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BIFURCATION OF LIMIT LOOPS

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Abstract

The unified method is used to deal with the bifurcation problems of limit loops generated from the singular points of centre type and from the closed orbits. Some known results and methods are generalized and improved.

§0. Introduction

A great deal of work has been carried on for the research of the problems of the Hopf bifurcation, the centre bifurcation and the closed orbit bifurcation. In this paper, such kinds of problems are considered by a unified method, and some known results and methods are extended and improved.

In section 1, the limit loop bifurcation is considered in which the loops come from the singular point with at least one couple of pure imaginary eigenvalues (simply called the singular point of centre type below). The Hopf bifurcation theorem is generalized. By using the Birkhoff normal form, the nonexistence and the uniqueness of the limit loop bifurcation are proved for 2-dimensional systems.

The problem of the Pioncaré closed orbit bifurcation is studied in the second section. The method initiated by Pioncaré is extended to a class of nonlinear systems and high dimensional sytems.

§1. Limit Loop Bifurcation on the "Singular Point of Centre Type"

In this section, we use a unified method to attack the problems of the Hopf bifurcation, the bifurcation of limit loops generated from foci and real centres.

Consider the system

$$\begin{aligned} \dot{x} &= y + f(x,y) + \lambda P(x,y,z), \\ \dot{y} &= -x + g(x,y) + \lambda Q(x,y,z), \\ \dot{z} &= Bz + h(x,y,z) + \lambda W(x,y,z), \end{aligned} \tag{1.1)$_\lambda$}$$

where $x, y \in \mathbf{R},\ z \in \mathbf{R}^{n-2}$, and B is a constant matrix of order $n-2$.

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Denote system $(1.1)_\lambda$ by $(1.1)^*_\lambda$ when it is restricted to the $x-y$ plane. Let $\sigma(B)$ be the spectral set of B, $i^2=-1$. Express

$$f=f_k+h.o.t,\quad g=g_k+h.o.t,\quad P=P_r+h.o.t.,$$
$$Q=Q_r+h.o.t,\quad W=W_s+h.o.t,\quad h=h_m+h.o.t.,$$
$$P_r=P_r^1(x,y)+P_r^2(x,y,z),,\qquad Q_r=Q_r^1(x,y)+Q_r^2(x,y,z),$$

where f_k,g_k,P_r,Q_r,h_m,W_s are homogeneous polynomials in x,y,z with degree k,r,m and s respectively. Assume

(H_1) $ki\notin\sigma(B)$, $\forall k\in\mathbf{Z}$.

(H_2) $k\geq 2,\ 1\leq r\leq k,\ m\geq 2,\ s\geq 2.$

When $m=2$ or $s=2$, we may further assume

(H_3) $h_2(x,y,z)$ and $W_2(x,y,z)$ do not contain the terms only in x and y.

In fact, by the first several steps in the process which is similar to that of the Birkhoff normalization, we can eliminate the terms in h_2 and W_2 which only consist of x and y.

For $\varepsilon>0$, let $x\to\varepsilon x,\ y\to\varepsilon y,\ z\to\varepsilon^2 z,\ \lambda=\varepsilon^{k-r}\delta$. Then system $(1.1)_\lambda$ becomes

$$\begin{aligned}\dot{x}&=y+\varepsilon^{k-1}f_k(x,y)+\varepsilon^{k-1}\delta P_r^1(x,y)+O(\varepsilon^k),\\ \dot{y}&=-x+\varepsilon^{k-1}g_k(x,y)+\varepsilon^{k-1}Q_r^1(x,y)+O(\varepsilon^k),\\ \dot{z}&=Bz+O(\varepsilon).\end{aligned}\tag{1.2}$$

Its solution with initial condition $x(0)=0,\ y(0)=u,\ z(0)=z_0$ is

$$\begin{aligned}x(t)&=u\sin t+\varepsilon^{k-1}\int_0^t[(f_k+\delta P_r^1)\cos(t-s)+(g_k+\delta Q_r^1)\sin(t-s)]ds+O(\varepsilon^k),\\ y(t)&=u\cos t+\varepsilon^{k-1}\int_0^t[-(f_k+\delta P_r^1)\sin(t-s)+(g_k+\delta Q_r^1)\cos(t-s)]ds+O(\varepsilon^k),\\ z(t)&=z_0e^{Bt}+O(\varepsilon).\end{aligned}\tag{1.3}$$

The necessary and sufficient condition for the solution (1.3) to be a periodic solution of system (1.2) with period $T=2\pi(1+\varepsilon^{k-1}\tau)$ is that

$$x(T)=0,\ y(T)-u=0,\ z(T)-z_0=0,$$

that is,

$$u\sin 2\pi\varepsilon^{k-1}\tau+\varepsilon^{k-1}(u^kE_k+\delta u^rG_r)+O(\varepsilon^k)=0,\tag{1.4}$$

$$u(\cos 2\pi\varepsilon^{k-1}\tau-1)+\varepsilon^{k-1}(u^kF_k+\delta u^rH_r)+O(\varepsilon^k)=0,\tag{1.5}$$

$$z_0(e^{2\pi B}-1)+O(\varepsilon)=0,\tag{1.6}$$

where

$$G_r=\int_0^{2\pi}[P_r^1(\sin t,\cos t)\cos t-Q_r^1(\sin t,\cos t)\sin t]dt,$$

$$H_r=\int_0^{2\pi}[P_r^1(\sin t,\cos t)\sin t+Q_r^1(\sin t,\cos t)\cos t]dt,$$

$$E_k=\int_0^{2\pi}[f_k(\sin t,\cos t)\cos t-g_k(\sin t,\cos t)\sin t]dt,$$

$$F_k=\int_0^{2\pi}[f_k(\sin t,\cos t)\sin t+g_k(\sin t,\cos t)\cos t]dt.$$

By (1.4) and (1.6), we get

$$\tau = \tau(u, \varepsilon, \delta) = -(u^{k-1}E_k + \delta u^{r-1}G_r)/2\pi + O(\varepsilon), \tag{1.7}$$

$$z_0 = z(u, \varepsilon, \delta) = O(\varepsilon). \tag{1.8}$$

Substituting (1.7) and (1.8) into (1.5), we have

$$M \triangleq \varepsilon^{1-k}(y(T) - u) = u^r[F_k u^{k-r} + \delta H_r] + O(\varepsilon). \tag{1.9}$$

