Chin. Ann. of Math.
14B: 2(1993),129–152.

VARIATIONAL FORMULATION OF STEADY FLOW IN A HOPPER

AN LIANJUN*

Abstract

A variational problem about maximal stable arches in a hopper is formulated. This problem idealizes an industrial problem related to guaranteering reliable flow of material out of a storage silo. To obtain existence, the generalized function spaces are introduced and studied. Specifically, functions in the spaces can be discontinuous in the interior of the domain as well as along the boundaries. For the von Mises type of material in two dimension, the limit load is estimated and its asymptotic behavior is investigated.

§0. Introduction

For flow of granular material under gravity in a hopper (Figure 0.1), information about the moment of collapse under increasing loads is very important for the design of a hopper. In this paper, we shall investigate this problem. Assume that the material is rigid-perfectly plastic. Based on the limit analysis, we shall formulate two variational problems, which we shall call the stress problem and the strain problem. Then we shall investigate solutions of these variational problems.

To study these problems, appropriate function spaces should be chosen, especially for the strain problem. Physically, deformations can be discontinuous inside plastic material. Mathematically, only an L^1 estimate can be obtained for the minimizing sequence in the strain problem. But the unit ball in L^1 is not compact either in the norm topology or in the weak topology. Strang[19] and Temam[17] introduced a new function space to allow strain discontinuities in the interior of a domain. Functions in this space assume their boundary values in a continuous fashion. We shall extend their function space to one whose functions are defined on the closure of a domain so that boundary can also be treated. Specifically, velocities are summable functions on the domain, and the entries of the strain rate tensor are bounded measures on the closure of the domain. In this space, functions can be discontinuous in the interior of the domain as well as the boundaries and the unit ball is compact in the weak topology.

Most previous work[1,3,17,19] on the variational formulation of plastic problems dealt with pressure insensitive material which is incompressible. When velocity discontinuities occur along a surface, the normal component of velocity along the surface remains continuous. But a granular material is pressure sensitive. Its deformation is accompanied by a change of volume, and a tangential discontinuity of velocity along a surface is accompanied by a

Manuscript received August 17, 1992.
*Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1 .

normal discontinuity. In general, abrupt thinning or abrupt thickening could happen near a slip surface.

In section 1, two variational problems are formulated. The stress problem is given in terms of statically admissible stress fields (§1.2); the strain problem is given in terms of kinematically admissible velocity fields (§1.4 and §1.5). These two problems are dual to each other under the Legendre transform and the duality relation is consistent with an associative flow rule. In §1.3, we also discuss the yield condition used for granular material.

In section 2, we shall study the generalized function spaces. A function in the space $\Sigma(\Omega)$, used for the stress problem, has L^∞ trace on the boundary and C^∞ functions are dense in some weak topology in this space (§2.1). A function in the space $\widetilde{BD}(\Omega)$, used for the strain problem, has a trace which is only a measure on the boundary (§2.2). Different topologies are discussed for the space $\widetilde{BD}(\Omega)$ as well as its subspace $\widetilde{BD}_0(\Omega)$ (§2.2.2). A generalized Green formula in the space $\Sigma(\Omega)\times\widetilde{BD}_0(\Omega)$ is proved. It is given (§2.3) that the product of a stress tensor and a strain rate tensor depends continuously on stress and strain rate tensors in the weak topology.

In section 3, existence of both problems are proved. Existence of the stress problem follows from the minimax theorem (§3.1). Existence of the strain problem is derived from the minimizing sequence which is weakly compact in the space $\widetilde{BD}(\Omega)$ (§3.2). Under a regularity assumption on the solution of the strain problem, the extremality relation between solutions of both problems are attained and both problems give us the same limit load (§3.3). Also, the feature of velocity discontinuity in granular material is addressed (§3.4).

In section 4, some explicit solutions are given in some cases and their asymptotic behavior is investigated. When gravity is assumed to be in the radial direction, exact solutions of both problems are found and the limit load is given in terms of parameters of both the geometry and the material (§4.1). Specifically, how the maximal stress depends asymptotically on the height of the hopper is estimated. When gravity is vertical, choosing specific stress and velocity, we obtain a lower bound and a upper bound of the limit load, and conclude that, in this case, the limit load has the same asymptotic behavior as in the radial case (§4.2). Also we investigate a linearly ill-posed free-boundary problem related to the stress problem (§4.3).

