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ON THE FIRST AND SECOND EIGENVALUES OF SCHRÖDINGER OPERATOR**

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Abstract

The above boundary of the gap between the 2nd and 1st eigenvalues of Schrödinger operator and the above boundary of the ratio of the 2nd and 1st eigenvalues of Laplace operator are given.

§1. Introduction

Let Ω be a bounded domain in $R^n, \triangle$ be the Laplace operator and let $V : \Omega \to R$ be a non-negative function. The eigenvalues of the equation

$$-\triangle f + Vf = \lambda f \tag{1.1}$$

with the condition

$$f = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots .$$

B. Wong, A. T. Yau and Stephep S. T. Yan[1] proved

$$\lambda_2 - \lambda_1 \leq \frac{4\pi^2}{D^2} + \frac{4M}{n}, \tag{1.3}$$

where D is the diameter of the largest inscribed ball in Ω and $M = \sup\limits_{\bar{\Omega}} V$. In this paper more accurate evaluation will be given.

For $n = 2$, and $V \equiv 0$ Payne, Pôyne and Weinberger proved [5]

$$\frac{\lambda_2}{\lambda_1} \leq 3 \tag{1.4}$$

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2(D)}{\lambda_1(D)} \simeq 2.539\cdots, \tag{1.5}$$

where $\lambda_1(D)$ and $\lambda_2(D)$ are the 1st and 2nd eigenvalues of unit disk in R^2. In 1980 G. H. Hile and M. H. Protter proved [4]

$$\frac{\lambda_2}{\lambda_1} < 2.686\cdots .$$

In this paper the above bound will be discussed.

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§2. Symbols and Lemmas

f —the first eigenfunction of equation (1.1) with the condition (1.2) satisfying

$$\int_\Omega f^2 dx = 1, \quad \text{and} \quad f(x) > 0^{[3]}, x \in \Omega;$$

x_i—the coordinate function in $R^n, i = 1, 2, \cdots, n$;

$$A_s = \int_\Omega f^{s+2} dx; X_s = \frac{A_s^2}{A_{2s}}; X_{pq} = \frac{A_{p+q}^2}{A_{2p}A_{2q}};$$

$$Y = \frac{\lambda_2}{\lambda_1} - 1; I_{s+t} = (s+t+1)Y - st.$$

Lemma 2.1. $\int_\Omega f^s |\nabla f|^2 dx = \frac{\lambda_1 A_s}{s+1} - \frac{\int V f^{s+2}}{s+1}$.

Proof. $f^s \triangle f^2 = 2f^s(|\nabla f|^2 + f\triangle f) = 2f^s|\nabla f|^2 - 2\lambda_1 f^{s+2} + 2Vf^{s+2}$.

Thus

$$\int_\omega f^s \triangle f^2 dx = 2\int_\Omega f^s |\nabla f|^2 dx - 2\lambda_1 A_s + 2\int_\Omega V f^{s+2} dx. \tag{2.1}$$

On the other hand,

$$\begin{aligned}\int_\Omega f^s \triangle f^2 dx &= -\int_\Omega \nabla f^s \nabla f^2 dx + \int_{\partial\Omega} f^s \frac{\partial f^2}{\partial \nu} dl \\ &= -\int_\Omega \nabla f^s \nabla f^2 dx \\ &= \int_\Omega f^2 \triangle f^s dx \\ &= s(s-1)\int_\Omega f^s |\nabla f|^2 dx - \lambda_1 s A_s + s\int_\Omega V f^{s+2} dx. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$(2 - s^2 + s)\int_\Omega f^s |\nabla f|^2 + (2-s)\lambda_1 A_s - (2-s)\int_\Omega V f^{s+2} dx,$$

i.e.,

$$\int_\Omega f^s |\nabla f|^2 dx = \frac{\lambda_1}{s+1} A_s - \frac{1}{s+1}\int_\Omega V f^{s+2} dx.$$

Lemma 2.2. *Suppose that* $u : \Omega \to R$ *is a* C^∞ *function satisfying* $\int_\Omega u f^2 dx = 0$. *Then*

$$Y \int_\Omega u^2 f^2 dx \le \frac{\int_\Omega f^2 |\nabla u|^2 dx}{\lambda_1}.$$