If we notice that: (i) $0(0,0,0)$ is always a solution of system $(1.1)_\lambda$, and it implies that $u = 0$ is always a solution of (1.9); (ii) In a sufficiently small neigborhood of the origin, every periodic solution intersects the $y - z$ plane at exactly two points (u, z_0) and (u_1, z_1), and $u_1 = -u + O(\varepsilon^{k-1})$; (iii) $G_k = H_k = E_k = F_k = 0$ when k is even; (iv) δ can be chosen such that $F_k + \delta H_r \neq 0$ when $H_r \neq 0$, then it is easy to get the following theorem.

Theorem 1.1. *Suppose $f, g \in \mathbf{C}^k$, $h, W \in \mathbf{C}^2$, $P, Q \in \mathbf{C}^r$, $H_r \neq 0$, and the conditions* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *are satisfied. Then, for fixed $\delta \neq 0$, there exist $\varepsilon_1 > \varepsilon_0 > 0$ such that the following conclusions hold when $0 < \varepsilon < \varepsilon_0$, $\lambda = \varepsilon^{k-r}\delta$.*

i) *If $k - r > 0$ and $\delta H_r F_k < 0$, then system $(1.1)_\lambda$ has at least one (its redius is approximately $(-\lambda H_r F_k^{-1})^{1/(k-r)}$) and at most $[(r+1)/2]$ limit loops situated in the ε_1-neighborhood of* $0(0,0,0)$, *where $[x]$ denotes the integral part of x.*

ii) *If $k - r > 0$ and $\delta H_r F_k \geq 0$ or $k - r = 0$, then system $(1.1)_\lambda$ has at most $[(r-1)/2]$ limit loops situated in the ε_1-neighborhood of* $0(0,0,0)$.

Remark 1.1. If $H_r = H_{r+1} = \cdots = H_{r+j-1} = 0$, $H_{r+j} \neq 0$, and $r_1 \triangleq r + j \leq k, j < (k-1)/2$, then we can rewrite $\lambda = \varepsilon^{k-r_1}\delta$ so that the above discussion is still true. Thus, if we replace r by r_1 in Theorem 1.1, then all the conclusions remain true. Similarly, when

$$H_r \neq 0, \ F_k = F_{k+1} = \cdots = F_{k+j-1} = 0, \ F_{k+j} \neq 0,$$

and $j < k-1$, $r < k+j$, the conclusion still hold if $k+j$ is substituted for k and $\lambda = \varepsilon^{k+j-r}\delta$.

Remark 1.2. $F_k = 0$ when $0(0,0,0)$ is a real centre.

Theorem 1.2. *Under the same conditions of Theorem 1.1, for any given $\bar{u} > 0$, there are $\varepsilon_0 > 0$, $u_0 > 0$ such that when $0 < \varepsilon < \varepsilon_0$, $|u - \bar{u}| < u_0$, there exist functions $\tau = \tau(\varepsilon, u)$, $\delta = \delta(\varepsilon, u)$ and $z_0 = z_0(\varepsilon, u)$, and when $\lambda = \varepsilon^{k-1}\delta$, system $(1.1)_\lambda$ has a periodic orbit passing through point $(0, \varepsilon u, \varepsilon^2 z_0)$ with period $T = 2\pi(1 + \varepsilon^{k-1}\tau)$, where $\tau \to \tau_0 = (2\pi H_r)^{-1}\bar{u}^{k-1}[F_k G_r - E_k H_r]$, $\delta \to \delta_0 = -\bar{u}^{k-r}F_k H_r^{-1}$, $z_0 \to 0$ as $\varepsilon \to 0$, $u \to \bar{u}$.*

Proof. Denote $L = \varepsilon^{1-k}x(T)$, $N = z(T) - z_0$. Since $L = M = N = 0$ for $\varepsilon = 0$, $z_0 = 0$, $u = \bar{u} > 0$, $\delta = \delta_0$, $\tau = \tau_0$, and since

$$\left.\frac{\partial(L, M, N)}{\partial(\tau, \delta, z_0)}\right|_{\varepsilon=0,\ u=\bar{u}} = 2\pi\bar{u}^{r+1}H_r(\exp\{2\pi B\} - 1)$$

is invertible, the theorem follows immediately from the implicit function theorem.

Remark 1.3. If $r = 1$ and take $P_r^1 = H_1 x/2\pi$, $Q_r^1 = H_1 y/2\pi$, then Theorem 1.2 becomes the Hopf bifurcation theorem.

Remark 1.4. When $H_r = 0$,

$$\tilde{G}_r = \int_0^{2\pi} [P_r^1(\cos t, -\sin t)\cos t - Q_r^1(\cos t, -\sin t)\sin t]dt \neq 0,$$

we may consider the solution with initial point $(u, 0, z_0)$ instead of solution (1.3), replace H_r by $\tilde{G}_r$ in Theorem 1.1 and Theorem 1.2, and then repeat the proof as above.