§1. Variational formulas

1.1. Notation

In what follows, some of the notations we shall use in this section are listed for reference.

(1) E: the space of symmetric tensors of order n;

(2) E^D: the subspace of E consisting of the tensors whose trace is zero;

(3) $|\xi|=\{\xi\cdot\xi\}^{\frac{1}{2}}=\left\{\sum\limits_{i,j=1}^{n}\xi_{ij}\xi_{ij}\right\}^{\frac{1}{2}}$: the norm of a tensor $\xi=(\xi_{ij})$;

(4) $u=(u_1,u_2,\dots,u_n)$: the velocity vector;

(5) $\sigma=(\sigma_{ij})$: the $n\times n$ symmetric stress tensor;

(6) $\xi^D = (\xi^D_{ij})$: the deviator of the tensor ξ,

$$\xi^D_{ij} = \xi_{ij} - \frac{1}{n}\xi_{kk}\delta_{ij} \qquad (\delta_{ij} \text{ the Kronecker delta});$$

(7) $\mathrm{tr}\sigma$: the trace of the tensor σ;

(8) $\epsilon(u) = (\epsilon_{ij}(u))$: the $n \times n$ symmetric strain rate tensor associated with u, which is given by

$$\epsilon_{ij}(u) = -\frac{1}{2}(\partial_i u_j + \partial_j u_i);$$

(9) $\sigma.\epsilon(u) = \sum_{i,j=1}^{n} \sigma_{ij}\epsilon_{ij}$: the scalar product of two tensors;

(10) $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$: the outward unit vector normal to a surface;

(11) $\sigma.\nu = \sum_{j=1}^{n} \sigma_{ij}\nu_j$: the surface traction;

(12) $(\sigma.\nu)_\nu = \sum_{i,j=1}^{n} \sigma_{ij}\nu_i\nu_j$: the normal traction component of $\sigma.\nu$;

(13) $(\sigma.\nu)_\tau = \sigma.\nu - (\sigma.\nu)_\nu\nu$: the tangential traction component of $\sigma.\nu$.

In our paper, we define the stress tensor in the compressive sense, which is convenient for studying granular material (a granular material can support only compressive stresses). Correspondingly, we put a minus sign in the definition of strain rate tensor.

1.2. Formulation of the stress problem

Let Ω be the region inside the hopper whose boundary consists of a lateral wall (two lateral walls in 2-dimensions), part of spherical surface with radius R on top (circular arc in 2-dimensions) and part of spherical surface with radius r_0 on bottom (circular arc in 2-dimensions). Suppose that a body force f, gravity in practice, is given in Ω. Even if the wall of the hopper is smooth, cohesive material in the hopper may remain at rest when the width of the exit is small. But as the width increases, the material will be unsupportable, and collapse will occur. Instead of finding the largest exit width such that collapse does not occur, we try to find the largest multiplier λ^* of f such that $\lambda^* f$ can be balanced by admissible stresses while the width of the exit is fixed. The latter is so called the problem of limit analysis. The stress problem will be set up based on the static principle of limit analysis in this subsection.

For rigid-perfectly plastic material, this collapse multiplier does not depend on the loading history[9]. Hence it is possible to find the largest multiplier λ^* of f without following the development in time.

The rigid-perfectly plastic structure means that the stress tensor σ must belong to a closed subset $B(x)$ of E at every point of Ω. When $\sigma \in \overset{\circ}{B}(x)$, the set of interior points of $B(x)$, the material is at the state of rigid motion and no deformation occurs. When σ reaches the boundary ∂B of B, the material begins to yield and plastic deformation occurs. Incidentally, slip between surfaces may happen in the region of plastic deformation.

