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STRIATED SOLUTIONS OF QUASILINEAR HYPERBOLIC TWO SPEED SYSTEMS**

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Abstract

It is proved that solutions of quasilinear hyperbolic 2×2 systems in general space dimension are striated if they are so at the initial time or in the past. Results about the propagation of singularities along characteristic curves for both striated and stratified solutions are given.

§1. Introduction

Striated solutions of semilinear systems were studied in [1]. Roughly speaking, striated solutions are those smooth except for two directions in space-time variables. They are characterized by iterated regularity with respect to a foliation of codimension two. The aim of this paper is to extend the result of [1] to quasilinear case. The difficulty for quasilinear problems comes from the situation that the characteristic surfaces depend on the solutions; consequently, the smoothness of the foliation is not known a priori.

Suppose $L=\sum\limits_{i=0}^{n} A_i(x,u)\partial_{x_i}$ is a strictly hyperbolic (with respect to the time variable x_0) 2×2 operator, where $A_0(x,u),\ldots,A_n(x,u)$ are 2×2 matrix-valued C^∞ functions. Consider the following Cauchy problem

$$\sum_{i=0}^{n} A_i(x,u)\partial_{x_i}u=f(x,u), \tag{1.1}$$

$$u|_{x_0=0}=g, \tag{1.2}$$

where $f(x,u)$ is a vector-valued C^∞ function. It is well known that locally there is a unique classical solution of (1.1) and (1.2) if $g\in H^s$ with $s>\frac{n}{2}+1$ (see [2]). Let $u={}^t(u_1(x),u_2(x))$ be such a C^1 solution and Ω be a domain of determinacy of $\omega=\Omega\cap\{x_0=0\}$. Suppose Ψ is a C^∞ foliation of codimension one on $\omega\subset R^n$ (its foliages are C^∞ hypersurfaces in ω). There are locally two families of characteristic surfaces through the foliages of Ψ, corresponding to the two characteristics of L. Assume Ω is suitably small in the x_0 direction, so that these two families form a C^1 foliation, which will be denoted by Φ, of codimension two on Ω. In general, the regularity of Φ depends on that of the solution u. This fact is reflected in the careful inductive definition in Section 2 below of the distribution spaces to describe the striated solutions. Those spaces $\mathcal{H}^{r,k}(\Omega,\Phi)$, $H^{t,k}(\Omega,\Phi)$, etc. are defined through Φ and they

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will be senseless without the intertwining regularity of Φ. In the terminology of the next section, the main result of this paper is

Theorem 1.1. *Suppose* $s > \frac{n+1}{2} + 5, g \in H^{s,k}(\omega, \Psi)$, *and* $u \in H^s(\Omega)$ *is a solution of the Cauchy problem* (1.1) *and* (1.2). *Then* $\mathcal{H}^{s-1,k}(\Omega, \Phi)$ *makes sense and* $u \in H^{s,k}(\Omega, \Phi)$.

Similar to the Cauchy problem, it can be shown that a solution striated in the past remains so in the future. To state it precisely, assume $u \in H^s(\Omega)$ is a solution of (1.1), Ω is a domain of determinacy of $\Omega^- = \Omega \cap \{x_0 < 0\}$, Φ is a foliation of codimension two on Ω whose foliages have the representations $\{x \in \Omega : \phi_1(x) = c_1, \phi_2(x) = c_2\}$, where $\{\phi_i(x) = \text{constant}\}$ are characteristic.

Theorem 1.2. *If* $\mathcal{H}^{s-1,k}(\Omega^-, \Phi)$ *makes sense and*

$$u|_{\Omega^-} \in H^{s,k}(\Omega^-, \Phi), \ \phi_i|_{\Omega^-} \in H^{s,k}(\Omega^-, \Phi),$$

then $\mathcal{H}^{s-1,k}(\Omega, \Phi)$ *makes sense and* $u \in H^{s,k}(\omega, \Phi)$.

Theorem 1.1 will be proved in Section 3 after the relevant distribution spaces are explained in the next section. Result similar to Theorem 1.2 still holds if Φ is replaced by a foliation of codimension one, and such solutions are called stratified solutions[3]. The proof of these things is somehow the same as that of Theorem 1.1, so it will not be given in this paper apart from some brief notes in Section 3. In Section 4, it is proved that the singularities of the striated solutions propagate along characteristic curves, which is not the case for general solutions in higher space dimension.

Although the results in this paper are formulated and proved for 2×2 systems, they can be readily extended to more general two speed quasilinear equations (as in [1]).

§2. The Distribution Spaces

The spaces of distributions describing the striated solutions for semilinear problems (see [1]) are analogies of the spaces of conormal distributions introduced in [4], while the spaces to be used in this paper are analogies of the spaces of conormal distributions for quasilinear problems (see [5,6]).

Suppose Θ is a C^1 foliation of codimension d on $M \subset R^m$. Locally the foliages have the representations $\{z \in M : \theta_i(z) = c_i, i = 1, \dots, d\}$, where θ_i are C^∞ functions and their Jacobian is of the rank d. A vector field V is called tangent (or parallel) to the foliation Θ if and only if $V\theta_i = 0, \dots, V\theta_d = 0$. Clearly it is independent of the choice of the representation functions $\theta_1, \dots, \theta_d$. Assume $B \subset \mathcal{D}'(M)$ is an algebra closed under C^∞ composition, and if there is a representation such that $\theta_1, \dots, \theta_d \in B$ (resp. $\partial\theta_1, \dots, \partial\theta_d \in B$), then we say $\Theta \in B$ (resp. $\partial\Theta \in B$) by an abuse of notations.