By the Lie transformation (usually be called the averaging method) or polynomial transformation, the 2-dimensional complex system with a fine focus can be transformed into the following Birkhoff normal form (c.f. [1, 3]):

$$\dot{z} = iz + \sum_{k=1}^{N} a_k z^{k+1}\bar{z}^k + h.o.t.,$$
$$\dot{\bar{z}} = -i\bar{z} + \sum_{k=1}^{N} a_k z^k \bar{z}^{k+1} + h.o.t.. \tag{1.10}$$

In paper [3], the author defined the i−th focus value of the singular point O as the quantity $f_i = \text{Re}a_i$, and called O the fine focus of order k when $f_1 = \cdots = f_{k-1} = 0,\ f_k \neq 0$.

Assume that O is a fine focus of order k of system (1.10), and $b_i = \text{Im}a_i$. Then, in the real coordinate system, (1.10) becomes

$$\dot{x} = -y(1 + \sum_{j=1}^{k} b_j \Delta^j) + f_k x \Delta^k + h.o.t.,$$
$$\dot{y} = x(1 + \sum_{j=1}^{k} b_j \Delta^j) + f_k y \Delta^k + h.o.t., \tag{1.11}$$

where $\Delta = x^2 + y^2$.

Now it is easy to see that $F_2 = \cdots = F_{2k} = 0,\ F_{2k+1} = 2\pi f_k \neq 0$. When $\lambda \neq 0$, by a similar Lie transformation, system $(1.1)^*_\lambda$ will take the following form

$$\dot{x} = -y(1 + \sum_{i=1}^{k} b_i \Delta^i) + f_k x \Delta^k + \mu p_j x \Delta^j + h.o.t.,$$
$$\dot{y} = x(1 + \sum_{i=1}^{k} b_i \Delta^i) + f_k y \Delta^k + \mu p_j y \Delta^j + h.o.t., \tag{1.12}$$

where b_i depends on λ for $[(r-1)/2] \leq i \leq j$, and is independent of λ for $i \notin [[(r-1)/2], j]$, and $\mu = \mu(\lambda),\ p_j = p_j(\lambda),\ \mu(0) = 0,\ p_j(0) \neq 0$.

Take $d > 0$ such that $\mu(\lambda) = \varepsilon^{2(k-j)}\delta_1 + o(\varepsilon^{2(k-j)})$ when $\lambda = \varepsilon^d \delta_1$. Let $x \to \varepsilon x,\ y \to \varepsilon y,\ dt \to (1 + \sum_{i=1}^{k} \varepsilon^{2i} b_i \Delta^i)^{-1} dt$. If we notice that the coefficients $b_i,\ f_i,\ p_i$ in (1.12) are polynomials of the coefficients of terms in $(1.1)^*_\lambda$ with degree$\leq i$, and that the degree of these polynomials increases with the increasing of the subscript of $b_i,\ f_i,\ p_i$, then it is easy to see that (1.12) now has the form

$$\dot{x} = -y + \varepsilon^{2k}[\delta_1 p_j x \Delta^j + f_k x \Delta^k] + o(\varepsilon^{2k}) R_1 \Delta^{j+1} + o(\varepsilon^N) W_1,$$
$$\dot{y} = x + \varepsilon^{2k}[\delta_1 p_j y \Delta^j + f_k y \Delta^k] + o(\varepsilon^{2k}) R_2 \Delta^{j+1} + o(\varepsilon^N) W_2 \tag{1.13}$$

when $(1.1)^*_\lambda$ is $\mathbf{C}^\infty$, where R_1, R_2 are polynomials in x and y, $W_1,\ W_2 \in \mathbf{C}^\infty$, and N is an arbitrarily given positive integer.

Now let $T = 2\pi(1 + \varepsilon^{2k}\tau)$ and $M = \varepsilon^{-2k}(y(T) - u)$. Then we have

$$M = 2\pi u^{2j+1}(f_k u^{2(k-j)} + \delta_1 p_j) + O(\varepsilon) u^{2j+1} + o(\varepsilon^{N-2k}).$$

Theorem 1.3. *Suppose $f, g, P, Q \in \mathbf{C}^\infty$, $0(0,0)$ is fine focus of $(1.1)^*_o$ with order k and focus value f_k, and O is a fine focus of system $(1.1)^*_\lambda$ with order $j(j \leq k)$ and focus value*

μp_j (or $\mu p_j + f_k$ as $j = k$) when $0 < |\lambda| \ll 1$. Then

i) when $j = k$ or $j < k$, $\mu p_j f_k > 0$, there is no new limit cycle generated in the sufficiently small neighborhood of O for system $(1.1)^*_\lambda$;

ii) when $j < k$, $\mu p_j f_k < 0$, a unique new limit cycle occurs in the sufficiently small neighborhood of O for system $(1.1)^*_\lambda$.

Proof. It suffices to make the following two remarks:

1) Each non-zero null point, which is generated from the $2j+1$-fold zero of the function

$$M_1 = 2\pi u^{2j+1}(f_k u^{2(k-j)} + \delta_1 p_j + O(\varepsilon))$$

under the perturbation $o(\varepsilon^{N_1})$ with arbitrarily high order $N_1 = N - 2k$, is an infinitesimal of arbitrarily high order. But the position of the limit cycles of $(1.1)^*_\lambda$ is fixed when ε is fixed, and hence the limit cycles of (1.13) can occur only at the outside of a certain neighborhood of the origin. This is because the polynomials transformation, which transforms $(1.1)^*_\lambda$ into (1.12),

$$x = x + \sum_{i+j=2}^{N} a_{ij}x^i y^j, \quad y = y + \sum_{i+j=2}^{N} b_{ij}x^i y^j$$

is sufficiently clsoed to the identity if x and y are sufficiently small. So, the limit cycles generated from O for $\lambda \neq 0$ can be decided only by the zero point of the function $M_2 = f_k u^{2(k-j)} + \delta_1 p_j + O(\varepsilon)$.