Proof. It is well-known that

$$\begin{aligned}\lambda_2 &= \inf_{g \perp f, g|_{\partial\Omega} = 0} \frac{\int_\Omega -g\triangle g dx + \int_\Omega V g^2 dx}{\int_\Omega g^2 dx} \\ &= \inf_{g \perp f, g|_{\partial\Omega} = 0} \frac{\int_\Omega |\nabla g|^2 dx + \int_\Omega V g^2 dx}{\int_\Omega g^2 dx}.\end{aligned}$$

Hence

$$\begin{aligned}\lambda_2 \int_\Omega u^2 f^2 dx &\le \int_\Omega |\nabla u f|^2 dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega |u\nabla f + f\nabla u|^2 dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega (u^2|\nabla f|^2 + f^2|\nabla u|^2 + \frac{1}{2}\nabla f^2 \cdot \nabla u^2) dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega u^2 |\nabla f|^2 dx + \int_\Omega f^2 |\nabla u|^2 dx - \frac{1}{2}\int_\Omega u^2 \Delta f^2 dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega u^2 |\nabla f|^2 dx + \int_\Omega f^2 |\nabla u| dx - \int_\Omega u^2 |\nabla f|^2 dx \\ &\quad - \int_\Omega u^2 f \Delta f dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega f^2 |\nabla u|^2 dx - \int_\Omega u^2 f^2 (-\lambda_1 + V) dx + \int_\Omega V u^2 f^2 dx \\ &= \int_\Omega f^2 |\nabla u|^2 dx + \lambda_1 \int_\Omega u^2 f^2 dx.\end{aligned}$$

Corollary 2.1. *Let $q \ge 0, a_i = \frac{\int_\Omega x_i f^{q+2} dx}{\int_\Omega f^{q+2} dx}$. Then*

$$I_{2q} \int_\Omega (a_i - x_i)^2 f^{2q+2} \le \frac{1+q}{\lambda_1} A_{2q} \ for\ any\ i = 1, 2, \cdots, n.$$

In particular, for $q = 0$,

$$Y \int_\Omega (a_i - x_i)^2 f^2 dx \le \frac{1}{\lambda_1} \text{ for any } i = 1, 2, \cdots, n.$$

Proof. Set $u = (a_i - x_i) f^q$ in Theorem 2. Then $\int u f^2 dx = 0$. Also

$$|\nabla u|^2 = q^2 (a_i - x_i)^2 f^{2q-2} |\nabla f|^2 + f^{2q} - 2q(a_i - x_i) f^{2q-1} \nabla f \cdot \nabla x_i$$

and

$$\begin{aligned}\int_\Omega f^2 |\nabla u|^2 dx &= A_{2q} + q^2 \int_\Omega (a_i - x_i)^2 f^{2q} |\nabla f|^2 dx + \frac{q}{2(q+1)} \int_\Omega \nabla f^{2q+2} \cdot \nabla (a_i - x_i)^2 dx \\ &= A_{2q} + q^2 \int_\Omega (a_i - x_i)^2 f^{2q} |\nabla f|^2 dx - \frac{q}{2(q+1)} \int_\Omega f^{2q+2} \cdot \Delta (a_i - x_i)^2 dx \\ &= \frac{A_{2q}}{q+1} + q^2 \int_\Omega (a_i - x_i)^2 f^{2q} |\nabla f|^2 dx. \qquad (2.3)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_\Omega (a_i - x_i)^2 \Delta f^{2q+2} \\ =&2(q+1)(2q+1) \int_\Omega (a_i - x_i)^2 f^{2q} |\nabla f|^2 dx - 2(q+1) \int_\Omega (a_i - x_i)^2 f^{2q+2} (\lambda_1 - V) dx, \qquad (2.4)\end{aligned}$$

and

$$\int_\Omega (a_i - x_i)^2 \Delta f^{2q+2} dx = \int_\Omega f^{2q+2} \Delta (a_i - x_i)^2 dx = 2A_{2q}. \qquad (2.5)$$

Thus it follows that

$$\int_\Omega (a_i - x_i)^2|\nabla f|^2 dx = \frac{A_{2q}}{(q+1)(2q+1)} + \frac{\lambda_1 \int_\Omega (a_i - x_i)^2 f^{2q+2}dx}{2q+1} - \frac{\int_\Omega V(a_i - x_i)^2 f^{2q+2}dx}{2q+1}. \tag{2.6}$$

