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THE GROWTH THEOREM FOR BIHOLOMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES***

GONG SHENG*　　WANG SHIKUN**　　YU QIHUANG**

Abstract

The Growth Theorem for normalized biholomorphic starlike and convex mappings on Reinhardt domains and classical domains are established.

§1. The Growth Theorem for Starlike Mappings on Reinhardt Domains

There are fruitful results on the geometrical function theory of one complex variable. But there exist a lot of counter-examples to show that the corresponding reseults in several complex variables are not true. In 1933, H.Cartan[1] already pointed out that the growth theorem of biholomorphic mappings on the unit ball of C^n is not true. In 1988, Carl H.Fitzgerald, Sheng Gong and Roger W.Barnard[2] gave the first affirmative result about the growth theorem in several complex variables. They proved the following result.

Let $B^n \equiv \{Z = (z_1, z_2, \cdots, z_n) \in C^n | \; |Z| < 1\}$ be the unit ball in C^n, where $|Z| = (\sum\limits_{i=1} |z_i|^2)^{1/2}$, and

$$f(Z) = (f_1(Z), f_2(Z), \cdots, f_n(Z)) = Z + (ZA^{(1)}Z^T, \cdots, ZA^{(n)}Z^T) + \cdots \tag{1.1}$$

be the normalized biholomorphic starlike mappings on B^n, where $A^{(j)} = (a_{kl}^{(j)})_{1\le l,k\le n}$, $j = 1, 2, \cdots, n$, are constant matrices, starlike mapping means the image of B^n by $f(Z)$ is starlike with respect to origin, and normalized mapping means $f(0) = 0$, $J_f(0) = I$, where J_f is the Jacobian of f, I is the identity matrix. Then

$$\frac{|Z|}{(1+|Z|)^2} \le |f(Z)| \le \frac{|Z|}{(1-|Z|)^2} \tag{1.2}$$

holds for every $z \in B^n$. The estimation is precise, but the extremal mapping is not unique. Obviously, $f(B^n) \supset \frac{1}{4}B^n$, that is, the Koebe constant of this family of mappings on B^n is $\frac{1}{4}$.

In 1989, Qihuang Yu, Shikun Wang and Sheng Gong[3] extended the Growth Theorem of normalized biholomorphic starlike mappings on ball to the Reinhardt domain B_p where $B_p \equiv \{Z \in C^n | \; \|Z\|_p = (\sum\limits_{i=1}^{n} |z_i|^p)^{\frac{1}{p}} < 1, p > 1\}$. We proved the following result.

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*Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, China.
**Institute of Applied Mathematics, Academia Sinica, Beijing, 100080, China.

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Let (1.1) be the normalized biholomorphic starlike mappings on B_p. Then

$$\frac{|Z|}{(1+\|Z\|_p)^2} \le |f(Z)| \le \frac{|Z|}{(1-\|Z\|_p)^2} \tag{1.3}$$

holds for every $Z \in B_p$. The estimation is precise, but the extremal mapping is not unique. As a consequence of (1.3), $f(B^p) \supset \frac{1}{4K}B^p$ where $K = 1$ if $p \ge 2$, $K = n^{\frac{2-p}{2p}}$ if $2 \ge p > 1$. That is, the Koebe constant of this family of mappings on B_p is $\frac{1}{4K}$.

On the other hand, using Loewner Chain's method, John A. Pfeltzgraff[4] proved that:

Let (1.1) be the normalized biholomorphic starlike mappings on B_p. Then

$$\frac{\|Z\|_p}{(1+\|Z\|_p)^2} \le \|f(Z)\|_p \le \frac{\|Z\|_p}{(1-\|Z\|_p)^2} \tag{1.4}$$

holds for every $Z \in B_p$. The estimation is precise, but the extremal mapping is not unique. As a consequence of (1.4), $f(B^p) \supset \frac{1}{4}B^p$. That is, the Koebe constant of this family of mappings on B_p is $\frac{1}{4}$ in the sense of p-norm.

Now we would like to point out that (1.3) and (1.4) are equivalent. That means they can imply each other.

Let $\rho(W) = \mathrm{tr} W\overline{W}^T$ where $W = f(Z)$, $Z \in B_p$. Then it is easy to verify

$$d\rho(W) = \frac{1}{2\rho}\sum_{i=1}^{n}(d\omega_i\overline{\omega_i} + \overline{d\omega_i}\omega_i).$$

and

$$d\|W\|_p = \frac{1}{2}\|W\|_p\sum_{i=1}^{n}\left(\frac{d\omega_i}{\omega_i} + \frac{\overline{d\omega_i}}{\overline{\omega_i}}\right),$$

where $\|W\|_p = (\sum\limits_{i=1}^{n}|\omega_i|^p)^{\frac{1}{p}}$, $p > 1$. We have $< d\|W\|_p, d\rho >= \frac{\|W\|_p}{\rho}$, where $<,>$ is the inner product of the cotangent space of C^n at the point W. On the other hand, $< d\|W\|_p, d\rho >= \frac{d\|W\|_p}{d\rho}$. We obtain

$$\frac{d\|W\|_p}{\|W\|_p} = \frac{d\rho}{\rho}. \tag{1.5}$$

Then we integrate (1.5) on both sides from $W = f(\epsilon Z)$ to $W = f(Z)$ where ϵ is a small positive number. We get

$$\frac{\|f(Z)\|_p}{\|f(\epsilon Z)\|_p} = \frac{|f(Z)|}{|f(\epsilon Z)|}. \tag{1.6}$$

Now we can prove that (1.3) imply (1.4). From (1.6) and (1.3), the following inequality holds

$$\frac{\|f(\epsilon Z)\|_p(1-\epsilon\|Z\|_p)^2}{(1+\|Z\|_p)^2\epsilon} \le \|f(Z)\|_p \le \frac{\|f(\epsilon Z)\|_p(1+\epsilon\|Z\|_p)^2}{(1-\|Z\|_p)^2\epsilon}. \tag{1.7}$$

Then letting $\epsilon \to 0$, we get (1.4) since $\lim\limits_{\epsilon\to 0}\frac{\|f(\epsilon Z)\|_p}{\epsilon} = \|Z\|_p$ due to the normalized condition of f.