2) When $j = k$, we can take δ_1 such that $\delta_1 \neq -p_j^{-1} f_k$, and hence M_2 has not any zero point.

Remark 1.5. Theorem 1.3 tells us that a fine focus of a $\mathbf{C}^\infty$ system can generate at most one limit cycle under the perturbation of form $(1.1)^*_\lambda$ which has only one parameter. But if the perturbation has two or more parameters, the situation will be completely different. It is well known that a fine focus with order 3 can yield three limit cycles under the perturbation with three parameters. It is easy to show that, under the smooth perturbation with two parameters and with the following form

$$\dot{x} = -y(1 + \sum_{i=1}^{k} b_i y\Delta^i) + f_3 x\Delta^3 + \lambda_1 P_{r_1}(x,y) + \lambda_2 P_{r_2}(x,y) + h.o.t.,$$

$$\dot{y} = x(1 + \sum_{i=1}^{k} b_i x\Delta^i) + f_3 y\Delta^3 + \lambda_1 Q_{r_1}(x,y) + \lambda_2 Q_{r_2}(x,y) + h.o.t.,$$

a fine focus with order three of a smooth system can produce at most two limit cycles, where the lowest degree of the Taylor expansions of P_k, Q_k at $0(0,0)$ is k , and $1 \leq r_1 \leq r_2 \leq 7$. In fact, if we write $\lambda_1 = \varepsilon^{7-r_1}\delta_1$, $\lambda_2 = \varepsilon^{7-r_2}\delta_2$, and notice that none of the four coefficients of a cubic function with three positive zero points can be null, then the claim easily follows from the preceding proof. But, with the same method, we can prove that under the two-parameter perturbation with the above form, a fine focus of order four (higher than four, resp.) can generate three (more than three, resp.) limit cycles simultaneously in the neighborhood of the focus.

Remark 1.6. When $0(0,0)$ is a real centre of system $(1.1)^*_0$ and system $(1.1)^*_\lambda$ is analytic, then $F_k = 0$ for all k. So, the center O cannot produce any limit cycle for $\lambda \neq 0$.

Remaek 1.7. Theorem 1.3 extends the results of [4].

§2. The Poincaré Bifurcation

In this section, the Poincaré method is used to study the bifurcation problem of closed orbits surrounding a centre, and the theorem in [2, Chapter 8.2] is extended to a class of nonlinear systems and high-dimensional systems.

Assume that O is a real centre of the system

$$\dot{x} = y + f(x,y), \quad \dot{y} = -x + g(x,y). \tag{2.1}$$

In the Birkhoff normal form, (2.1) becomes

$$\begin{aligned} \dot{x} &= y + y\sum_{k=1}^{n} a_k \Delta^k + h.o.t., \\ \dot{y} &= -x - \sum_{k=1}^{n} a_k \Delta^k + h.o.t.. \end{aligned} \tag{2.2}$$

Therefore, when consider the bifurcation problem of closed orbits in a small neighborhood of a real centre, we may as well assume (2.1) has the following form

$$\dot{x} = y + yf(x,y), \quad \dot{y} = -x - xf(x,y), \tag{2.3}$$

and

(H_4) $1 + f(x,y) > 0$.

Now consider the perturbed system

$$\begin{aligned} \dot{x} &= y + yf(x,y) + \lambda P(x,y), \\ \dot{y} &= -x - xf(x,y) + \lambda Q(x,y), \end{aligned} \tag{2.4}$$

with condition

(H_5) $f, P, Q \in \mathbf{C}^1$, $f(0,0) = P(0,0) = Q(0,0) = 0$.

Let

$$\theta = -\int_0^t (1 + f(x(s), y(s))ds, \tag{2.5}$$

where $(x(t), y(t))$ is the solution of system (2.4).

Obviously, the period of the periodic orbit of system (2.3) passing through point $(x_0, 0)$ is

$$T(x_0) = \int_0^{2\pi} (1 + f(x_0 \cos\theta, x_0 \sin\theta))^{-1} d\theta. \tag{2.6}$$

Under the transformation (2.5), (2.4) now takes the form

$$\begin{aligned} \frac{dx}{d\theta} &= -y - \lambda P(x,y)(1 + f(x,y))^{-1}, \\ \frac{dy}{d\theta} &= x - \lambda Q(x,y)(1 + xf(x,y))^{-1}. \end{aligned} \tag{2.7}$$

Its solution with initial point $(x_0 + v, 0)$ is

$$\begin{aligned} x(\theta) &= (x_0 + v)\cos\theta + \lambda \int_0^\theta F(-P\cos(\theta - u) + Q\sin(\theta - u))du, \\ y(\theta) &= (x_0 + v)\sin\theta - \lambda \int_0^\theta F(P\sin(\theta - u) + Q\cos(\theta - u))du, \end{aligned} \tag{2.8}$$

where F denotes $(1 + f)^{-1}$.