If $\partial\Theta \in B$, then there exist vector fields $V_{d+1}, \dots, V_m$ with coefficients in B, satisfying the following property:

$$\text{Any vector field } V \text{ tangent to } \Theta \text{ and with coefficients in } B \text{ can be written as } V = \sum_{j=d+1}^{m} a_j V_j, \text{ where } a_j \in B. \tag{2.1}$$

In fact, we can choose $V_j = \sum_{l=1}^{m} b_{jl}\partial_{z_l}$, where $\{(b_{j1}, \dots, b_{jm}) : j = d+1, \dots, m\}$ is a basic set of the solutions of the linear algebraic system $\sum_{l=1}^{m} \frac{\partial\theta_i}{\partial z_l} b_l = 0, i = 1, \dots, d$.

Definition 2.1. *Assume $\partial\Theta \in H^{s-1}(M), s > \frac{m}{2}+2$. For $0 \le r \le s-1$, define $\mathcal{H}^{r,0}(M,\Theta) = H^r(M)$. If for integer $k \ge 1$, $\mathcal{H}^{r,k-1}(M,\Theta)$, $0 \le r \le s-1$, have been defined and $\partial\Theta \in \mathcal{H}^{s-1,k-1}(M,\Theta)$ (otherwise $\mathcal{H}^{r,k}(M,\Theta)$, etc. may have no sense), then define inductively*

$$\mathcal{H}^{r,k}(M,\Theta) = \{v \in \mathcal{H}^{r,k-1}(M,\Theta) : Vv \in \mathcal{H}^{r,k-1}(M,\Theta) \text{ for any vector field } V \text{ tangent to } \Theta \text{ and with } \mathcal{H}^{s-1,k-1} \text{ coefficients}\}, 0 \le r \le s-1.$$

Once $\mathcal{H}^{r,k}$ make sense, define for any $t \ge 0$

$$H^{t,k}(M,\Theta) = \{v \in H^t(M) : \partial^\alpha v \in \mathcal{H}^{t-\mu,k} \text{ for any } |\alpha| \le \mu\},$$

where μ is a non-negative integer such that $t-\mu \le s-1$.

Similar to the spaces $\mathcal{H}^{r,k}(M,\mathcal{V})$,$H^{t,k}(M,\mathcal{V})$ defined in [6] for a C^∞ module $\mathcal{V}$ of vector fields, $\mathcal{H}^{r,k}(M,\Theta)$ or $H^{t,k}(M,\Theta)$ defined here will not always have sense with respect to k. To make sure that the definition is well established, we show

Lemma 2.1. *Assume $\partial\Theta \in \mathcal{H}^{s-1,k-1}(M,\Theta)$, $s > \frac{m}{2}+2$. Then there are vector fields $V_{d+1}, \dots, V_m$, tangent to Θ and with $\mathcal{H}^{s-1,k-1}(M,\Theta)$ coefficients, satisfying the property (2.1) where $B = \mathcal{H}^{s-1,k-1}(M,\Theta)$. If $\mathcal{V}$ is the $C^\infty(M)$-module generated by any set $\{V_{d+1}, \dots, V_m\}$ as above, then*

$$\mathcal{H}^{r,l}(M,\Theta) = \mathcal{H}^{r,l}(M,\mathcal{V}), 0 \le r \le s-1, 0 \le l \le k; \tag{2.2}$$

consequently $H^{t,l}(M,\Theta) = H^{t,l}(M,\mathcal{V}), t \ge 0, 0 \le l \le k$.

Proof. The conclusion is proved by induction on k, so without loss of generality, we can assume $\mathcal{H}^{r,l}(M,\Theta) = \mathcal{H}^{r,l}(M,\mathcal{V})$, $0 \le r \le s-1, 0 \le l \le k-1$, where $\mathcal{V}$ is a $C^\infty(M)$-module generated by any set of vector fields tangent to Θ and with coefficients in $\mathcal{H}^{s-1,k-2}$, having the property (2.1) where $B = \mathcal{H}^{s-1,k-2}$. By Lemma 2.3 of [6] we know that $\mathcal{H}^{s-1,k-1}$ is an algebra, so in view of the condition of the lemma and the discussion before Definition 2.1, we do have such $V_{d+1}, \dots, V_m$ as stated in the lemma. The conclusion that (2.2) holds for C^∞-module $\mathcal{V}$ generated by any of such set $\{V_{d+1}, \dots, V_m\}$ follows from Definition 2.1 and the definition of $\mathcal{H}^{r,l}(M,\mathcal{V})$ in [6] together with the algebraic properties ([6, Lemma 2.2]).

We content ourself with the main point of the inductive argument, the full induction can easily be completed.

Thus we have ordinary properties for $\mathcal{H}^{r,k}(M,\Theta)$ inherited from $\mathcal{H}^{r,k}(M,\mathcal{V})$ in [6]. They will not be repeated here.