Similarly, using (1.6) and (1.4), we can prove (1.3).

Note that the methods used to prove (1.3) in [3] and (1.4) in [4] are essentially different, but (1.3) and (1.4) can imply each other. In other words, they are two different methods to prove both (1.3) and (1.4) from [3] and [4].

§2. The Growth Theorems for Starlike Mappings on Classical Domains

As a part of his doctor thesis, in 1989, Taishun Liu[5] extended the result of [2] in another direction. He got the growth theorems of normalized biholomorphic starlike mapping on classical domains. The four types of classical domains are defined as follows[6]:

$$\mathcal{R}_I(m,n) = \{Z \in C^{m\times n} | I - Z\overline{Z}^T > 0\},$$

$$\mathcal{R}_{II}(n) = \{Z \in C^{n\times n} | I - Z\overline{Z}^T > 0, Z = Z^T\},$$

$$\mathcal{R}_{III}(n) = \{Z \in C^{n\times n} | I - Z\overline{Z}^T > 0, Z = Z^T\},$$

$$\mathcal{R}_{IV}(n) = \{Z \in C^{n} | 1 - 2Z\overline{Z}^T + |ZZ^T|^2 > 0, |Z\overline{Z}^T| < 1\}.$$

He proved that: Let $f(Z)$ be the normalized biholomorphic starlike mappings on classical domains $\mathcal{R}_K$, $(K = I, II, III, IV)$ which map $\mathcal{R}_K$ to $C^{\dim \mathcal{R}_K}$. Then

$$\frac{\|Z\|_K}{(1+\|Z\|_K)^2} \le \|f(Z)\|_K \le \frac{\|Z\|_K}{(1-\|Z\|_K)^2} \tag{2.1}$$

holds for every $Z \in \mathcal{R}_K$, where $\| \ \|_K$ is the norm of $\mathcal{R}_K$ (cf. [7]), that is, $\|Z\|_K$ is a positive square root of the largest eigenvalue $\lambda(Z)$ of $Z\overline{Z}^T$ for $K = I, II, III$ and $\|Z\|_{IV} = (|Z|^2 + (|Z|^4 - |ZZ^T|^2)^{\frac{1}{2}}$. As the corollary of (2.1), he got $f(\mathcal{R}_K) \supset \frac{1}{4}\mathcal{R}_K$, $K = I, II, III, IV$.

All these estimation are precise, and the extremal mapping is not unique.

In this paper, we establish another growth theorem of normalized biholomorphic starlike mapping on classical domains.

Theorem. *Let $W = f(Z)$ be the normalized biholomorphic starlike mappings on classical domains $\mathcal{R}_K$, $(K = I, II, III, IV)$, which map $\mathcal{R}_K$ to $C^{\dim \mathcal{R}_K}$. Then*

$$\frac{|Z|}{(1+\|Z\|_K)^2} \le |f(Z)|_K \le \frac{|Z|}{(1-\|Z\|_K)^2} \tag{2.2}$$

holds for every $Z \in \mathcal{R}_K$.

The estimations are precise, but the extremal mapping is not unique.

Finally, we prove that (2.1) and (2.2) are equivalent, that is, (2.1) and (2.2) can imply each other. In other words, we have two different methods, one is Taishun Liu's in [5] and the other is that we use in this paper, they can prove both (2.1) and (2.2). Actually, in the case of the domain is $\mathcal{R}_{IV}$, Taishun Liu already proved that (2.1) implies (2.2), and it is easy to prove that (2.2) implies (2.1) in the same way as he used. So, in this paper, we only need to prove the theorem in the cases that the domains are $\mathcal{R}_I$, $\mathcal{R}_{II}$ and $\mathcal{R}_{III}$, and then prove that (2.1) and (2.2) are equivalent in these domains.

§3. Some Lemmas

To prove the theorem, we need some lemmas.

Lemma 3.1. *Let $\Phi(t)$ be the holomorphic mappings which map $\mathcal{R}_K$ into $\mathcal{R}_K$ and $\Phi(0) = 0$. Then*

$$\|\Phi(Z)\|_K \le \|Z\|_K, \quad K = I, II, III, IV,$$

holds for every $Z \in \mathcal{R}_K$.

This lemma can be deduced from Schwartz Lemma of the normal linear space by Lawrence A. Harris[8]. Using the Schwartz Lemma, we have

Lemma 3.2. *Let $W = f(Z)$ be the normalized biholomorphic starlike mappings on $\mathcal{R}_K$ which map $\mathcal{R}_K$ to $C^{\dim \mathcal{R}_K}$. Then*

$$\|f^{-1}(rW)\|_K \leq \|f^{-1}(W)\|_K$$

holds for every $Z \in \mathcal{R}_K$ and every $r \in [0,1]$.