The necessary and sufficient condition for (2.8) being a periodic solution with period $\phi = 2\pi + \varepsilon\tau$ is that

$$(x_0 + v)(\cos\varepsilon\tau - 1) + \lambda\int_0^{\phi} F(-P\cos(\varepsilon\tau - u) + Q\sin(\varepsilon\tau - u))du = 0,$$
$$(x_0 + v)\sin\varepsilon\tau - \lambda\int_0^{\phi} F(P\sin(\varepsilon\tau - u) + Q\cos(\varepsilon\tau - u))du = 0. \tag{2.9}$$

Let $\lambda = \varepsilon\delta$. Then, from the second formula of (2.9), we get

$$\begin{aligned}\tau =&\tau(\varepsilon,\delta,x_0,v)\\ =&\delta(x_0+v)^{-1}\int_0^{2\pi} F(-P\sin u + Q\cos u)|_{L_v}du + O(\varepsilon),\end{aligned} \tag{2.10}$$

where L_v is the loop $\{(x,y) : x = (x_0+v)\cos u,\ y = (x_0+v)\sin u,\ 0 \le u \le 2\pi\}$.

Putting (2.10) into the first formula of (2.9), we have

$$\int_0^{2\pi} F(P\cos u + Q\sin u)|_{L_v}du + O(\varepsilon) = 0.$$

Now set $L(x_0) = \int_0^{2\pi} F(P\cos u + Q\sin u)|_{L_v}du$.

Theorem 2.1. *Suppose that conditions* (H_4) *and* (H_5) *are satisfied. Then*

i) *if $L(x)$ has k zero points (taking the multiple into account), then system (2.4) has at most $[(k-1)/2]$ limit cycles when $0 < |\lambda| \ll 1$;*

ii) *if there is an $x_0 > 0$, such that $L(x_0) = 0$, $L'(x_0) \neq 0$, then, for any given $\delta_0 \neq 0$, there exists $\varepsilon_0 > 0$ and a unique differentiable function $v(\varepsilon)$, so that system (2.4) has a periodic orbit $\Gamma(x_0)$ passing tgrough $(x_0 + v(\varepsilon), 0)$ with period $T_1(x_0) = \int_0^{\phi} F(x(\theta), y(\theta))d\theta$ when $|\varepsilon| < \varepsilon_0$, $\lambda = \varepsilon\delta_0$, where τ is given by (2.10), $x(\theta), y(\theta)$ are defined by (2.8), $v(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\Gamma(x_0)$ is an unstable (stable) limit cycle when $\lambda L'(x_0) < 0 (> 0)$.*

Proof. The conclusion i) is obvious.

Since

$$\varepsilon^{-1}(x(2\pi + \varepsilon\tau) - (x_0 + v)) = -\delta_0 L(x_0 + v) + O(\varepsilon) = 0$$

when $\varepsilon = 0$, $v = 0$ and $L(x_0) = 0$, the existence of the $\mathbf{C}^1$ function $v(\varepsilon)$ comes from the implicit function theorem. Considering the sign of $x(2\pi + \varepsilon\tau + \Delta\tau) - (x_0 + v(\varepsilon) + \Delta v)$, we obtain the criterion of the stability of limit cycles.

Remark 2.1. From the definition of $L(x)$, it is easy to see that, when $x_0 = 0$, the limit cycle bifurcation guaranteed by Theorem 2.1 is just the Hopf bifurcation.

Example 2.1. Consider the system

$$\begin{aligned}\dot{x} =& y(1 + f(x,y)) + \lambda bx(x^2 + y^2),\\ \dot{y} =& -x(1 + f(x,y)) + \lambda ey(x^2 + y^2)^2,\end{aligned} \tag{2.11}$$

where $f(x,y) = a(x^2 + y^2)$, $a > 0$, $be > 0$.

$$\begin{aligned}L(x) =& \int_0^{2\pi}(1 + ax^2)^{-1}(bx^3\cos^2 u + ex^5\sin^2 u)du\\ =& \pi x^3(b + ex^2)(1 + ax^2)^{-1}.\end{aligned}$$

Since $L(x_0) = 0$ and $L'(x_0) = 2\pi e x_0^4(1 + ax_0^2)^{-1} \neq 0$ for $x_0 = (-be^{-1})^{1/2}$, system (2.11) has a unique limit cycle in the small neighborhood of $\Gamma_0 : \{(x,y) : x^2 + y^2 = x_0^2\}$ when

$0 < |\lambda| \ll 1$, and the cycle is unstable (stable) for $\lambda e < 0$ (> 0).

Theorem 2.1 gives the sufficient conditions for system (2.4) to have the closed-orbit bifurcation. But, in general case, the function $L(x)$ is rather complex, for its positive zero points are difficult to get. In this case, an alternative way is to use the Poincaré-Bendixson theorem which in turn produces the following theorem.

Theorem 2.2. *Suppose that* (H_4) *holds,* $f(0,0) = P(0,0) = Q(0,0) = 0$, $f, P, Q \in \mathbf{C}^0$ *and guarantee the existence and uniqueness of the solutions of (2.4) with initial values, and the zero points of* $L(x)$ *are isolated. If there are* $x_1 > x_2 > 0$*such that* $L(x_1)L(x_2) < 0$, *then there exist* $\lambda_0 > 0$ *and* $x_0(\lambda)$ *such that system (2.4) possesses a limit cycle passing through* $(x_0, 0)$ *when* $0 < |\lambda| < \lambda_0$, *where* $x_2 < x_0(\lambda) < x_1$.