If $\{\theta_1(z), \dots, \theta_d(z)\}$ is a set of representation functions of Θ, we can add C^∞ functions $\theta_{d+1}(z), \dots, \theta_m(z)$ to them so that the mapping

$$\chi : z \to \theta = (\theta_1(z), \dots, \theta_m(z)) \tag{2.3}$$

is locally a diffeomorphism. Let $\overline{M} = \chi(M)$ and $\overline{\Theta}$ be the induced foliation from Θ, i.e., $\overline{\Theta}$ is the foliation on $\overline{M}$ whose foliages are $\{\theta \in \overline{M} : \theta_1 = c_1, \dots, \theta_d = c_d\}$ in new coordinates $\theta_1, \dots, \theta_m$, so $\overline{\Theta}$ is a C^∞ foliation. It is clear that $\mathcal{H}^{r,k}(\overline{M},\overline{\Theta})$ makes sense for any $r \ge 0, k \ge 0$, and in fact

$$\begin{aligned}\mathcal{H}^{r,k}(\overline{M},\overline{\Theta}) = H^{r,k}(\overline{M},\overline{\Theta}) = \{\overline{v} \in H^r(\overline{M}) : \\ \partial_{\theta_{d+1}}^{\beta_{d+1}} \cdots \partial_{\theta_m}^{\beta_m} \overline{v} \in H^r(\overline{M}) \text{ for } \beta_{d+1} + \cdots + \beta_m \le k\}\end{aligned} \tag{2.4}$$

which are the spaces in [1] for semilinear problems.

Lemma 2.2. *If $\mathcal{H}^{s-1,k}(M,\Theta)$ makes sense, i.e.,*

$$\partial\Theta \in \mathcal{H}^{s-1,k-1}(M,\Theta) \text{ or } \Theta \in H^{s,k-1}(M,\Theta)$$

with $s > \frac{m}{2}+2$, then the diffeomorphism defined by (2.3) is in $H^{s,k-1}(M,\Theta)$ and

$$\mathcal{H}^{r,l}(M,\Theta) = \chi^*(H^{s,k-1}(\overline{M},\overline{\Theta})), 0 \le r \le s-1, 0 \le l \le k,$$

where $\chi^* : \overline{v} \to \overline{v}\circ\chi$.

We omit the proof, which is similar to that of [6, Lemma 3.4].

§3. Proof of The Main Theorem

We proceed to prove Theorem 1.1. Since it is local in character, we can assume without loss of generality that the foliages of Ψ are given by $\{x_1 = \text{const}\}$. Let

$$p(x,u,\xi) = \det(\sum_{i=0}^{n} A_i(x,u)\xi_i)$$

and $\lambda_j(x,u,\xi'), j=1,2$, denote the two characteristic roots in ξ_0 of $p(x,u,\xi)=0$, where $\xi' = (\xi_1,\dots,\xi_n)$. The strict hyperbolicity of L implies $\lambda_1 \ne \lambda_2$ when $\xi' \ne 0$, and λ_j are C^∞ functions of their arguments for $\xi' \ne 0$.

Consider the following first order characteristic equation

$$p(x,u(x),\partial\phi_j) = 0 \tag{3.1}$$

with data

$$\phi_j|_{x_0=0} = x_1, \partial_{x_0}\phi_j|_{x_0} = \lambda_j(0,x';g(x');1,0,\dots,0), \tag{3.2}$$

for $j=1,2$. Noting that ϕ_j exist by the assumption on the solution u of (1.1), and that $u(\cdot)\in H^s$, we are ready to see that $\phi_j \in H^s(\Omega)$. Obviously the foliages of the foliation Φ on Ω are $\{x\in\Omega : \phi_1(x)=c_1, \phi_2(x)=c_2\}$, so $\Phi \in H^s(\Omega)$.

Let $u^{(\mu)} = \{\partial^\alpha u : |\alpha| \le \mu\}$ where μ is an integer. Differentiating (1.1) and performing the coordinate change $\chi : x \to \overline{x}$ defined by

$$\overline{x}_0 = \frac{\phi_1(x)-\phi_2(x)}{2}, \overline{x}_1 = \frac{\phi_1(x)+\phi_2(x)}{2}, \overline{x}_l = x_l, 2\le l\le n, \tag{3.3}$$

we will have

$$\begin{pmatrix} \overline{L} & & \\ & \ddots & \\ & & \overline{L} \end{pmatrix} \overline{\mathcal{U}} + \overline{\mathcal{A}}\,\overline{\mathcal{U}} = \overline{\mathcal{F}}, \quad \overline{\mathcal{U}}|_{\overline{x}_0=0} = \overline{\mathcal{G}}, \tag{3.4}$$

where

$$\overline{L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_{\overline{x}_0} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\partial_{\overline{x}_1} + \sum_{l=2}^{n} \overline{A}_l \partial_{\overline{x}_l}, \tag{3.5}$$

$$\overline{\mathcal{U}} = \overline{\mathcal{B}}\, u^{(3)}\circ\chi^{-1}, \tag{3.6}$$

and matrix-valued $\overline{A}_l$ are C^∞ functions of $\phi_j^{(1)}\circ\chi^{-1}, u\circ\chi^{-1}$, $\overline{\mathcal{A}}$ is a C^∞ function of $\phi_j^{(2)}\circ\chi^{-1}, u^{(1)}\circ\chi^{-1}$, $\overline{\mathcal{B}}$ is a C^∞ function of $\phi_j^{(1)}\circ\chi^{-1}, u\circ\chi^{-1}$, vector-valued $\overline{\mathcal{F}}$ is a C^∞ function of $\chi^{-1}, u^{(2)}\circ\chi^{-1}$, $\overline{\mathcal{G}}$ is a C^∞ function of $\overline{x}', g^{(3)}(\overline{x}')$, where $\overline{x}' = (\overline{x}_1,\dots,\overline{x}_n)$.