Therefore, our stress problem can be formulated as follows

$$\sup\left\{\lambda : \begin{array}{ll} \exists\sigma \in K(\Omega) \text{ s.t. } \mathrm{div}\sigma = \lambda f \text{ in } \Omega; & \sigma.\nu = 0 \text{ on the bottom,} \\ |(\sigma.\nu)_\tau| \le \mu(\sigma.\nu)_\nu \text{ on the sides;} & (\sigma.\nu)_\nu \ge 0 \text{ on the top} \end{array}\right\},$$

where $\mathrm{div}\sigma = \nabla \cdot \sigma = \sum_{i=1}^{n} \partial_i \sigma_{ij}$, and μ the coefficient of friction between the material and the wall of the hopper and

$$K(\Omega) = \left\{ \sigma \in E : \sigma \in B(x) \quad \text{a.e. } x \in \Omega \right\}$$

is the yield set whose choice depends on the material (§1.3). The equation $\mathrm{div}\sigma = \lambda f$ comes from the conservation of momentum. The boundary conditions on the bottom come from the fact that no surface forces are imposed on it. Since a granular material is, unlike metal, not ductile and can support only compressive stresses, we have

$$(\sigma.\nu)_\nu \geq 0$$

along any surface. In particular, $(\sigma.\nu)_\nu \geq 0$ on the top. The conditions on the sides express the simple mechanics model for friction. If the wall of the hopper is smooth, then the friction coefficient $\mu = 0$ and the conditions on the sides become

$$(\sigma.\nu)_\nu \geq 0, \qquad (\sigma.\nu)_\tau = 0. \tag{1.1}$$

1.3. The yield condition

As we know, the yield condition tells us when the internal stresses can support the external force so that no plastic deformation occurs and when a material begins to yield and possibly collapses. So the type of yield set we should take depends on the nature of the material we are dealing with. Here we shall discuss the characteristics of granular material, and give the yield sets adopted in our paper.

When we study pressure-insensitive materials, like metals, the yield set can be written as

$$K = K^D \oplus \mathbb{R}I, \tag{1.2}$$

where K^D is a non-vacuous closed convex subset of E^D. In other words, when the material yields, the yield strength $|\sigma^D|$ is independent of the total pressure $\mathrm{tr}\sigma$, the trace of stress tensor. If we assume that the plastic deformation takes place in the normal direction of the yield surface ∂K (the associative flow rule), we have

$$\sum_i \epsilon_{ii}(u) = -\mathrm{div}u = 0.$$

So the material is incompressible[4]. But a granular material, based on the experimental facts[15,16], is pressure-sensitive. The yield set cannot be written in the form (1.2). The yield strength σ^D depends on the total pressure. In terms of the associative flow rule, which will be discussed in § 1.5, a pressure-sensitive material undergoes volume changes. As a result, a tangential discontinuity is accompanied by a normal discontinuity along the discontinuity surface in velocity (see §3.4).

In our paper, only cohesive materials are considered. So the yield set $B(x)$ must contain a neighborhood of 0 in E for almost all $x \in \Omega$. In this case, from the following equivalent form of the original stress problem

$$\sup \left\{ \lambda : \begin{array}{l} \exists \sigma \text{ s.t. } \lambda\sigma \in K(\Omega) \text{ and } \mathrm{div}\sigma = f \text{ in } \Omega, \quad \sigma.\nu = 0 \text{ on bottom,} \\ |(\sigma.\nu)_\tau| \leq \mu(\sigma.\nu)_\nu \text{ on sides,} \quad (\sigma.\nu)_\nu \geq 0 \text{ on top} \end{array} \right\},$$

we can obtain that, when there is a $\sigma \in L^\infty(\Omega, E)$ satisfying the equation and the boundary conditions, we can choose $\lambda > 0$ such that $\lambda\sigma \in K(\Omega)$. As expected, a cohesive material could stay at rest.