Example 2.2 Consider the system

$$\begin{aligned}\dot{x} =& y(1+f(x,y)) + \lambda x y^2 g(x,y)\exp\{(b-2c)x^2 - cy^2\},\\ \dot{y} =& -x(1+f(x,y)) + \lambda a x^2 y g(x,y)\exp\{bx^2 + cy^2\},\end{aligned} \tag{2.12}$$

where $b > c > 0$, $-1 < a < 0$, $f(0,0) = 0$, $f(x,y) + 1 > 0$, $g(x,y) \ge 0$, $g(x,y) \not\equiv 0$, f, g being continuous and guaranteeing the existence and uniqueness of the solutions of system (2.12) with initial values.

Now

$$L(x) = x^3 \exp\{-cx^2\}(1 + a\exp\{2cx^2\}) \int_0^{2\pi} G(t)dt,$$

where

$$G(t) = g(x\cos t, x\sin t)\exp\{(b-c)x^2\cos^2 t\}\sin^2 t\cos^2 t/(1 + f(\cos t, x\sin t)).$$

It is easy to see that $L(x_1) < 0$, $L(x_2) > 0$ when $x_1 \gg 1$, $0 < x_2 \ll 1$. So, when $0 < |\lambda| \ll 1$ and $\lambda > 0 (< 0)$, the focus O of system (2.12) is unstable (stable) and there exists at least one stable (unstable) limit cycle Γ, $\Gamma \cap \{(x,0) : x > 0\} \in (x_2, x_1)$.

Next we consider the high dimensional system

$$\begin{aligned}\dot{x} =& y + yf(x,y)) + \lambda P(x,y,z),\\ \dot{y} =& -x - xf(x,y)) + \lambda Q(x,y,z),\\ \dot{z} =& Bz + \lambda R(x,y,z),\end{aligned} \tag{2.13}$$

where $x, y \in \mathbf{R}$, $z \in \mathbf{R}^n$.

Assume that, for system (2.13), (H_4) and the following hypothesis are valid:

$$(\mathrm{H}_6): \quad f, P, Q, R \in \mathbf{C}^1,\ f(0,0) = P(0,0,0) = Q(0,0,0) = R(0,0,0) = 0.$$

Similar to above, make transformation

$$\theta(t) = -\int_0^t (1 + f(x(s), y(s)))ds,$$

where $(x(t), y(t))$ is the solution of system (2.13).

System (2.13) now is transformed into

$$\frac{dx}{d\theta} = -y - \lambda PF,$$
$$\frac{dy}{d\theta} = x - \lambda QF,$$
$$\frac{dz}{dt} = Bz + \lambda R. \tag{2.14}$$

The solution of (2.14) with initial value $(x_0 + v, 0, w)$ is

$$x(\theta) = (x_0 + v)\cos\theta + \lambda \int_0^\theta F(-P\cos(\theta - u) + Q\sin(\theta - u))du,$$
$$y(\theta) = (x_0 + v)\sin\theta - \lambda \int_0^\theta F(P\sin(\theta - u) + Q\cos(\theta - u))du,$$
$$z(t) = e^{Bt}w + \lambda \int_0^t e^{B(t-s)}R(x(\theta(s)), y(\theta(s)), z(s))ds, \tag{2.15}$$

where $t(\theta) = -\int_0^\theta F(x(\theta), y(\theta))d\theta$ is the inverse function of $\theta(t)$.

Let $\lambda = \varepsilon\delta$ and $\phi = 2\pi + \varepsilon\tau$ again.

$$T = T(x_0) = \int_0^{2\pi} F(x_0\cos\theta, x_0\sin\theta)d\theta, \tag{2.16}$$

$$T_1 = T_1(x_0) = \int_0^\phi F(x(\theta), y(\theta))d\theta.$$

The necessary and sufficient condition for system (2.13) to have periodic solution with initial value $(x_0 + v, 0, w)$ and period T_1 is that

$$x(\phi) - (x_0 + v) = (x_0 + v)(\cos\varepsilon\tau - 1) + \varepsilon\delta \int_0^\phi F(-P\cos t + Q\sin t)du = 0,$$
$$y(\phi) = (x_0 + v)\sin\varepsilon\tau - \varepsilon\delta \int_0^\phi F(P\sin t + Q\cos t)du = 0,$$
$$z(T_1) - w = (e^{BT_1} - 1)w + \varepsilon\delta \int_0^{T_1} e^{B\theta}Rds = 0,$$

where $t = \varepsilon\tau - u,\ \theta = T_1 - s$.

From the second formula above, we get

$$\begin{aligned}\tau =& \tau(\varepsilon, \delta, x_0, v, w)\\ =& \delta(x_0 + v)^{-1} \int_0^{2\pi} F(-P\sin u + Q\cos u)du + O(\varepsilon),\end{aligned} \tag{2.17}$$

where

$$x = (x_0 + v)\cos u,\ y = (x_0 + v)\sin u,\ z = w\exp\{Bt(u)\}$$

in the integrand.

Denote

$$M_1 = \varepsilon^{-1}(x(\phi) - (x_0 + v)),\quad M_2 = \varepsilon^{-1}y(\phi),$$
$$M_3 = z(T_1) - w,\qquad \tau_0 = \tau(0, \delta, x_0, 0, 0)$$

and

$$L(x)=\int_0^{2\pi} F(P\cos u+Q\sin u)du,$$

$$H(x)=\int_0^{2\pi} F(P\sin u-Q\cos u)du. \tag{2.18}$$

The above two integrals are calculated along the circle: $x=x_0\cos u,\ y=x_0\sin u,\ z=0$.