Theorem 1.1 will be proved if we show that under the hypothesis of the theorem, $\mathcal{H}^{s-1,k}(\Omega,\Phi)$ makes sense and

$$u,\phi_1,\phi_2 \in H^{s,k}(\Omega,\Phi). \tag{3.7$_\mathrm{k}$}$$

Obviously $(3.7)_0$ holds. We are going to show $(3.7)_\mathrm{k}$ assuming $(3.7)_{\mathrm{k}-1}$. The inductive assumption $(3.7)_{\mathrm{k}-1}$ implies $\mathcal{H}^{s-1,k}(\Omega,\Phi)$ makes sense and χ defined by (3.3) belongs to $H^{s,k-1}(\Omega,\Phi)$. Therefore $\chi^{-1}\in H^{s,k-1}(\overline{\Omega},\overline{\Phi})$ in view of Lemma 2.2, where $\overline{\Omega}=\chi(\Omega)$ and the foliages of $\overline{\Phi}$ are $\{\overline{x}\in\overline{\Omega}:\overline{x}_0-\overline{x}_1=c_1,\overline{x}_0+\overline{x}_1=c_2\}$. Using again Lemma 2.2, we see that

$$\begin{cases}\overline{A}_l,\overline{\mathcal{B}}\in H^{s-1,k-1}(\overline{\Omega},\overline{\Phi}),\ \overline{\mathcal{A}},\overline{\mathcal{F}}\in H^{s-2,k-1}(\overline{\Omega},\overline{\Phi}),\\ \quad\overline{\mathcal{U}}\in H^{s-3,k-1}(\overline{\Omega},\overline{\Phi}),\ \overline{\mathcal{U}}|_{\overline{x}_0=0}\in H^{s-3,k}(\overline{\omega},\overline{\Psi})\end{cases} \tag{3.8}$$

(Note that $\overline{\omega}=\omega,\overline{\Psi}=\Psi$). Using the expression of

$$H^{r,k}(\overline{\Omega},\overline{\Phi})=\{\overline{v}\in H^r(\overline{\Omega}):\partial_{\overline{x}_2}^{\beta_2}\cdots\partial_{\overline{x}_n}^{\beta_n}\overline{v}\in H^r(\overline{\Omega}),\beta_2+\cdots+\beta_n\le k\}$$

corresponding to (2.4) and the particular form of (3.5), we can deduce from (3.4) that $\overline{\mathcal{U}}_k=\{\partial_{\overline{x}_2}^{\beta_2}\cdots\partial_{\overline{x}_n}^{\beta_n}\overline{\mathcal{U}}:\beta_2+\cdots+\beta_n\le k\}$ satisfies

$$\begin{pmatrix}\overline{L}&&\\&\ddots&\\&&\overline{L}\end{pmatrix}\overline{\mathcal{U}}_k+\overline{\mathcal{A}}_k\overline{\mathcal{U}}_k\in H^{s-3},\quad \overline{\mathcal{U}}_k|_{\overline{x}_0=0}\in H^{s-3},$$

where $\overline{\mathcal{A}}_k\in H^{s-3}$ and $\begin{pmatrix}\overline{L}&&\\&\ddots&\\&&\overline{L}\end{pmatrix}$ is a diagonal hyperbolic operator with H^{s-2} coefficients. We can apply the regularity theorem of [2] to conclude that $\overline{\mathcal{U}}_k\in H^{s-3}(\overline{\Omega})$. It follows from (3.6) and Lemma 2.2 that $u\in H^{s,k}(\Omega,\Phi)$.

To prove the remaining part of $(3.7)_\mathrm{k}$, we differentiate (3.1) and get the diagonal system

$$\sum_{i=0}^{n}\frac{\partial p}{\partial\xi_i}(x,u(x),\partial\phi_j)\partial_{x_i}(\partial\phi_j)+h(x,u^{(1)}(x),\partial\phi_j)=0, \tag{3.9}$$

where h is a vector-valued C^∞ function of its arguments and $j=1,2$.

Lemma 3.1. *Let* $Z_j=\sum\limits_{i=0}^{n}\frac{\partial p}{\partial\xi_i}(x,u(x),\partial\phi_j)\partial_{x_i}$ *and*

$$\overline{Q}_j=\partial_{\overline{x}_0}+(-1)^j\partial_{\overline{x}_1}+\sum_{l=2}^{n}\overline{a}_{jl}\partial_{\overline{x}_l},j=1,2,$$

where $\begin{pmatrix}\overline{a}_{1l}&\overline{a}_{3l}\\\overline{a}_{4l}&\overline{a}_{2l}\end{pmatrix}=\overline{A}_l$ *in* (3.5). *Then we have*

$$\chi_*(Z_j)=(-1)^{j-1}\overline{b}\,\overline{Q}_j,\ j=1,2, \tag{3.10}$$

where χ_* *is the tangential map induced by* χ, $\overline{b}$ *is a* C^∞ *function of* $\phi_1^{(1)},\phi_2^{(1)},u\circ\chi^{-1}$.

Proof. Let $\overline{p}(\overline{x},\overline{\xi})=p(\chi^{-1}(\overline{x}),u\circ\chi^{-1}(\overline{x}),{}^t(d\chi)\overline{\xi})$. Then

$$\chi_*(Z_j)=\sum_{i=0}^{n}\frac{\partial\overline{p}}{\partial\overline{\xi}_i}(\overline{x};(-1)^{j-1},1,0,\dots,0)\partial_{\overline{x}_i}$$

and

$$\overline{p}(\overline{x},\overline{\xi})=\frac{1}{2}\overline{b}\det\begin{pmatrix}\overline{\xi}_0-\overline{\xi}_1+\sum\limits_{l=2}^{n}\overline{a}_{1l}\overline{\xi}_l & \sum\limits_{l=2}^{n}\overline{a}_{3l}\overline{\xi}_l\\ \sum\limits_{l=2}^{n}\overline{a}_{4l}\overline{\xi}_l & \overline{\xi}_0+\overline{\xi}_1+\sum\limits_{l=2}^{n}\overline{a}_{2l}\overline{\xi}_l\end{pmatrix}$$

for some $\overline{b}(\neq 0)$ as stated in the lemma. A direct calculation will give (3.10).

