_1} \wedge \cdots \wedge x_{\beta_j}) = \sum_{1 \le r < \le s \le j} (-1)^{r+s} [x_{\beta_r}, x_{\beta_s}] \wedge \cdots \wedge x_1 \wedge \cdots \wedge \hat{x}_{\beta_r} \wedge \cdots \wedge \hat{x}_{\beta_s} \wedge \cdots \wedge x_{\beta_j} = 0.$$

Since $H_j(\boldsymbol{n}, F) \simeq \oplus_{w \in W_j} F_{\rho-w\rho}$ and $H^j(\boldsymbol{n}, F) = H_j(\boldsymbol{n}, F)$, we obtain

Corollary 5.1. *Let charF* $= p > q'$. *For each* $j \in \mathbb{Z}_+$,

$$H^j(\boldsymbol{n}, F) = \bigoplus_{w \in W_j} F_{w\rho-\rho}.$$

Remark 5.1. Let $\langle\Phi_+\rangle = \sum_{i=1}^{l} n_i\alpha_i$, $q = \max\{q', n_1, \cdots, n_l\}$ and char$F = p > q$. Obviously, if $w_1\rho - \rho = w_2\rho - \rho$, for $w_1, w_2 \in W$, then $w_1 = w_3$. But if $p < q$, then the conclusiuon is not always true.

§6. The Cohomology of g with Coefficients in $Z(\lambda)$

In this section, we assume that char$F = p > q$. First, we compute $H^*(\boldsymbol{b}, F_\mu)$ for $\mu \in \Lambda$. By (2.1) and Corollary 5.1, we have

$$E_2^{i,j} \simeq \Lambda^i(\boldsymbol{h}) \otimes_F (\bigoplus_{w \in W_j} F_{w\rho-\rho})_{-\mu}.$$

Thus we obtain two cases:

(1) If $\mu \neq \rho - w\rho$, for all $w \in W$, then $E_2^{i,j} = 0$, for $p, q \geq 0$. It implies that

$$H^k(\boldsymbol{b}, F_\mu) = E_2^{k,0} = 0 \ \ \forall k \in \mathbb{Z}_+.$$

(2) If $\mu = \rho - w_0\rho$, for some $w_0 \in W_j$, then w_0 is unique and

$$E_2^{i,j} \simeq \begin{cases} 0, & \text{for } i \geq 0 \text{ and } j \neq l(w_0), \\ \Lambda^j(\boldsymbol{h}), & \text{for } i \geq 0 \text{ and } j = l(w_0). \end{cases}$$

Thus we have

$$H^k(\boldsymbol{b}, F_\mu) = E_2^{k-l(w_0),l(w_0)} = \Lambda^{k-l(w_0)}\boldsymbol{h}.$$

Then we have shown

Proposition 6.1. *Let $\mu \in \Lambda$. Then for each $k \in \mathbb{Z}_+$,*

$$H^k(\boldsymbol{b}, F_\mu) = \begin{cases} \Lambda^{k-l(w_0)}\boldsymbol{h}, & \text{if } \mu = \rho - w_0\rho, \text{ for some } w_0 \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Next, since $\sigma|_h = -\langle\Phi_+\rangle = -2\rho$, by Proposition 6.1, Lemma 2.1 and (2.1), we have

Theorem 6.1. *Let $\lambda \in \Lambda$. Then for each $k \in \mathbb{Z}_+$,*

$$H^k(\boldsymbol{g}, Z(\lambda)) \simeq \begin{cases} \bigoplus\limits_{i+j=k} \Lambda^i(\boldsymbol{g}/\boldsymbol{b}) \otimes_F \Lambda^{j-l(w_0)}\boldsymbol{h}, & \text{if } \lambda = -w_0\rho - \rho, \text{ for some } w_0 \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 6.1. If $\lambda = (p-1)\rho$, then $Z((p-1)\rho)$ is called the Steinberg module. Since $(p-1)\rho \neq -w\rho - \rho$ for any $w \in W$, by Theorem 6.1, $H^*(\boldsymbol{g}, Z((p-1)\rho)) = 0$. On the other hand, by [9, Corollary 5,4], $Z((p-1)\rho)$ is irreducible and projective, hence is injective (since $\mathcal{U}(\boldsymbol{g})$ is a symmetric algebra). It also implies that $H^*(\boldsymbol{g}, Z((p-1)\rho)) = 0$, which is compatible with the above fact.

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