If a yield function $F : E \times \Omega \to I\!R$ is introduced, then the yield set can be expressed as

$$K(\Omega) = \{\sigma : F(\sigma, x) \leq 0 \quad \text{a.e. } x \in \Omega\}.$$

The yield function we are considering in this paper has the following form

$$F = f\left(\sigma^D\right) - k\left(\frac{\text{tr}\sigma}{n} + c\right), \tag{1.3}$$

where $f\left(\sigma^D\right)$ satisfies (1) $f\left(\sigma^D\right) \geq 0$ for all $\sigma^D \in E^D$, (2) $f(0) = 0$, (3) f is convex, (4) f is homogeneous of degree one in σ^D. The scalars k and c, related to the internal friction and the cohesion of the material respectively, are given positive functions of x (deformation history may not the same for different points). In fact, if $c = 0$, the yield set does not contain a neighborhood of 0 in E, and such a material is not cohesive. Note that the yield set (1.3) is a cone in E. Yield functions of the form (1.3) include the Coulomb type in the two dimensional case, the von Mises type and the Tresca type in the three dimensional case[10]. For instance, for the von Mises yield condition,

$$f(\sigma^D) = |\sigma^D|. \tag{1.4}$$

Remark 1.1. Another type of yield condition that can be used for granular material comes from Critical State Soil Mechanics[16], in which the yield set is a bounded convex set in E. It would be better to include hardening effects in this type of model. But it is difficult to formulate them in variational forms. We will study these types of models in further publications.

1.4. Derivation of the strain problem

The strain problem, which is formulated in terms of kinematically admissible velocity fields, can be found through a minimax theorem. The variables u and σ are connected by a Legendre transformation.

From §1.2, the admissible fields are denoted by $\Phi = \bigcup_{\lambda > 0} \Phi_\lambda$, where

$$\Phi_\lambda = \left\{\sigma : \begin{array}{ll} \text{div}\sigma = \lambda f \text{ in } \Omega, & |(\sigma.\nu)_\tau| \leq \mu(\sigma.\nu)_\nu \text{ on sides,} \\ \sigma.\nu = 0 \text{ on bottom,} & (\sigma.\nu)_\nu \geq 0 \text{ on top} \end{array}\right\}.$$

From the following argument, we shall see that the admissible velocity fields would be

$$\Psi = \left\{u : \int_\Omega f u \, dx = 1, \quad \begin{array}{l} u_\nu \leq 0, \ u_\tau = 0 \text{ on top} \\ u_\nu + \mu|u_\tau| \leq 0 \text{ on sides} \end{array}\right\}.$$

We define a Lagrange function $L(\sigma, u)$ as

$$L(\sigma, u) = \int_\Omega \sigma.\epsilon(u) \, dx.$$

We claim that the stress problem can be written as a maxmin

$$\sup_{\lambda \in \Lambda} \{\lambda\} = \sup_{\sigma \in K} \inf_{u \in \Psi} L(\sigma, u),$$

where $\Lambda = \{\lambda : \ \exists \sigma \in K \cap \Phi_\lambda\}$. By Green's formula, we have

$$\sup_{\lambda \in \Lambda} \{\lambda\} = \sup_{\sigma \in K} \inf_{u \in \Psi} \left\{\int_\Omega u.\text{div}\sigma \, dx - \int_{\partial\Omega} (\sigma.\nu).u \, ds\right\}. \tag{1.5}$$

For the integral over Ω, it is easy to see that the inner minimum is $-\infty$ unless $\operatorname{div}\sigma$ is proportional to f.

$$\inf\left\{\int_\Omega u.\operatorname{div}\sigma\,dx, \qquad \int_\Omega f\,u\,dx=1\right\}=\begin{cases}\lambda & \text{if } \operatorname{div}\sigma=\lambda f,\\ -\infty & \text{otherwise.}\end{cases}$$

For the integral over $\partial\Omega$,

$$\inf\left\{-\int_{\partial\Omega}(\sigma.\nu).u\,dx: \begin{array}{l} u_\nu\le 0, u_\tau=0 \text{ on top}\\ u_\nu+\mu|u_\tau|\le 0 \text{ on sides}\end{array}\right\}$$

$$=\begin{cases}0 & \text{if } \sigma.\nu=0 \text{ on bottom, } (\sigma.\nu)_\nu\ge 0 \text{ on top}\\ & \qquad \text{and } |(\sigma.\nu)_\tau|\le\mu(\sigma.\nu)_\nu \text{ on sides,}\\ -\infty & \text{otherwise.}\end{cases}$$

Therefore,

$$\inf_{u\in\Psi}\left\{\int_\Omega u.\operatorname{div}\sigma\,dx-\int_{\partial\Omega}(\sigma.\nu).u\,ds\right\}$$

$$=\begin{cases}\lambda & \text{if } \sigma\in\Phi,\\ -\infty & \text{otherwise.}\end{cases}$$

The outer minimum of (1.5) is obtained only in the case $\sigma\in\Phi$. The proof of the claim is complete.