In view of the fact that $Z_j\phi_j\equiv 0$ which follows from (3.1) and the homogeneity of p with respect to ξ, we get by differentiating (3.9) further and performing the coordinate change (3.3) that $\overline{\varphi}_j=\phi_j^{(2)}\circ\chi^{-1}$ satisfies

$$\begin{pmatrix}\overline{Q}_j & & \\ & \ddots & \\ & & \overline{Q}_j\end{pmatrix}\overline{\varphi}_j+\overline{h}_j(\chi^{-1},u^{(2)}\circ\chi^{-1},\overline{\varphi}_j)=0,$$

$$\overline{\varphi}_j|_{\overline{x}_0=0}=\overline{\Lambda}_j(\overline{x}',g^{(1)}(\overline{x}')),$$

where $\overline{h}_j,\overline{\Lambda}_j$ are C^∞ functions of their arguments. Since Ω is a domain of determinacy with respect to L, so is Ω with respect to both Z_1 and Z_2. Since, obviously, $\overline{Q}_j$ are hyperbolic operators of first order, we can reason just as before for u that $\overline{\varphi}_j\in\mathcal{H}^{s-2,k}(\overline{\Omega},\overline{\Phi})$. The proof of $(3.7)_k$ is completed.

The overall argument is valid equally well for Theorem 1.2, where the regularity theorem for Cauchy problem with data given in the past $\{x_0<0\}$ should be used (e.g. the result in [7]) instead of that with data given on the initial plane. In the case of stratified solutions (i.e., Φ is a foliation of codimension one), a result similar to Theorem 1.2 holds for general $N\times N$ systems, which cannot be expected in the case of striated solutions when $N>2$.

Theorem 3.1. *Suppose $\sum\limits_{i=0}^{n}A_i(x,u)\partial_{x_i}$ is a strictly hyerbolic $N\times N$ operator and $u\in H^s(\Omega)$, $s>\frac{n+1}{2}+5$, is a solution of*

$$\sum_{i=0}^{n}A_i(x,u)\partial_{x_i}u=F(x,u)$$

and Ω is a domain of determinacy of $\Omega^-=\Omega\cap\{x_0<0\}$, Φ is a foliation of codimension one on Ω and its foliages $\{x\in\Omega:\phi(x)=0\}$ are characteristic. If $\mathcal{H}^{s-1,k}(\Omega^-,\Phi)$ makes sense (i.e., $\phi\in H^{s,k-1}(\Omega^-,\Phi)$) and $u|_{\Omega^-}\in H^{s,k}(\Omega^-,\Phi)$, then $\mathcal{H}^{s-1,k}(\Omega,\Phi)$ makes sense and $u\in H^{s,k}(\Omega,\Phi)$.

We only point out the main difference of the proof. The coordinate change can be taken as $x\to\overline{x}=(\phi(x),\psi_1(x),\dots,\psi_n(x))$ and the problem can be similarly reduced to $\begin{pmatrix}\overline{L} & & \\ & \ddots & \\ & & \overline{L}\end{pmatrix}\overline{\mathcal{U}}+\overline{\mathcal{A}}\overline{\mathcal{U}}=\overline{\mathcal{F}}$, where

$$\overline{L}=\begin{pmatrix}0 & \cdots \\ \vdots & I_{n-1}\\ 0 & \end{pmatrix}\partial_{\overline{x}_0}+\sum_{l=1}^{n}\overline{A}_l\partial_{\overline{x}_l}. \tag{3.11}$$

Here $\overline{x}_0$ would no longer be a time variable. The foliages of the induced foliation $\overline{\Phi}$ become $\{\overline{x}\in\overline{\Omega}:\overline{x}_0=c\}$ and we have

$$H^{r,k}(\overline{\Omega},\overline{\Phi})=\{\overline{v}\in H^r(\overline{\Omega}):\partial_{\overline{x}_1}^{\beta_1}\cdots\partial_{\overline{x}_n}^{\beta_n}\overline{v}\in H^r(\overline{\Omega}),\beta_1+\cdots+\beta_n\le k\}.$$

The succeeding argument in the proof of Theorem 1.1 would work in favour of the special form (3.11) of the operator.

§4. Propagation of Singularities

From the definition of $\overline{Q}_j$ in Lemma 3.1, it is easy to see that the integral curves of $\overline{Q}_j$ ($j = 1, 2$) in the coordinates $\overline{x}$ are just the projection curves of the bicharacteristic of $\overline{p}$ (whose fibre variables $\overline{\xi} = ((-1)^{j-1}, 1, 0, \dots, 0)$), so given a foliation Φ of codimension two as in Theorem 1.1, we can have the concept of characteristic curves—integral curves of Z_j in Lemma 3.1—associated with Φ. There are exactly two such curves passing through any given point.