Clearly, we have $M_1=M_2=M_3=0$, and

$$\frac{\partial(M_1,M_2,M_3)}{\partial(v,\tau,w)}=-\delta_0x_0L'(x_0)(e^{BT}-1)$$

is invertible when $\varepsilon=w=v=L(x_0)=0,\ \tau=\tau_0,\ \delta=\delta_0\neq 0,\ x_0>0,\ L'(x_0)\neq 0$ and $2k\pi T^{-1}(x_0)i\notin\sigma(B)$ for any $k\in\mathbf{Z}$.

Thus, the following theorem is an immediate corollary of the inverse function theorem.

Theorem 2.3. *Suppose that conditions* (H_4) *and* (H_6) *are satisfied. Then the following are true:*

i) *If $L(x)$ has k zeros (taking account of the multiple), then system* (2.13) *has at most $[(k-1)/2]$ limit cycles when $0<|\lambda|\ll 1$;*

ii) *If there is an $x_0>0$, such that $L(x_0)=0,\ L'(x_0)\neq 0$, and $2k\pi T^{-1}(x_0)i\notin\sigma(B)$ for any $k\in\mathbf{Z}$, then for any given $\delta_0\neq 0$, there exist $\varepsilon_0>0$ and a unique set of functions $v(\varepsilon),\ w(\varepsilon)$ and $\tau(\varepsilon)=\tau(\varepsilon,\delta_0,x_0,v(\varepsilon),w(\varepsilon))$ guaranteeing system* (2.13) *has a limit loop $\Gamma:\{(x,y,z): x=x(\theta(t)),\ y=y(\theta(t)),\ z=z(t)\}$ with period $T_1=\int_0^{2\pi+\varepsilon\tau}F(x(\theta),y(\theta))d\theta$ when $|\varepsilon|<\varepsilon_0$ and $\lambda=\varepsilon\delta_0$, where $x(\theta),y(\theta),z(t),\tau(\varepsilon),L(x)$ are determined by* (2.15), (2.17) *and* (2.18) *respectively, $v(\varepsilon)\to 0,\ w(\varepsilon)\to 0$ as $\varepsilon\to 0$. Moreover, when $\sigma(B)\cap\{z\in\mathbf{C}:\mathrm{Re}z\geq 0\}=\emptyset$ and λ is sufficiently small, Γ is stable for $\lambda L'(x_0)>0$ and unstable for $\lambda L'(x_0)<0$.*

Example 2.3. Suppose $R\in\mathbf{C}^1$, $a>0$, $eb<0$, the constant matrix B of order n has not any pure imaginary eigenvalue. Then the following system

$$\dot{x}=y(1+a(x^2+y^2))+\lambda bx(x^2+y^2),$$
$$\dot{y}=-x(1+a(x^2+y^2))+\lambda ey(x^2+y^2)^2,$$
$$\dot{z}=Bz+\lambda R(x,y,z)$$

has a unique limit loop near the circle $\{(x,y,0): x^2+y^2=-be^{-1}\}$ for $0<|\lambda|\ll 1$.

Now consider the $2n$-dimension system

$$\dot{x}_k=p_ky_k+\lambda P_k(x,y),$$
$$\dot{y}_k=-p_kx_k+\lambda Q_k(x,y), \tag{2.19$_n$}$$

where $k=1,2,\cdots,n$ $p_k\in\mathbf{Z}-\{0\},\ x=(x_1,x_2,\cdots,x_n),\ y=(y_1,y_2,\cdots,y_n)$. Assume

(H_7) : $P_k,\ Q_k\in\mathbf{C}^1,\ P_k(0,0)=Q_k(0,0)=0,\ k=1,2,\cdots,n$.

Since the method used is basically the same, for simplicity, we assume $n=2$ and

(H_8) : $(p_1,p_2)=1$.

Under the hypothesis (H_8), Theorem 2.3 gives the sufficient condition for the existence of periodic orbit with period sufficiently close to $2\pi p_1^{-1}(2\pi p_2^{-1})$ for $p_1\neq 1(p_2\neq 1)$. Now we consider the existence of periodic orbit with long period sufficiently near 2π.

The solution of system (2.19) with initial value $(x_{10},y_{10},x_{20},y_{20})=(\bar{x}_1+u,0,\bar{x}_2+v,\bar{y}_2+w)$ can be denoted as follows.

$$
\begin{aligned}
x_1(t) &= x_{10}\cos p_1 t + \lambda\int_0^t (P_1\cos p_1(t-s) + Q_1\sin p_1(t-s))ds,\\
y_1(t) &= -x_{10}\sin p_1 t + \lambda\int_0^t (-P_1\sin p_1(t-s) + Q_1\cos p_1(t-s))ds,\\
x_2(t) &= x_{20}\cos p_2 t + y_{20}\sin p_2 t + \lambda\int_0^t (P_2\cos p_2(t-s)) + Q_2\sin p_2(t-s))ds,\\
y_2(t) &= -x_{20}\sin p_2 t + y_{20}\cos p_2 t + \lambda\int_0^t (-P_2\sin p_2(t-s) + Q_2\cos p_2(t-s))ds. \qquad (2.20)
\end{aligned}
$$

The necessary and sufficient condition for (2.20) to be a periodic solution with period $2\pi + \varepsilon\tau$ is that $M_1 = M_2 = M_3 = M_4 = 0$, where

$$
\begin{aligned}
\varepsilon M_1 &= x_1(T) - x_{10}, \quad \varepsilon M_2 = y_1(T),\\
\varepsilon M_3 &= x_2(T) - x_{20}, \quad \varepsilon M_4 = y_2(T) - y_{20},
\end{aligned}
$$

and $T = 2\pi + \varepsilon\tau$.