Proof. Let $f(\mathcal{R}_K)$ denote the image of $\mathcal{R}_K$ under f. For a fixed $Z \in \mathcal{R}_K$, the subset of $\mathcal{R}_K$ is defined as

$$\mathcal{E}_a = \{Y \in \mathcal{R}_K | \ \|Y\|_K < \|Z\|_K = a\}.$$

Obviously, $\mathcal{E}_a$ is an open set. Because a holomorphic mapping is an open mapping, $f(\mathcal{E}_a)$ is also an open set and $\{f(\overline{\mathcal{E}_a})\} = \overline{\{f(\mathcal{E}_a)\}}$. By the starlike hypothesis for the mapping f, where $W = f(Z)$, we have $rW \in f(\mathcal{R}_K)$. That means $f^{-1}(rW) \in \mathcal{R}_K$ for all $0 \leq r \leq 1$. We claim that

$$rW \in f(\overline{\mathcal{E}_a}). \tag{3.1}$$

If it is true, then $f^{-1}(rW) \in \overline{\mathcal{E}_a}$, which implies the lemma holds. To show that (3.1) is true we suppose there is such an $r_0 < 1$ that $r_0 W \notin \{f(\overline{\mathcal{E}_a})\}$, i.e., $\|f^{-1}(r_0 W)\|_K > a$. We define a new mapping $K(Z)$ from $\mathcal{R}_K$ to $\mathcal{R}_K$ by $K(Z) = f^{-1}(r_0 f(Z))$. Since f is a biholomorphic starlike mapping, $K(Z)$ is well defined and is holomorphic with $K(0) = 0$. By Lemma 3.1, we have

$$\|Z\|_K \geq \|K(Z)\|_K = \|f^{-1}(r_0 W)\|_K > a.$$

This contradicts $\|f^{-1}(W)\|_K = \|Z\|_K = a$. Thus (3.1) is true. The lemma is proved.

Let $W = f(Z)$ be the biholomorphic starlike mapping on $\mathcal{R}_K$. Then $g(W) = \|f^{-1}(W)\|_K$ is a continuous function of each entry of W. For a fixed $W^0 \in f(\mathcal{R}_K)$, let $\sigma(W^0) \equiv \{W = rW^0, 0 \leq r \leq 1\}$, this is a closed set. Thus $g(W)$ is absolute continuous on $W \in \sigma(W^0)$, and then $g(W)$ is differentiable with respect to W almost everywhere. We have

Lemma 3.3. *Let $W = f(Z)$ be the biholomorphic starlike mappings on $\mathcal{R}_K$ which map $\mathcal{R}_K$ to $C^{\dim \mathcal{R}_K}$. Then $\|f^{-1}(W)\|_K$ is differentiable with respect to W almost everywhere on the $\sigma(W^0) \equiv \{W = rW^0, 0 \leq r \leq 1\}$ where $W^0 = f(Z^0)$ is a fixed point in $f(\mathcal{R}_K)$ and $K = I, II, III, IV$.*

§4. The Proof of the Growth Theorem

For $\mathcal{R}_{IV}$, (2.2) was proved by Taishun Liu[5]. Here we only give the proof of the theorem in the case that the domain is $\mathcal{R}_I$, and the proofs of the theorem are similar in the cases that the domains are $\mathcal{R}_{II}$ and $\mathcal{R}_{III}$.

Let $W \in f(\mathcal{R}_I(m,n))$ and denote $\rho^2(W\overline{W}^T)$, $\lambda(Z) = \|Z\|_I^2$. When $\lambda \circ f^{-1}$ is constrained on the segment $\sigma(W^0)$ joining the origin and $W^0 = f(Z^0)$, for a fixed $Z^0 \in \mathcal{R}_I(m,n)$, the directional derivative of $\lambda \circ f^{-1}$ along the direction $d\rho$, that is $\frac{d(\lambda \circ f^{-1})}{d\rho}$, exists almost everywhere on the segment $\sigma(W^0)$ by Lemma 3.3. By the definition of $\lambda(Z)$, $\lambda(Z)$ satisfies the following equation

$$\det(\lambda(Z)I - Z\overline{Z}^T) = 0.$$

That is,

$$\sum_{k=0}^{m}(-1)^k \mathrm{tr}^{(k)} Z\overline{Z}^T \lambda^{m-k} = 0, \tag{4.1}$$

where $\mathrm{tr}^{(k)} Z\overline{Z}^T$ is the sum of all the $k \times k$ principal minors of $Z\overline{Z}^T$. Differentiating (4.1) on both sides, we have

$$\sum_{k=0}^{m-1}(-1)^k (m-k)\mathrm{tr}^{(k)} Z\overline{Z}^T \lambda^{m-k-1} d\lambda + \sum_{k=1}^{m}(-1)^k d(\mathrm{tr}^{(k)} Z\overline{Z}^T)\lambda^{m-k} = 0.$$

Thus $d\lambda = \frac{P}{Q}$ where

$$P = \sum_{k=1}^{m}(-1)^{k+1} d(\mathrm{tr}^{(k)} Z\overline{Z}^T)\lambda^{m-k}, \tag{4.2}$$

$$Q = \sum_{k=0}^{m-1}(-1)^k (m-k)\mathrm{tr}^{(k)} Z\overline{Z}^T \lambda^{m-k-1}. \tag{4.3}$$