Now, the strain problem is defined as a minmax.

$$\inf_{u\in\Psi}\int_\Omega D(u)\,dx=\inf_{u\in\Psi}\sup_{\sigma\in K}L(\sigma,u), \tag{1.6}$$

where $D(u)=\sup\limits_{\tau\in K}\tau\cdot\epsilon(u)$. More precisely,

$$\inf\left\{\int_\Omega D(u)\,dx: \int_\Omega f\,u\,dx=1, \quad \begin{array}{l} u_\nu\le 0, u_\tau=0 \text{ on top}\\ u_\nu+\mu|u_\tau|\le 0 \text{ on sides}\end{array}\right\},$$

where $D(u)$ can be expressed explicitly when a proper yield set is chosen (§1.6).

Essentially, the stress σ in the stress problem is related to the velocity u in the strain problem through a Legendre transformation; the duality relation. If we can prove

$$\sup_{\sigma\in K}\inf_{u\in\Psi}L(\sigma,u)=\inf_{u\in\Psi}\sup_{\sigma\in K}L(\sigma,u),$$

then we can find the largest multiplier by solving the strain problem, the dual of the stress problem. But the order of optimization is not always reversible. A "duality gap" exists in some situations[13]. Also whether or not the extreme solution can be attained depends on the choice of function space. We shall answer these questions in sections 2 and 3.

Remark 1.2. In mechanics, the strain rate tensor is usually found by using its relation to the stress tensor at yield — the plastic potential flow rule. It can be proved that the duality relation discussed in the last subsection is equivalent to the associative flow rule in plasticity.

1.5. Explicit formula for $D(u)$

By definition, the dissipation function

$$D(u)=\sup_{\sigma\in K}\sigma.\epsilon(u).$$

$$= \sup_{\sigma \in K} \left\{ \frac{1}{n} \mathrm{tr}\epsilon(\mathrm{tr}\sigma + cn) - c\mathrm{tr}\epsilon + \epsilon^D . \sigma^D \right\}.$$

Let $s = \frac{\mathrm{tr}\sigma}{n} + c$, then the yield condition can be written $f(\sigma^D) \le ks$, $s \ge 0$. Therefore,

$$D(u) = \sup_{s \ge 0} \{ \mathrm{tr}\epsilon\, s + \sup_{f(\sigma^D) \le ks} \sigma^D \cdot \epsilon^D \} - c\mathrm{tr}\epsilon$$

$$= \sup_{s \ge 0} \{ \left(\mathrm{tr}\epsilon + k \sup_{f(\sigma^D)=1} \sigma^D \cdot \epsilon^D \right) s \} - c\mathrm{tr}\epsilon,$$

here the homogeneity and convexity of $f(\sigma^D)$ are used. Thus

$$D(u) = \begin{cases} -c\,\epsilon_{ii} & \text{if } \epsilon_{ii} + k \sup_{f(\sigma^D)=1} \epsilon_{ij}^D \cdot \sigma_{ij}^D \le 0, \\ +\infty & \text{if } \epsilon_{ii} + k \sup_{f(\sigma^D)=1} \epsilon_{ij}^D \cdot \sigma_{ij}^D \le 0. \end{cases}$$

In order to get the infimum of the integral

$$\int_\Omega D(u)\, dx \qquad \forall u \in \Psi,$$

admissible strain rates must satisfy

$$K^* = \left\{ \epsilon : \epsilon_{ii} \ge k \sup_{f(\sigma^D)=1} \epsilon^D \cdot \sigma^D \right\}$$

which is the dual cone of the yield set K. It implies that under the yield condition in our paper, plastic deformation is always accompanied by dilation. It follows from the formula

$$\inf_u \left\{ c \int_\Omega \mathrm{div} u\, dx : \mathrm{div} u \ge k \sup_{f(\sigma^D)=1} \epsilon^D \cdot \sigma^D,\ u \in \Psi \right\} \tag{1.7}$$

that, physically, the collapse solution of the strain problem corresponds to the least dilation in volume.