Substitution (2.6) to (2.5), we have

$$\int_\Omega f^2|\nabla u|^2 dx = \frac{q+1}{2q+1}A_{2q} + \frac{\lambda_1 q^2}{2q+1}\int_\Omega (a_i - x_i)^2 f^{2q+2}dx - \frac{q^2}{2q+1}\int_\Omega V(a_i - x_i)^2 f^{2q+2}dx.$$

By Lemma 2.2, we obtain

$$I_{2q}\int_\Omega (a_i - x_i)^2 f^{2q+2}dx \le \frac{q+1}{\lambda_1}A_{2q} - \frac{q^2}{\lambda_1}\int_\Omega V(a_i - x_i)^2 f^{2q+2}dx.$$

Corollary 2.2. *If $p, q \ge 0$, then*

$$\frac{2A_pA_qI_{p+q}}{p+q+1}A_{p+q} \ge \frac{A_q^2 I_{2p}}{2p+1}A_{2p} + \frac{A_p^2 I_{2q}}{2q+1}A_{2q};$$

in particular, for $q = 0$

$$X_p \ge \frac{I_{2p}}{(2p+1)Y}.$$

Proof. Set $u = A_q f^p - A_p f^q$ in Lemma 2.2. Then

$$|\nabla u|^2 = p^2A_q^2 f^{2p-2}|\nabla f|^2 + q^2A_p^2 f^{2q-2}|\nabla f|^2 - 2pqA_pA_q f^{p+q-2}|\nabla f|^2.$$

By Lemma 2.1 we have

$$\begin{aligned}\int_\Omega f^2|\nabla u|^2 dx &= p^2A_q^2\int_\Omega f^{2p}|\nabla f|^2dx + q^2A_p^2\int_\Omega f^{2q}|\nabla f|^2dx - 2pqA_pA_q\int_\Omega f^{p+q}|\nabla f|^2 \\ &= \frac{\lambda_1 p^2A_q^2}{2p+1}A_{2p} + \frac{\lambda_1 q^2A_p^2}{2q+1}A_{2q}\frac{2pqA_pA_q}{p+q+1}A_{p+q} - B(p,q),\end{aligned}$$

where

$$\begin{aligned}B(p,q) &= \int_\Omega V\left(\frac{p^2A_q^2}{2p+1}f^{2p+2} + \frac{q^2A_p^2}{2q+1}f^{2q+1} - \frac{2pqA_pA_q}{p+q+1}f^{p+q+2}\right)dx \\ &\ge \int_\Omega 2VpqA_pA_q f^{p+q+2}\left(\frac{1}{\sqrt{(2p+1)(2q+1)}} - \frac{1}{p+q+1}\right)dx \\ &\ge 0.\end{aligned}$$

The inequalities are from inequalities of average. From Lemma 2.2, it follows that

$$\lambda_1 Y\int_\Omega (A_qf^p - A_pf^q)f^2dx \le \frac{\lambda_1 p^2A_q^2}{2p+1}A_{2p} + \frac{\lambda_1 q^2A_p^2}{2q+1}A_{2q} - \frac{2pqA_pA_q}{p+q+1}A_{p+q}.$$

On thte other hand,

$$Y\int_\Omega (A_qf^p - A_pf^q)^2f^2dx = YA_q^2A_{2p} + YA_p^2A_{2q} - 2YA_pA_qA_{p+q}.$$

Hence

$$\frac{2A_pA_qI_{p+q}}{p+q+1}A_{p+q} \ge \frac{A_q^2 I_{2p}}{2p+1}A_{2p} + \frac{A_p^2 I_{2q}}{2q+1}A_{2q}.$$

§3.The Above Bound of $\lambda_2 - \lambda_1$

Theorem 3.1. *If λ_1 and λ_2 are the 1st and 2nd eigenvalues of the equation* (1.1) *with the condition* 2, *then*

$$\lambda_2 - \lambda_1 \leq \frac{(\sqrt{(n+1)(n+9)}+3-n)\pi^2}{2D^2} + \frac{\sqrt{(n+1)(n+9)}+3-n}{2n}M,$$

where D is the diameter of the largest inscribed ball in Ω and $M = \sup\limits_{\bar{\Omega}} V$.