Obviously, P is equal to

$$2\mathrm{Re}\sum_{k=1}^{m}(-1)^{k+1}\lambda^{m-k}\sum_{i,j}\frac{\partial}{\partial \omega_{ij}}(\mathrm{tr}^{(k)} Z\overline{Z}^T) d\omega_{ij}. \tag{4.4}$$

Since $d\rho^2 = 2\mathrm{Re}(\sum\limits_{i,j}\omega_{ij}\overline{d\omega_{ij}})$, we obtain

$$< d\lambda \circ f^{-1}, d\rho^2 > |_{W=f(Z)} = \frac{P_1}{Q} \tag{4.5}$$

where $<,> |_W$ is the inner product of contangent space of $C^{m\times n}$ at the point W, and

$$P_1 = 4\mathrm{Re}\sum_{k=1}^{m}(-1)^{k+1}\lambda^{m-k}\sum_{i,j}\frac{\partial}{\partial \omega_{ij}}(\mathrm{tr}^{(k)} Z\overline{Z}^T)\omega_{ij}. \tag{4.6}$$

It is easy to evaluate, for $1, 2, \cdots, m$,

$$\begin{aligned}
&\frac{\partial}{\partial \omega_{ij}}(\mathrm{tr}^{(k)} Z\overline{Z}^T) \\
=&\frac{\partial}{\partial \omega_{ij}}\sum_{l_1<\cdots<l_k}\begin{vmatrix}\sum\limits_{s=1}^{k} z_{l_1 s}\bar{z}_{l_1 s}, \cdots, \sum\limits_{s=1}^{k} z_{l_1 s}\bar{z}_{l_k s} \\ \vdots \quad \cdots, \quad \vdots \\ \sum\limits_{s=1}^{k} z_{l_k s}\bar{z}_{l_1 s}, \cdots, \sum\limits_{s=1}^{k} z_{l_k s}\bar{z}_{l_k s}\end{vmatrix} \\
=&\frac{\partial}{\partial \omega_{ij}}\sum_{l_1<\cdots<l_k}\sum_{p_1,\cdots,p_k}\delta^{1,\cdots,k}_{p_1,\cdots,p_k}\left(\sum_s z_{l_1 s}\bar{z}_{l_{p_1} s}\right)\left(\sum_s z_{l_2 s}\bar{z}_{l_{p_2} s}\right)\cdots\left(\sum_s z_{l_k s}\bar{z}_{l_{p_k} s}\right) \\
=&\sum_{l_1<\cdots<l_k}\sum_{p_1,\cdots,p_k}\delta^{1,\cdots,k}_{p_1,\cdots,p_k}\sum_{q=1}^{k}\left(\prod_{j=1,j\neq q}^{k}\left(\sum_s z_{l_j s}\bar{z}_{l_{p_j} s}\right)\left(\sum_s \frac{\partial z_{l_q s}}{\partial \omega_{ij}}\bar{z}_{l_{p_q} s}\right)\right),
\end{aligned} \tag{4.7}$$

where $(p_1, p_2, \cdots, p_k)$ is a permutation of $(1, 2, \cdots, k)$ and $\delta^{1,2,\cdots,k}_{p_1,\cdots,p_k} = 1$ if this permutation is even, $= -1$ if this permutation is odd, and the second summation in the right hand side of (4.7) is over all the permutations of $(1,2,\cdots,k)$, and $l_1,l_2,\cdots,l_k$ are K integers of $1,2,\cdots,m$, $1\le k\le m$.

Substituting (4.7) to (4.6), we have

$$P_1=4\mathrm{Re}\{\sum_{k=1}^{m}(-1)^{k+1}\lambda^{m-k}\sum_{i,j}\omega_{ij}$$

$$\cdot\sum_{l_1<\cdots<l_k}\sum_{p_1,\cdots,p_k}\delta^{1,\cdots,k}_{p_1,\cdots,p_k}\sum_{g=1}^{k}(\prod_{i=1,j\ne q}^{k}(\sum_{s}z_{l_js}\bar z_{l_{p_j}s})(\sum_{s}\frac{\partial z_{l_qs}}{\partial\omega_{ij}}\bar z_{l_{p_q}s}))\},\tag{4.8}$$

for a $Z\in\mathcal{R}_I(m,n)$. Let $Y=\frac{tZ}{\|Z\|_I}$ where t is a complex number with $|t|<1$. Then

$$I-Y\overline{Y}^T=I-\frac{(t)^2}{\lambda(Z)}\cdot Z\overline{Z}^T>0.$$

That means $Y=\frac{tZ}{\|Z\|_I}\in\mathcal{R}_I(m,n)$, and obviously $\|Y\|_I=|t|$, $\lambda(Y)=|t|^2$. Let $W^0=f(Y)$. By Lemma 3.2,

$$\lambda(f^{-1}(W_0))-\lambda(f^{-1}((1-r)W_0)\ge 0$$

for every $1\ge r\ge 0$, and

$$\frac{\lambda(f^{-1}(W_0))-\lambda(f^{-1}((1-r)W_0))}{r}\ge 0$$

for every $1\ge r>0$. By the definition of directional derivative

$$\left.\frac{d(\lambda\circ f^{-1})}{d\rho}\right|_{W=W_0}\ge 0,\tag{4.9}$$

if the directional derivative at $W=W^0$ exists. But by Lemma 3.3, it exists almost everywhere on the segment $\sigma(W^0)$. On the other hand