Theorem 4.1. *Assume u is a striated solution of* (1.1) *and* (1.2) *as in Theorem 1.1 where $k = \infty$. For $y \in \Omega$, let γ_1, γ_2 be the two characteristic curves through y, and y^j be the intersection points of γ_j with $\{x_0 = 0\}$. If $g \in C^\infty$ at y^1 and y^2, then $u \in C^\infty$ at y.*

Proof. Usual regularity theorem for hyperbolic systems guarantees the existence of neighbourhoods $D_j \subset \Omega$ of y^j such that $u \in C^\infty(D_j)$. Come back to the coordinates $\overline{x}$ in the proof of Theorem 1.1 and let $\overline{\Omega}_j \subset \overline{\Omega}$ be domains of determinacy of $\overline{D}_j = \chi(D_j)$, i.e., for any $\overline{x} \in \overline{\Omega}_j$ there is a characteristic curve $\overline{\gamma}$ of $\overline{Q}_j$ such that $\overline{x} \in \overline{\gamma}, \overline{\gamma} \subset \overline{\Omega}_j$ and $\overline{\gamma} \cap \overline{D}_j \neq \emptyset$. Surely $\overline{\Omega}_1 \cap \overline{\Omega}_2$ is a neibourhood of $\overline{y}$ and the system in (3.4) can be written as the following coupled diagonal system

$$\begin{pmatrix} \overline{Q}_1 & & \\ & \ddots & \\ & & \overline{Q}_1 \end{pmatrix} \overline{\mathcal{U}}^1 + \sum_{l=2}^{n} \overline{\mathcal{A}}_l^1 \partial_{\overline{x}_l} \overline{\mathcal{U}}^2 + \overline{\mathcal{A}}^1 \overline{\mathcal{U}} = \overline{\mathcal{F}}^1, \tag{4.1}$$

$$\begin{pmatrix} \overline{Q}_2 & & \\ & \ddots & \\ & & \overline{Q}_2 \end{pmatrix} \overline{\mathcal{U}}^2 + \sum_{l=2}^{n} \overline{\mathcal{A}}_l^2 \partial_{\overline{x}_l} \overline{\mathcal{U}}^1 + \overline{\mathcal{A}}^2 \overline{\mathcal{U}} = \overline{\mathcal{F}}^2. \tag{4.2}$$

Because $\overline{\mathcal{U}}$ and the coefficients in (4.1) and (4.2) belong to $H^{s-3,\infty}(\overline{\Omega}, \overline{\Phi})$, $(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})\overline{\mathcal{U}}^1 \in H^{s-3,\infty}(\overline{\Omega}, \overline{\Phi})$. Differentiating (4.2) we can get

$$\overline{Q}_2(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})\overline{\mathcal{U}}^2 + \overline{\mathcal{C}}(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})\overline{\mathcal{U}}^2 \in H^{s-3,\infty}(\overline{\Omega}, \overline{\Phi}), \tag{4.3}$$

where $\overline{\mathcal{C}} \in H^{s-3,\infty}(\overline{\Omega}, \overline{\Phi})$. Since $\overline{\Omega}_2$ is a domain of determinacy of $\overline{D}_2$ with respect to $\overline{Q}_2$, we can deduce from (4.3) that

$$(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})\overline{\mathcal{U}}^2 \in H^{s-3,\infty}(\overline{\Omega}_2, \overline{\Phi}).$$

Thus $(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})\overline{\mathcal{U}} \in H^{s-3,\infty}(\overline{\Omega}_2, \overline{\Phi})$. A repeated argument will give

$$(\partial_{\overline{x}_0} - \partial_{\overline{x}_1})^{\alpha_1} \partial_{\overline{x}_2}^{\alpha_2} \cdots \partial_{\overline{x}_n}^{\alpha_n} \overline{\mathcal{U}}|_{\overline{\Omega}_2} \in H^{s-3}(\overline{\Omega}_2), |\alpha| \geq 0.$$

Similarly

$$(\partial_{\overline{x}_0} + \partial_{\overline{x}_1})^{\alpha_1} \partial_{\overline{x}_2}^{\alpha_2} \cdots \partial_{\overline{x}_n}^{\alpha_n} \overline{\mathcal{U}}|_{\overline{\Omega}_1} \in H^{s-3}(\overline{\Omega}_1), |\alpha| \geq 0,$$

so $\overline{\mathcal{U}}|_{\overline{\Omega}_1 \cap \overline{\Omega}_2} \in C^\infty$.

Let $\mathcal{V}$ be the C^∞-module of vector fields generated by $Z_1, Z_2, \chi_*^{-1}(\partial_{\overline{x}_2}), \dots, \chi_*^{-1}(\partial_{\overline{x}_n})$. Then we can show by induction and making use of Lemma 2.2 that $\mathcal{H}^{s-1,\infty}(\Omega_1 \cap \Omega_2, \mathcal{V})$

makes sense (i.e., the coefficients of any vector field in $\mathcal{V}$ belong to $\mathcal{H}^{s-1,\infty}(\Omega_1 \cap \Omega_2, \mathcal{V})$). The conclusion of Theorem 4.1 follows from the next lemma.

Lemma 4.1. *Suppose $\mathcal{H}^{s-1,\infty}(M,\mathcal{V})$ makes sense and $\mathcal{V}$ generates $H^{s-1}(M,TM)$. Then $\mathcal{H}^{s-1,\infty}(M,\mathcal{V}) = C^\infty(M)$.*

Proof. We prove inductively on k that

$$\mathcal{H}^{s-1,k}(M,\mathcal{V}) \subset H^{s-1+k}(M). \qquad (4.4)_{k-1}$$

It obviously holds when $k=0$. Assume $(4.4)_{k-1}$ holds. Then the coefficients of any vector field in $\mathcal{V}$ belong to H^{s+k-2}. If $\{V_1,\dots,V_m\} \subset \mathcal{V}$ is a basis in $H^{s+k-2}(M,TM)$, we will have

$$\partial_{z_i} = \sum_{l=1}^{m} c_{il} V_l, 1 \le i \le m, c_{il} \in H^{s+k-2}.$$

Hence

$$\begin{aligned}\mathcal{H}^{s-1,k}(M,\mathcal{V}) &= \{v \in \mathcal{H}^{s-1,k-1} : V_i v \in \mathcal{H}^{s-1,k-1}, 1 \le i \le m\}\\ &\subset \{v \in H^{s+k-2} : V_i v \in H^{s+k-2}, 1 \le i \le m\}\\ &= \{v \in H^{s+k-2} : \partial_{z_i} v \in H^{s+k-2}, 1 \le i \le m\} = H^{s-1+k}(M),\end{aligned}$$

so $(4.4)_{k-1}$ holds.