Proof. By Schwartz' inequality, we have

$$\int_\Omega f^{2p}(\nabla f \cdot \nabla X_i)^2 dx \int_\Omega (a_i - x_i)^2 f^2 dx \geq \left[\int_\Omega (a_i - x_i) f^{p+1} \nabla f \cdot \nabla x_i dx\right]^2$$

$$= \left[-\frac{1}{2(p+2)}\int_\Omega \nabla f^{p+2} \cdot \nabla (a_i - x_i)^2 dx\right]^2 = \frac{A_p^2}{(p+2)^2}.$$

Thus

$$\sum_{i=1}^n \int_\Omega f^{2p}(\nabla f \cdot \nabla x_i)^2 dx \int_\Omega (a_i - x_i)^2 f^2 dx \geq \frac{nA_p^2}{(p+2)^2}.$$

From Corollary 2.1 and Lemma 2.1, it follows that

$$\frac{1}{\lambda_1 Y}\int_\Omega f^{2p}|\nabla f|^2 \geq \frac{nA_p^2}{(p+2)^2}$$

and

$$\frac{1}{\lambda_1 Y}\left[\frac{\lambda_1 A_{2p}}{2p+1} - \frac{\int_\Omega V f^{2p+2} dx}{2p+1}\right] \geq \frac{nA_p^2}{(p+2)^2}.$$

By Corollary 2.2, we have

$$1 \geq \frac{n(2p+1)YX_p}{(p+2)^2} \geq \frac{nI_{2p}}{(p+2)^2},$$

i. e.,

$$[(p+2)^2 + np^2]\lambda_1 \geq n(2p+1)(\lambda_2 - \lambda_1). \tag{3.1}$$

Now suppose that g and $\lambda_1(V \equiv 0)$ are the 1st eigenfunction and the 1st eigenvalue of the equation

$$-\Delta g = \lambda g$$

respectively, and satisfy the condition $g|_{\partial\Omega} = 0$. Then the formulas

$$\lambda_1 = \inf_{h|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla h|^2 dx + \int_\Omega V h^2 dx}{\int_\Omega h^2 dx}$$

$$= \frac{\int_\Omega |\nabla g|^2 dx + \int_\Omega V g^2 dx}{\int_\Omega g^2 dx} \leq \lambda_1(V \equiv 0) + M$$

hold. According to the Theorem in [2], $\lambda_1(V \equiv 0) \geq \frac{n\pi^2}{D^2}$. Hence

$$\lambda_1 \leq \frac{n\pi^2}{D^2} + M. \tag{3.2}$$

Setting

$$p = \frac{1}{2}\left(\sqrt{1 + \frac{8}{n+1}} - 1\right) \tag{3.3}$$

and substituting (3.2) and (3.3) to (3.1), we complete the proof of Theorem 3.1.

§4. The Case for $n=2, V\equiv 0$

For $n=2, V\equiv 0$ we can obtain more precise estimate than Theorem 1.

Theorem 4.2 *Let $\Omega\subset R^2$ be a bounded domain, and let λ_1 and λ_2 be the first and second eigenvalues of the Laplacian with zero boundary. Then*

$$\frac{\lambda_2}{\lambda_1}\le 2.618.$$

Proof. If Theorem 4.2 is not true i.e., $\frac{\lambda_2}{\lambda_1}>2.618$, then $Y=\frac{\lambda_2}{\lambda_1}-1>1.618$. By [5], we have $Y\le 2$. Set $p\cdot q\in[0,1],\quad 0\le q<p\le 1$. Then

$$I_{2p}>0I_{2q}>0,\quad I_{p+q}>0.$$

From Lemma 2.2, it follows that

$$\begin{aligned}\frac{2A_pA_qI_{p+q}}{p+q+1}A_{p+q}&\ge\frac{A_q^2I_{2p}}{2p+1}A_{2p}+\frac{A_p^2I_{2q}}{2q+1}A_{2q}\\&\ge\frac{2A_pA_q\sqrt{I_{2p}I_{2q}}\sqrt{A_{2p}A_{2q}}}{\sqrt{(2p+1)(2q+1)}},\end{aligned}$$

i.e.,

$$\sqrt{X_{pq}}\ge\frac{(p+q+1)\sqrt{I_{2p}I_{2q}}}{\sqrt{(2p+1)(2q+1)}I_{p+q}}. \tag{4.1}$$

For a given $q,\ 0\le q\le 1$, we can assume that

$$\int_\Omega f^{2q}\langle\nabla f\cdot\nabla x_i\rangle^2dx=\frac{1}{2}\int_\Omega f^{2q}|\nabla f|^2dx\quad\text{for } i=1,2.$$

If not, we can obtain the equality by rotation of axes.