$$\langle d(\lambda\circ f^{-1}),d\rho^2\rangle|_{W=W_0}=2\rho\left.\frac{d(\lambda\circ f^{-1})}{d\rho}\right|_{W=W_0},$$

hence (4.9) is equivalent to

$$\langle d(\lambda\circ f^{-1}),d\rho^2\rangle|_{W=W_0}\ge 0.$$

By (4.3),(4.5),(4.8) and (4.9), we have

$$\frac{P_2}{Q_2}\ge 0,$$

where

$$P_2=4\mathrm{Re}\left\{\sum_{k=1}^{m}\frac{(-1)^{k+1}|t|^{2(m-k)}|t|^{2k}}{t\|Z\|_I^{2k-1}}\sum_{i,j}\omega_{ij}\right|_{W=f(\frac{tZ}{\|Z\|_I})}\sum_{l_1<\cdots<l_k}$$

$$\left.\sum_{p_1,\cdots,p_k}\sum_{q=1}^{k}\delta^{1,\cdots,k}_{p_1,\cdots,p_k}\left[\prod_{j=1,j\ne q}^{k}(\sum_{s}z_{l_js}\bar z_{l_{p_j}s})(\sum_{s}\left.\frac{\partial z_{l_qs}}{\partial\omega_{ij}}\right|_{W=f(\frac{tZ}{\|Z\|_I})}\bar z_{l_{p_q}s})\right]\right\}$$

and

$$Q_2=\sum_{k=0}^{m-1}\frac{(-1)^k(m-k)|t|^{2(m-1)}}{\lambda^{(k)}(Z)}\mathrm{tr}^{(k)}Z\overline{Z}^T.$$

It is easy to observe that we can rewrite P_2/Q_2 as $4|t|^2\mathrm{Re}\frac{P_3}{Q_3}$, where

$$P_3(t) = \|Z\|_I \sum_{k=1}^{m} \frac{(-1)^{k+1}\lambda(Z)^{m-k}}{t} \sum_{i,j} \omega_{ij}\Big|_{W=f(\frac{tZ}{\|Z\|_I})} \cdot \sum_{l_1<\cdots<l_k}$$

$$\sum_{p_1,\cdots,p_k} \delta^{1,\cdots,k}_{p_1,\cdots,p_k} \sum_{q=1}^{k} \left\{ \prod_{j=1,j\neq q}^{k} (\sum_s z_{l_j s}\bar{z}_{l_{p_j} s})(\sum_s \frac{\partial}{\partial \omega_{ij}} z_{l_q s}\Big|_{W=f(\frac{tZ}{\|Z\|_I})} \bar{z}_{l_{p_q} s}) \right\} \tag{4.10}$$

and

$$Q_3 = \sum_{k=0}^{m-1} (-1)^k (m-k)\mathrm{tr}^{(k)} Z\overline{Z}^T \lambda(Z)^{m-k}. \tag{4.11}$$

By (4.9), $\mathrm{Re}\frac{P_3}{Q_3} \geq 0$. Obviously, $C(t) = \frac{P_3(t)}{Q_3}$ is an analytic function of t when $0 < |t| < 1$ and $\mathrm{Re}C(t) \geq 0$. We have

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t}\omega_{ij}\left(\frac{tZ}{\|Z\|_I}\right) \\ =& \frac{\partial \omega_{ij}}{\partial t}\Big|_{t=0} = \sum_{k,l} \frac{\partial \omega_{ij}}{\partial z_{kl}} \frac{\partial z_{kl}}{\partial t} \\ =& \sum_{k,l} \delta^{ij}_{kl} \frac{z_{kl}}{\|Z\|_I} = \frac{z_{ij}}{\|Z\|_I} \end{aligned} \tag{4.12}$$

where δ^{ij}_{kl} are Kronecker symbols and

$$\lim_{t\to 0} \frac{\partial z_{kl}}{\omega_{ij}}\Big|_{W=f(\frac{tZ}{\|Z\|_I})} = \delta^{ij}_{kl}. \tag{4.13}$$

Using (4.12) and (4.13), we get

$$\begin{aligned} &\lim_{t\to 0} P_3(t) \\ =& \|Z\|_I \sum_{k=1}^{m} (-1)^{k+1}\lambda(Z)^{m-k} \sum_{ij} \frac{z_{ij}}{\|Z\|_I} \sum_{l_1<\cdots<l_k} \sum_{p_1,\cdots,p_k} \delta^{1,\cdots,k}_{p_1,\cdots,p_k} \cdot \\ & \sum_{q=1}^{k} \left\{ \prod_{j=1,j\neq q}^{k} (\sum_s z_{l_j s}\bar{z}_{l_{p_j} s})(\delta^{ij}_{l_q s}\bar{z}_{l_{p_q} s}) \right\} \\ =& \sum_{k=1}^{m} (-1)^{k+1}\lambda(Z)^{m-k} \sum_{l_1<\cdots<l_k} \sum_{p_1,\cdots,p_k} \delta^{1,\cdots,k}_{p_1,\cdots,p_k} \cdot \\ & \sum_{q=1}^{k} \left\{ \prod_{j=1,j\neq q}^{k} (\sum_s z_{l_j s}\bar{z}_{l_{p_j} s})(\sum_s z_{l_q s}\bar{z}_{l_{p_q} s}) \right\} \\ =& \sum_{k=1}^{m} (-1)^{k+1}\lambda(Z)^{m-k} \sum_{l_1<\cdots<l_k} \sum_{p_1,\cdots,p_k} \delta^{1,\cdots,k}_{p_1,\cdots,p_k} k \left\{ \prod_{j=1}^{k} (\sum_s z_{l_j s}\bar{z}_{l_{p_j} s}) \right\} \\ =& \sum_{k=1}^{m} (-1)^{k+1}\lambda(Z)^{m-k} k\mathrm{tr}^{(m-k)} Z\overline{Z}^T. \end{aligned}$$