Let $\lambda = \varepsilon\delta,\ \bar{x} = (\bar{x}_1, \bar{x}_2),\ \bar{y} = (0, \bar{y}_2)$,

$$
\begin{aligned}
\bar{x}(t) &= (\bar{x}_1\cos p_1 t, \bar{x}_2\cos p_2 t + \bar{y}_2\sin p_2 t),\\
\bar{y}(t) &= (-\bar{x}_1\sin p_1 t, -\bar{x}_2\sin p_2 t + \bar{y}_2\cos p_2 t),
\end{aligned}
$$

$$
\begin{aligned}
L_k(\bar{x},\bar{y}) &= \int_0^{2\pi} (P_k(\bar{x}(s),\bar{y}(s))\cos p_k s - Q_k(\bar{x}(s),\bar{y}(s))\sin p_k s)ds,\\
F_k(\bar{x},\bar{y}) &= \int_0^{2\pi} (P_k(\bar{x}(s),\bar{y}(s))\sin p_k s + Q_k(\bar{x}(s),\bar{y}(s))\cos p_k s)ds, \qquad (2.21)
\end{aligned}
$$

$$\tau = \delta p_1^{-1}(\bar{x}_1 + u)^{-1}F_1 + O(\varepsilon), \qquad (2.22)$$

$$\tau_0 = \delta p_1^{-1}\bar{x}_1^{-1}F_1(\bar{x},\bar{y}).$$

When $\varepsilon = u = v = w = 0,\ \tau = \tau_0,\ \delta = \delta_0 \neq 0$, and take $\bar{x}, \bar{y}$ such that

$$L_1(\bar{x},\bar{y}) = 0,\ p_1\bar{x}_1 L_2 + p_2\bar{y}_2 F_1 = 0,\ p_1\bar{x}_1 F_2 - p_2\bar{x}_2 F_1 = 0, \qquad (2.23)$$

we have $M_i = 0$ for $i = 1, 2, 3, 4$.

Denote $D(\varepsilon, u, v, w) = \delta^{-3}p_1^2\bar{x}_1^2\frac{\partial(M_1,M_3,M_4)}{\partial(u,v,w)}$, $D = D(0,0,0)$.

Theorem 2.4. *Suppose hypotheses* (H_7) *and* (H_8) *hold. If there are* $\bar{x}_1 > 0$ *and* $\bar{x}_2, \bar{y}_2$ *such that (2.23) is valid and* D *is invertible, then, for any given* $\delta_0 \neq 0$*, There exist* $\varepsilon_0 > 0$ *and a unique set of functions* $u(\varepsilon), v(\varepsilon), w(\varepsilon), \tau(\varepsilon)$*, such that (2.19) has a limit loop passing through* $(\bar{x}_1 + u, 0, \bar{x}_2 + v, \bar{y}_2 + w)$ *with period* $2\pi + \varepsilon\tau$ *when* $|\varepsilon| < \varepsilon_0,\ \lambda = \varepsilon\delta_0$*, where* $u(\varepsilon) \to 0,\ v(\varepsilon) \to 0,\ w(\varepsilon) \to 0$ *and* $\tau(\varepsilon) \to \tau_0 = \delta_0 p_1^{-1}\bar{x}_1^{-}F_1$*, as* $\varepsilon \to 0$.

Example 2.4. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= 2y_1 + \lambda(x_1 r^2 - x_1 r^4 + a_1 x_2 + b_1 y_2),\\
\dot{y}_1 &= -2x_1 + \lambda(y_1 r^2 - y_1 r^4 + a_2 x_2 + b_2 y_2),\\
\dot{x}_2 &= 3y_2 + \lambda(a x_2 + b y_2 + a_3 x_1 + b_3 y_1),\\
\dot{y}_2 &= -3x_2 + \lambda(c x_2 + d y_2 + a_4 x_1 + b_4 y_1), \qquad (2.24)
\end{aligned}
$$

where $r^2 = x_1^2 + y_1^2$, a_i, b_i are constants, $(a+d)^2 + (b-c)^2 \neq 0$.

We have

$$L_1(x,y) = 2\pi x_1^3(1 - x_1^2),$$
$$L_2(x,y) = \pi((a+d)x_2 + (b-c)y_2),$$
$$F_1(x,y) = 0,$$
$$F_2(x,y) = \pi(-(b-c)x_2 + (a+d)y_2).$$

If we take $\bar{x}_1 = 1, \bar{x}_2 = \bar{y}_2 = 0$, then (2.23) holds and

$$\det D = -4\pi^3 p_1^2((a+d)^2 + (b-c)^2) \neq 0.$$

When $|\varepsilon| \ll 1$, by Theorem 2.4, there exists functions $u(\varepsilon), v(\varepsilon), w(\varepsilon)$ and $\tau(\varepsilon)$, such that system (2.24) has a periodic orbit Γ_ε with initial point $(1 + u(\varepsilon), 0, v(\varepsilon), w(\varepsilon))$ and period $2\pi + \varepsilon\tau$, and

$$\Gamma_\varepsilon \to \Gamma_0 : \{(x,y) : x_1^2 + y_1^2 = 1,\ x_2 = y_2 = 0\} \text{ as } \varepsilon \to 0.$$

Remark 2.2. Example 2.4 tells us that, similar to the limit loop bifurcation taking place in Theorem 2.3, the limit loop bifurcation with long period sufficiently close to 2π can also be produced by the interaction of the closed orbit bifurcation which is confined to the $x - y$ plane and the Hopf bifurcation in the $(2n-2)$-dimensional complementary space.

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