By Schwartz' inequality it follows that

$$\begin{aligned}&\sum_{i=1}^2\int_\Omega f^{2q}\langle\nabla f\cdot\nabla x_i\rangle^2dx\int_\Omega[(a_i-x_i)f^{q+1}-\epsilon f^p\nabla f\cdot\nabla x_i]^2dx\\&\ge\sum_{i=1}^2\left[\int_\Omega(a_i-x_i)^{2q+1}\nabla f\cdot\nabla x_idx-\epsilon\int_\Omega f^{p+q}\langle\nabla f\cdot\nabla x_i\rangle^2dx\right]^2\\&=\sum_{i-1}^2\left[\frac{A_{2p}}{2(q+1)}-\epsilon\int_\Omega f^{p+q}\langle\nabla f\cdot\nabla x_i\rangle^2dx\right]^2,\end{aligned}\tag{4.2}$$

where

$$\epsilon=\frac{(p-q)I_{p+q}\sqrt{I_{2p}I_{2q}}}{(q+1)(p+q+2)[I_{p+q}^2-I_{2p}I_{2q}]}\sqrt{\frac{2p+1}{2q+1}}\sqrt{\frac{A_{2q}}{A_{2p}}}. \tag{4.3}$$

Since $g(\xi)=\xi^2$ is a convex function, $\frac{1}{n}\sum g(\xi_i)\ge g(\frac{\sum\xi_i}{n})$. Hence have

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega f^{2q}\langle\nabla f\cdot\nabla x_i\rangle^2dx\left\{\int_\Omega(a_i-x_i)^2f^{2q+2}dx\right.\\
&\left.+\epsilon^2\int_\Omega f^{2p}\langle\nabla f\cdot\nabla x_i\rangle^2dx-\int_\Omega 2\epsilon(a_i-x_i)f^{q+p+1}\langle\nabla f\cdot\nabla x_i\rangle dx\right\}\\
\geq&\sum_{i=1}^{2}\left[\frac{A_{2q}}{2(q+1)}-\epsilon\int_\Omega f^{p+q}\langle\nabla f\cdot\nabla x_i\rangle^2dx\right]^2\\
\geq&2\left\{\frac{A_{2q}}{2(q+1)}-\frac{\epsilon}{2}\int_\Omega f^{p+q}|\nabla f|^2dx\right\}^2\\
=&2\left\{\frac{A_{2\epsilon}}{2(q+1)}-\frac{\epsilon A_{p+q}}{2(p+q+1)}\right\}^2.
\end{aligned}\tag{4.4}$$

Using the equality

$$\int_\Omega f^{2p}\langle\nabla f\cdot\nabla x_i\rangle^2dx=\frac{1}{2}\int_\Omega f^{2p}|\nabla f|^2dx=\frac{A_{2p}}{2(2p+1)},\quad i=1,2\tag{4.5}$$

and (4.4), we have

$$\begin{aligned}
&\frac{A_{2q}}{2(2q+1)}\left[\sum_{i=1}^{2}\int_\Omega(a_i-x_i)^2f^{2q+2}dx+\frac{\epsilon^2A_{2p}}{2p+1}-\frac{4\epsilon A_{p+q}}{p+q+2}\right]\\
\geq&2\left\{\frac{A_{2q}}{2(q+1)}-\frac{\epsilon A_{p+q}}{2(p+q+1)}\right\}.
\end{aligned}\tag{4.6}$$

By Lemma 2.2, we have

$$\int_\Omega(a_i-x_i)^2f^{2q+2}dx\leq\frac{q+1}{I_{2q}}A_{2q},\quad i=1,2.$$

The inequality (4.6) can be rewritten as

$$\begin{aligned}
&\frac{q+1}{2(2q+1)I_{2q}}-\frac{1}{4(q+1)^2}\geq\frac{(p-q)\sqrt{X_{pq}}}{2(q+1)(2q+1)(p+q+1)(p+q+2)}\sqrt{\frac{A_{2p}}{A_{2q}}}\epsilon\\
&-\frac{(p+q+1)^2-(2p+1)(2q+1)X_{pq}}{4(2p+1)(2q+1)(p+q+1)^2}\cdot\frac{A_{2p}}{A_{2q}}\epsilon^2.
\end{aligned}\tag{4.7}$$