In the case of stratified solutions, there is also a concept of invariantly defined characteristic curves, and there is exactly one such curve through any given point, and we have a theorem on propagation of singularities along characteristic curves, which is true for general $N \times N$ system.

Theorem 4.2. *Assume u is a stratified solution as in Theorem 3.1, γ is a characteristic curve (associated with the foliation Φ in Theorem 3.1). If $u \in C^\infty$ at some point of γ, then $u \in C^\infty$ on the whole γ.*

The proof is omitted, as it is similar in spirit to that of Theorem 4.1.

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ON COUNTABLE σ-PRODUCT SPACES **

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Abstract

Let $X = \sigma\{X_i : i \in \omega\}$. It is proved that if every finite subproduct of X is normal, then X is normal if and only if X is countably paracompact.

This note is prompted by [5], where Zenor proved that if every finite subproduct of a given countable product is normal, then it is normal if and only if it is countably paracompact. It is known that the σ-product of a family of spaces is a dense subspace of the Tychonoff product of them. So it is natural to raise the question whether the foregoing result still holds for this kind of subspaces. In this note we give a positive answer to it.

Let λ be an infinite cardinal. $A(\lambda)$ denotes the one-point compactification of the discrete space $D(\lambda)$. Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces. The σ-product space of the spaces $\{X_\alpha : \alpha \in A\}$ with a base point $x^* = (x^*_\alpha)_{\alpha \in A}$ is the subspace $\sigma\{X_\alpha : \alpha \in A\}$ of $\prod\{X_\alpha : \alpha \in A\}$ such that for every $(x_\alpha)_{\alpha \in A} \in \sigma\{X_\alpha : \alpha \in A\}$,

$$|\{\alpha \in A : x_\alpha \neq x^*_\alpha\}| < \aleph_0.$$

Lemma 1[1]. *A space X is countably paracompact and λ-collectionwise normal if and only if $X \times A(\lambda)$ is normal.*

Lemma 2[2]. *Let $X = \sigma\{X_\alpha : \alpha \in A\}$. If every finite subproduct of X is countably paracompact and X is normal, then X is countably paracompact.*

Lemma 3[3]. *Let $X = \sigma\{X_\alpha : \alpha \in A\}$. If every finite subproduct of X is collectionwise normal and X is normal, then X is collectionwise normal.*

In fact, Lemma 3 is true for λ-collectionwise normality.

Lemma 4. *Let $X = \sigma\{X_\alpha : \alpha \in \lambda\}$, where each X_α contains at least two points. Then $A(\lambda)$ can be embedded in X.*

Proof. Let $x^* = (x^*_\alpha)_{\alpha<\lambda}$ be the base point of X. For each $\beta < \lambda$, choose $y_\beta = (y^\beta_\alpha)_{\alpha<\lambda}$ such that $y^\beta_\alpha \neq x^*_\alpha$ when $\beta = \alpha$ and $y^\beta_\alpha = x^*_\alpha$ otherwise. Let

$$Y = \{y_\beta : \beta < \lambda\} \cup \{x^*\}.$$

We prove that Y is the one-point compactification of the space $\{y_\beta : \beta < \lambda\}$ and the latter is a discrete space. For any open neighborhood U of x^*, there is a finite set a of λ and for each $\alpha \in a$ there is an open neighborhood U_α of x^*_α such that

$$\prod\{U_\alpha : \alpha \in a\} \times \sigma\{X_\alpha : \alpha \in \lambda \backslash a\} \subseteq U.$$

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It is easy to see that if $\beta \notin a$, then $y_\beta \in U$. So Y is compact. For each $\beta < \lambda$, let V_β be an open neighborhood of y_β^β such that $x_{\beta}^* \notin V_\beta$. Then

$$V_\beta \times \sigma\{X_\alpha : \alpha \in \lambda\backslash\{\beta\}\}$$

is an open neighborhood of y_β and contains no y_ν's for $\nu \neq \beta$. This shows that the space $\{y_\beta : \beta < \lambda\}$ is discrete.

Corollary 1. *If $X = \sigma\{X_\alpha : \alpha < \lambda\}$ is normal, then X is countably paracompact and λ-collectionwise normal.*

Proof. Assume that X is normal. For any finite subset a of λ, homeomorphically,

$$X = \prod\{X_\alpha : \alpha \in a\} \times \sigma\{X_\alpha : \alpha \in \lambda\backslash a\}.$$

By Lemma 4, $A(\lambda)$ can be embedded in $\sigma\{X_\alpha : \alpha \in \lambda\backslash a\}$. So

$$\prod\{X_\alpha : \alpha \in a\} \times A(\lambda)$$

is normal. By Lemma 1, $\prod\{X_\alpha : \alpha \in a\}$ is countably paracompact and λ-collectionwise normal. By Lemmas 2 and 3, X is countably paracompact and λ-collectionwise normal.

Now we prove the converse of the above statement for countable σ-products.

Theorem 1. *Let $X = \sigma\{X_i : i \in \omega\}$. If every finite subproduct of X is normal, then X is normal if and only if X is countably paracompact.*

Proof. The "only if" part follows directly from Corollary 1.