Thus

$$\lim_{t\to 0}\frac{P_3}{Q_3}=\frac{\sum\limits_{k=1}^{m}(-1)^{k+1}\lambda(Z)^{m-k}\mathrm{tr}^{(k)}Z\overline{Z}^T}{\sum\limits_{k=0}^{m-1}(m-k)(-1)^k\lambda(Z)^{m-k}\mathrm{tr}^{(k)}Z\overline{Z}^T}.$$

This is equal to 1 by (4.1).

The analytic function $C(t)$ of one complex variable t where t is in the unit disc of C satisfies: $C(0)=1$ and $\mathrm{Re}C(t)\geq 0$. By a theorem of analytic function with real part in the unit disc, we have

$$\frac{1-|t|}{1+|t|}\geq \mathrm{Re}C(t)\geq\frac{1+|t|}{1-|t|}. \tag{4.14}$$

Taking $t=\|Z\|_I$ in (4.14), we get

$$\frac{1-\|Z\|_I}{1+\|Z\|_I}\geq \mathrm{Re}C(\|Z\|_I)\geq\frac{1+\|Z\|_I}{1-\|Z\|_I}.$$

But it is easy to verify

$$C(\|Z\|_I)=\frac{1}{4\lambda(Z)}\langle d\lambda\circ f^{-1},d\rho^2\rangle|_{W=f(Z)}=\frac{\rho}{2\lambda(Z)}\cdot\frac{d\lambda\circ f^{-1}}{d\rho}\bigg|_{W=f(Z)}.$$

Thus the inequality

$$\frac{2\lambda(Z)(1-\sqrt{\lambda(Z)})}{(1+\sqrt{\lambda(Z)})}\leq\langle d(\lambda\circ f^{-1}),d\rho^2\rangle|_{W=f(Z)}\leq\frac{2\lambda(Z)(1+\sqrt{\lambda(Z)})}{(1-\sqrt{\lambda(Z)})} \tag{4.15}$$

holds almost everywhere on the segment $\sigma(W)$.

We take a sufficient small positive number $s\leq 1$ such that the small hyperball

$$B(s)=\{W\in C^{m\times n}|\ \mathrm{tr}W\overline{W}^T=\mathrm{tr}(f(sZ)\overline{f(sZ)}^T)\}$$

intersects $\sigma(Z)$. The intersection point is

$$Q=\sqrt{\frac{\mathrm{tr}(f(sZ)\overline{f(sZ)}^T}{\mathrm{tr}W\overline{W}^T}}\cdot W.$$

Obviously $\rho(Q)=\rho(f(sZ))$ and (4.15) is equivalent to the following two inequalities

$$\frac{(1-\sqrt{\lambda(Z)})}{2\lambda(Z)(1+\sqrt{\lambda(Z)})}\cdot\frac{d\lambda(Z)}{d\rho}\bigg|_{Z=f^{-1}(W)}\leq\frac{1}{\rho} \tag{4.16}$$

and

$$\frac{(1+\sqrt{\lambda(Z)})}{2\lambda(Z)(1-\sqrt{\lambda(Z)})}\cdot\frac{d\lambda(Z)}{d\rho}\bigg|_{Z=f^{-1}(W)}\geq\frac{1}{\rho}. \tag{4.17}$$

Integrating (4.16) with respect to $d\rho$ along the path $\sigma(W)$ from Q to W gives

$$\ln\rho(W)\geq\ln\frac{\sqrt{\lambda(Z)}(1+\sqrt{\lambda\circ f^{-1}(Q)})^2}{(1+\sqrt{\lambda(Z)})^2}+\ln\frac{\rho(Q)}{\sqrt{\lambda\circ f^{-1}(Q)}}. \tag{4.18}$$

It follows by the normalized assumption for f that

$$\rho^2(Q)=\rho^2(f(sZ))=\mathrm{tr}(f(sZ)\overline{f(sZ)}^T=s^2(\mathrm{tr}Z\overline{Z}^T+O(s)). \tag{4.19}$$

The normalized conditions shows f^{-1} is also normalized. That is $\partial z_{ij}/\partial \omega_{kl} = \delta_{kl}^{ij}$. Then we have

$$f^{-1}(Q) = f^{-1}\left(\sqrt{\frac{\mathrm{tr}(f(sZ)\overline{f(sZ)}^T)}{\mathrm{tr}W\overline{W}^T}}\cdot W\right)$$
$$= f^{-1}\left(s\left(\sqrt{\frac{\mathrm{tr}Z\overline{Z}^T}{\mathrm{tr}W\overline{W}^T}}+O(s)\right)W\right)$$
$$= s\left(\sqrt{\frac{\mathrm{tr}Z\overline{Z}^T}{\mathrm{tr}W\overline{W}^T}}W+O(s)\right). \tag{4.20}$$

Thus we have

$$\lambda\circ f^{-1}(Q) = s^2(\mathrm{tr}Z\overline{Z}^T+O(s)). \tag{4.21}$$

Substituting (4.19),(4.21) into (4.18) and letting $s\to 0$ in (4.18), we get

$$|f(Z)| = \rho(W) \geq \frac{\sqrt{\mathrm{tr}Z\overline{Z}^T}}{(1+\sqrt{\lambda(Z)})^2} = \frac{|Z|}{1+\|Z\|_I)^2} \tag{4.22}$$

holds almost everywhere.