From (4.1), (4.3) and (4.7), it follows that

$$\frac{1+q}{I_{2q}}-\frac{2q+1}{2(q+1)^2}\geq\frac{(p-q)^2I_{2p}I_{2q}}{2(q+1)^2(p+q+2)^2(2q+1)(I_{p+q}^2-I_{2p}I_{2q})}.\tag{4.8}$$

It is easy to see that

$$I_{p+q}^2-I_{2p}I_{2q}=(p-q)^2Y(Y+1).\tag{4.9}$$

Substituting (4.9) to (4.8), we have

$$\begin{aligned}
0\geq&\frac{2q+1}{2(q+1)^2}-\frac{1+q}{I_{2q}}+\frac{I_{2p}I_{2q}}{2(q+1)^2(p+q+2)^2(2q+1)Y(Y+1)}\\
=&\frac{2q+1}{2(q+1)^2}-\frac{1+q}{(2q+1)Y-q^2}\\
&+\frac{(2p+1)(2q+1)Y^2-[2pq(p+q)+p^2+q^2]Y+p^2q^2}{2(q+1)^2(p+q+2)^2(2q+1)Y(Y+1)}.
\end{aligned}$$

Define a real function $F:[0,1]\times[0,1]\times[\frac{3}{2},2]\to R$ as follows:

$$\begin{aligned}F(p,q,\xi)=&\frac{2q+1}{2(q+1)^2}-\frac{1+q}{(2q+1)\xi-q^2}\\&+\frac{(2p+1)(2q+1)\xi^2-[2pq(p+q)+p^2+q^2]\xi+p^2q^2}{2(q+1)^2(p+q+2)^2(2q+1)\xi(\xi+1)}\\=&\frac{2q+1}{2(q+1)^2}-\frac{1+q}{(2q+1)\xi-q^2}+\frac{(2p+1)(2q+1)\xi}{2(q+1)^2(p+q+2)^2(2q+1)(\xi+1)}\\&-\frac{2pq(p+q)}{2(q+1)^2(p+q+2)^2(2q+1)(\xi+1)}\\&-\frac{(p^2+q^2)\xi+p^2q^2}{2(q+1)^2(p+q+2)^2(2q+1)(\xi+1)\xi}.\end{aligned}$$

For $0\le q<p\le 1$ and $3/2\le\xi\le 2$, we have

$$\frac{\partial}{\partial\xi}\left[-\frac{1+q}{(2q+1)\xi-q^2}\right]>0,$$

$$\frac{\partial}{\partial\xi}\left[\frac{(2p+1)(2q+1)\xi}{2(q+1)^2(p+q+2)^2(2q+1)(\xi+1)}\right]>0,$$

$$\frac{\partial}{\partial\xi}\left[\frac{-2pq(p+q)}{2(q+1)^2(p+q+2)^2(2q+1)(\xi+1)}\right]>0,$$

and

$$\begin{aligned}&\frac{\partial}{\partial\xi}\left[-\frac{(P^2+q^2)\xi-p^2q^2}{\xi(\xi+1)}\right]\\=&\frac{1}{\xi^2(\xi+1)^2}[\xi^2(p^2+q^2)-p^2q^2(2\xi+1)]\\=&\frac{1}{\xi^2(\xi+1)^2}\left[p^2(\xi^2-q^2\xi-\frac{q^2}{2})+q^2(\xi-p^2\xi-\frac{p^2}{2})\right]\\=&\frac{1}{\xi^2(\xi+1)^2}\left\{\left[p^2(\xi-\frac{q^2}{2})^2-\frac{p^2}{2}-\frac{p^4}{4}\right]+\left[q^2(\xi-\frac{p^2}{2})^2-\frac{q^2}{2}-\frac{q^4}{4}\right]\right\}\ge 0.\end{aligned}$$

Thus $F(p,q,\xi)$ is a monotone increasing function of ξ, and if $Y>1.618$, we should have

$$0\ge F(p,q,\xi)|_{\xi=Y}>F(p,q,\xi)|_{\xi=1.618}. \tag{4.10}$$

(4.10) is valid for $0\le q<p\le 1$. Now setting $q=0.19,\ p=0.505625$, we obtain

$$F(p,q,\xi)|_{\xi=1.618}>0.000296. \tag{4.11}$$

(4.11) contradicts (4.10). Hence $Y\le 1.618$, i.e., $\frac{\lambda_2}{\lambda_1}\le 2.618$.

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