The "if" part. Assume that each finite subproduct of X is normal and X is countably paracompact. It is known that a space X is normal iff for every open cover of X consisting of two open sets, there is a countable open cover of X such that the closure of each member of it is contained in one of the two given open sets. Let $\{U_1, U_2\}$ be an open cover of X. For each $n \in \omega$ (Note that n is considered as the set of its precedents), p_n denotes the projection from X to $\prod\{X_i : i \in n\}$ defined as the restriction of the projection from $\prod\{X_i : i \in \omega\}$ to $\prod\{X_i : i \in n\}$. For each $n \in \omega$, let

$$W_{n,i} = \cup\{W : W \text{ is open in } \prod\{X_i : i \in n\} \text{ and } p_n^{-1}(W) \subseteq U_i\},$$

where $i = 1, 2$. Let

$$W_n = W_{n,1} \cup W_{n,2}.$$

Then it is easy to see that $\{p_n^{-1}(W) : n \in \omega\}$ is an increasing open cover of X. By the assumption, there is an increasing open cover $\{G_n : n \in \omega\}$ such that $\overline{G}_n \subseteq p_n^{-1}(W_n)$. For each $n \in \omega$, let

$$H_n = \cup\{H : H \text{ is open in } \prod\{X_i : i \in n\} \text{ and } p_n^{-1}(H) \subseteq G_n\}.$$

Then we can prove that $\{p_n^{-1}(H_n) : n \in \omega\}$ is an open cover of X and $\overline{H}_n \subseteq W_n$. As $\prod\{X_i : i \in n\}$ is normal, there are open sets $V_{n,1}$ and $V_{n,2}$ of $\prod\{X_i : i \in n\}$ such that

$$\overline{H}_n \subseteq V_{n,1} \cup V_{n,2}$$

and $\overline{V}_{n,i} \subseteq W_{n,i}$ for $i = 1, 2$. Thus

$$\overline{p_n^{-1}(V_{n,i})} = p_n^{-1}(\overline{V}_{n,i}) \subseteq p_n^{-1}(W_{n,i}) \subseteq U_i$$

for $i = 1, 2$. Since

$$\{p_n^{-1}(V_{n,i}) : n \in \omega \text{ and } i = 1, 2\}$$

is a countable open cover of X such that the closure of each member of it is contained in U_1 or U_2, X is normal.

We give an application of our results. In [4], K. Chiba proved

$$(*)\left\{\begin{array}{l}\text{Let } X=\sigma\{X_\alpha:\alpha\in A\}. \text{ If } X \text{ is normal and every finite subproduct of}\\ X \text{ is countably paracompact and para-Lindelöf, then } X \text{ is countably}\\ \text{paracompact and para-Lindelöf.}\end{array}\right.$$

By Corollary 1, the condition "countably paracompact" is superfluous.

Let D be a directed set. A collection $\{F_\lambda : \lambda \in D\}$ is said to increase with respect to the order of D if, whenever $\lambda \leq \mu$,

$$F_\lambda \subseteq F_\mu.$$

We do not know whether Theorem 1 is true for arbitrary σ-products. But a partial result is proved.

Theorem 2. *Let $X=\sigma\{X_\alpha : \alpha \in \lambda\}$. If every finite subproduct of X is normal and X is λ-paracompact, then X is normal.*

Proof. Let $\{U_1, U_2\}$ be an open cover of X. It is sufficient to prove that there is a locally finite open cover of X such that the closure of every member of it is contained in U_1 or U_2. Let $[\lambda]^{<\omega}$ be the set of all the finite subsets of λ with a partial order "inclusion". Then $[\lambda]^{<\omega}$ is a directed set. For each $a \in [\lambda]^{<\omega}$ and $i=1,2$, let

$$W_{a,i} = \cup\{W : W \text{ is open in } \textstyle\prod\{X_\alpha : \alpha \in a\} \quad \text{and} \quad p_a^{-1}(W) \subseteq U_i\},$$

where p_a is defined as p_n. Let

$$W_a = W_{a,1} \cup W_{a,2}.$$

Then

$$\{p_a^{-1}(W_a) : a \in [\lambda]^{<\omega}\}$$

is an open cover of X increasing with respect to the order on $[\lambda]^{<\omega}$. Since X is λ-paracompact, there is an open cover $\{G_a : a \in [\lambda]^{<\omega}\}$ of X increasing with respect to the order on $[\lambda]^{<\omega}$ such that $\overline{G}_a \subseteq p_a^{-1}(W_a)$. Let

$$H_a = \cup\{H : H \text{ is open in } \textstyle\prod\{X_\alpha : \alpha \in a\} \quad \text{and} \quad p_a^{-1}(H) \subseteq G_a\}.$$

Then

$$\{p_a^{-1}(H_a) : a \in [\lambda]^{<\omega}\}$$

is an open cover of X and $\overline{H}_a \subseteq W_a$ for each $a \in [\lambda]^{<\omega}$. By the hypothesis, there are open sets $V_{a,1}$ and $V_{a,2}$ such that

$$\overline{H}_a \subseteq V_{a,1} \cup V_{a,2}$$

and $\overline{V}_{a,i} \subseteq W_{a,i}$ for $i=1,2$. Let $\{S_a : a \in [\lambda]^{<\omega}\}$ be a locally finite open refinement of

$$\{p_a^{-1}(H_a) : a \in [\lambda]^{<\omega}\}.$$

Then it is easy to see that

$$\{S_a \cap p_a^{-1}(V_{a,i}) : a \in [\lambda]^{<\omega} \quad \text{and} \quad i=1,2\}$$

is a locally finite open cover of X such that the closure of every member of it is contained in one of U_1 and U_2. This shows that X is normal.

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