We have almost proved the left hand side inequality of (2.2) is true almost everywhere. Based on the fact that $\rho(f(Z))$ and $\lambda(Z)$ are continuous functions of Z on $\mathcal{R}_I(m,n)$ and the measure of the excluded set is zero, we see that the left hand side inequality of (2.2) holds for all $z\in\mathcal{R}_I(m,n)$. Similarly, we can prove the right hand side inequality of (2.2) holds for all $z\in\mathcal{R}_I(m,n)$.

Using the same method, we also can prove the theorem in the case that the domains are $\mathcal{R}_K$, $K = II, III, IV$. We omit the details of the proof. Obviously, the method used by Taishun Liu[4] to prove (2.1) are different from the one used by us in this paper.

The following mappings show (2.2) is precise, and the extremal mappings are not unique.

$$\mathcal{R}_I: \frac{1}{(1-z_n)}Z,\quad Z[I-\begin{pmatrix}Z\\0\end{pmatrix}]^{-2};$$
$$\mathcal{R}_{II}: \frac{1}{(1-z_n)}Z, Z=Z^T;\quad Z[I-Z]^{-2}, Z=Z^T;$$
$$\mathcal{R}_{IV}: \frac{1}{(1-z_n)}Z, Z=-Z^T;\quad Z[I-AZ]^{-2}, Z=Z^T;$$

where $\begin{pmatrix}Z\\0\end{pmatrix}$ is a square matrix and

$$A = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0&1\\-1&0\end{pmatrix} \text{ if } n \text{ is even,}$$
$$A = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0&1\\-1&0\end{pmatrix}\dotplus 0 \text{ if } n \text{ is odd.}$$

§5. The Equivalence of Two Growth Theorems for Starlike Mappings on Classical Domains

Now we prove (2.1) and (2.2) are equivalent. As did in the previous section, we only prove this conclusion in the case that the domains is $\mathcal{R}_I$. And the proofs for the other two cases are similar.

If $\lambda(W)$ is the largest eigenvalue of $W\overline{W}^T$, then $\|W\|_I^2 = \lambda(W)$, where $W = f(Z)$, $Z \in \mathcal{R}$. $\lambda(Z)$ satisfies the equation

$$\det(\lambda(W)I - W\overline{W}^T) = 0,$$

which is equivalent to

$$\sum_{k=0}^{m}(-1)^k\lambda^{m-k}\mathrm{tr}^{(k)}W\overline{W}^T = 0, \tag{5.1}$$

where $\mathrm{tr}^{(0)}W\overline{W}^T = \mathrm{tr}W\overline{W}^T$ and $\mathrm{tr}^{(m)}W\overline{W}^T = \det W\overline{W}^T$. Differentiating on both sides of (5.1), we have

$$\left\{\sum_{k=0}^{m-1}(-1)^k(m-k)\lambda^{m-k-1}\mathrm{tr}^{(k)}W\overline{W}^T\right\}d\lambda + \sum_{k=1}^{m}(-1)^k\lambda^{m-k}d\mathrm{tr}^{(k)}W\overline{W}^T = 0. \tag{5.2}$$

We can easily verify

$$\langle d(\mathrm{tr}^{(k)}W\overline{W}^T), d\rho\rangle = \frac{2k}{\rho}\mathrm{tr}^{(k)}W\overline{W}^T \tag{5.3}$$

because of $d\rho = \frac{1}{2\rho}\sum\limits_{ij}(\omega_{ij}d\bar{\omega}_{ij} + \overline{\omega_{ij}}d\omega_{ij})$. From (5.2) and (5.3), we get

$$\left\{\sum_{k=0}^{m-1}(-1)^k(m-k)\lambda^{m-k-1}\mathrm{tr}^{(k)}W\overline{W}^T\right\}\langle d\lambda, d\rho\rangle = \frac{-2}{\rho}\sum_{k=1}^{m}(-1)^k k\lambda^{m-k}\mathrm{tr}^{(k)}W\overline{W}^T. \tag{5.4}$$

Using (5.1), we have

$$\begin{aligned}
&\sum_{k=0}^{m-1}(-1)^k(m-k)\lambda^{m-k-1}\mathrm{tr}^{(k)}W\overline{W}^T\\
=&\frac{m}{\lambda}(-1)^{m+1}\det W\overline{W}^T - \{\sum_{k=1}^{m-1}(-1)^k k\lambda^{m-k-1}\mathrm{tr}^{(k)}W\overline{W}^T\}\\
=&-\frac{1}{\lambda}\sum_{k=1}^{m}(-1)^k k\lambda^{m-k}\mathrm{tr}^{(k)}W\overline{W}^T.
\end{aligned}$$

Substituting the previous equality into (5.4), we obtain

$$\langle d\lambda, d\rho\rangle = \frac{2\lambda}{\rho}.$$

That is $\langle d\|Z\|_I, d\rho\rangle = \frac{\|Z\|_I}{\rho}$. Thus

$$\frac{d\|Z\|_I}{\|Z\|_I} = \frac{d\rho}{\rho} \tag{5.5}$$

holds.

As we did in proving that (1.3) and (1.4) are equivalent, we integrate (5.5) on both sides from $W = f(\epsilon Z)$ to $W = f(Z)$ where ϵ is a small positive number, and then let $\epsilon \to 0$.

According to the definition of $\|Z\|_{I,\rho}$ and the normalization conditions of f, we immediately see that (2.1) and (2.2) are equivalent in the case that the domain is $\mathcal{R}_I$.

Similarly, we can prove

$$\frac{d\|Z\|_K}{\|Z\|_K} = \frac{d\rho}{\rho} \tag{5.6}$$

where $\rho = (\sum_{ij} \omega_{ij}\overline{\omega_{ij}})^{1/2}$, $W = f(Z)$, $Z \in \mathcal{R}_K$ if $K = II$ or $K = III$.

Using (5.6) we can also prove that (2.1) and (2.2) are equivalent in the cases that the domains are $\mathcal{R}_{II}$ and $\mathcal{R}_{III}$.

In the case that the domain is $\mathcal{R}_{IV}$, Taishun Liu[5] already proved that (2.1) implies (2.2), but it is easy to prove that (2.2) implies (2.1) if we use the same process.

§6. The Growth Theorems of Convex Mappings on Several Complex Variables

In 1989, Ted Suffridge, Caroly Thomas and Taishun Liu, using three different methods, independently proved the following Growth Theorem of normalized convex biholomorphic mappings on the unit ball in C^n.

Let $f(Z)$ be the normalized convex biholomorphic mapping on unit ball B^n in C^N. Then

$$\frac{|Z|}{1+|Z|} \leq |f(Z)| \leq \frac{|Z|}{1-|Z|} \tag{6.1}$$

holds for every $X \in B^n$. The estimation is precise, but the extremal mapping is not unique. As a consequence, $f(B^n) \supset \frac{1}{2}B^n$, that is, the Koebe constant of this family of mappings is $\frac{1}{2}$.

In 1989, Taishun Liu and Sheng Gong[9] proved one side of Growth Theorem of normalized convex biholomorphic mappings on the Reinhardt domains B_p, $B_p \equiv \{Z \in C^n | \ \|Z\|_p < 1, p \geq 1\}$, in p-norm.

Let $f(Z)$ be the normalized convex biholomorphic mapping on the Reinhardt domains B_p. Then

$$\|f(Z)\|_p \leq \frac{\|Z\|_p}{1-\|Z\|_p} \tag{6.2}$$

holds for all $Z \in B_p$. The estimation is precise, but the extremal mapping is not unique.

Now we can use equality (1.5) to get

$$|f(Z)| \leq \frac{|Z|}{1-\|Z\|_p} \tag{6.3}$$

from (6.2), and conversely, we can get (6.2) from (6.3).

As the other part of his doctor thesis, Taishun Liu[5] got the Growth Theorems of normalized convex biholomorphic mappings on classical domains.

Let $f(Z)$ be the normalized convex biholomorphic mapping on classical domains $\mathcal{R}_K(I, II, III, IV)$. Then

$$\frac{\|Z\|_K}{1+\|Z\|_K} \leq \|f(Z)\|_K \leq \frac{\|Z\|_K}{1-\|Z\|_K} \tag{6.4}$$

holds for every $Z \in \mathcal{R}_K$. The estimation is precise, but the extremal mapping is not unique. As a consequence, $f(\mathcal{R}_K) \supset \frac{1}{2}\mathcal{R}_K$, that is, the Koebe constant of this family of mappings is $\frac{1}{2}$.

Now we can use equalities (5.5) and (5.6) to get

$$\frac{|Z|}{1+\|Z\|} \leq |f(Z)| \leq \frac{|Z|}{1-\|Z\|} \tag{6.5}$$

from (6.4) if $K = I, II, III$ and in the case that the domain is $\mathcal{R}_{IV}$, Taishun Liu already proved (6.4). Conversely, we can get (6.4) from (6.5).

References

[1] Cartan, H., Sur La possibilityé d'étendre aux fonctions de plusieurs variable complexes la theoarié de fonction univalentes, Note added to P. Montel, Lecons sur les fonctions Univalentes ou multivalents,, Gauthier-Villars, Paris, 1933,

[2] FitzGerald, Carl H., Gong Sheng & Barnard, Roger W., The Growth and 1/4-theorems for starlike mappings in C^n, *Chinese Science Bulletin,* **34**:3 (1988), 161-162.

[3] Yu Qihuang, Wang Shikun & Gong Sheng, The Growth and 1/4-theorems for starlike mappings in B^p, *Chinese Annals of Mathematics,* **11B**:1 (1990), 100-104.

[4] Pfaltzgraff, J.A., Personal communication.

[5] Liu Taishun, The Growth and 1/4-theorems for starlike mappings on classical domains, Doctor Thesis, 1989.

[6] Hua, L.K., Harmonic analysis of functions of several complex variables in the classical domain, Science Press, Peking, 1958, Amer. Math. Soc, Transl. Math. Monograph, 1963.

[7] Harris, L.A., Bounded symmetric homogeneous domains in infinite-dimensional space, *Lecture Notes in Math.,* **364,** Springer, Berlin, 1974, 13-40.

[8] Harris, L.A., Schwartz's lemma in normal linear spaces, *Proc. Natl. Acd. Sci. U.S.A.,* **64(4)**:9 (1969), 1014-1017.

[9] Liu Taishun & Gong Sheng, The Growth theorem for convex mappings in B_p, (to appear in *Chinese Quarterly Journal).*