Proposition 1.1. *In the space* $\left\{ \epsilon \in E : \ -\mathrm{tr}\epsilon \ge k \sup_{f(\sigma^D)=1} \epsilon^D \cdot \sigma^D \right\}$, *there exist positive constants* C_1 *and* C_2, *such that*

$$C_2 |\epsilon| \le D(\epsilon) \le C_1 |\epsilon|. \tag{1.8}$$

The proof of Proposition 1.1 is a straightforward observation of our formulation (1.7).

Remark 1.3. For other yield sets, for instance, a bounded yield set in E, the material could (1) dilate (2) consolidate (3) neither dilate nor consolidate. Depending on the corresponding state of stress σ, if the projection of the normal direction at σ of the yield surface on the trace direction of stresses is positive, the material undergoes consolation; if the projection is negative, the material undergoes dilation; if the projection is zero, the material are neither dilate nor consolidate. Incidentally, consolation makes the material stronger and dilation makes the material weaker. This point can be recognized from the formulation of the strain problem for a material with a bounded yield condition.

§2. The Generalized Spaces and Green's Formula

Throughout this section, we assume that Ω is open, bounded domain with loclly Lipschitz continuous boundary.

2.1. The space $\Sigma(\Omega)$

The space

$$\Sigma(\Omega)=\left\{\sigma\in L^\infty(\Omega;E):\operatorname{div}\sigma\in L^p(\Omega)^n\quad(p>n)\right\}$$

with the norm

$$\|\sigma\|_{\Sigma(\Omega)}=\|\sigma\|_{L^\infty(\Omega;E)}+\|\operatorname{div}\sigma\|_{L^p(\Omega)^n}$$

is a Banach space. The trace operator is established in the following proposition and the density of C^∞ functions in $\Sigma(\Omega)$ with a weak topology is proved.

Proposition 2.1. *There exists a continuous linear operator, $\gamma_\nu:\Sigma(\Omega)\to L^\infty(\partial\Omega)$ such that*

$$\gamma_\nu(\sigma)=\sigma.\nu\big|_{\partial\Omega}\qquad\text{for all }\sigma\in\Sigma(\Omega)\cap C(\bar\Omega;E).\tag{2.1}$$

Green's formula

$$\langle\gamma_\nu(\sigma),u\rangle=\int_\Omega\operatorname{div}\sigma.u\,dx+\int_\Omega\sigma.\operatorname{grad}u\,dx$$

holds for every $u\in W^{1,1}(\Omega)^n$.

Proof. Since there exists[7] a lifting operator $\ell:L^1(\partial\Omega)^n\to W^{1,1}(\Omega)^n$, $\gamma_\nu(\sigma)$ can be defined by

$$\langle\gamma_\nu(\sigma),\phi\rangle=\int_\Omega\operatorname{div}\sigma.\ell(\phi)\,dx+\int_\Omega\sigma.\operatorname{grad}\ell(\phi)\,dx$$

for all $\phi\in L^1(\partial\Omega)^n$ and fixed $\sigma\in\Sigma(\Omega)$. Clearly, since $\ell(\phi)\in W^{1,1}(\Omega)^n$ and

$$\|\ell(\phi)\|_{L^{\frac{n}{n-1}}(\Omega)^n}\le\|\ell(\phi)\|_{W^{1,1}(\Omega)^n}\le C\|\phi\|_{L^1(\partial\Omega)^n},$$

we have

$$\begin{aligned}|\langle\gamma_\nu(\sigma),\phi\rangle|\le&C\|\operatorname{div}\sigma\|_{L^p(\Omega)^n}\|\ell(\phi)\|_{L^{\frac{n}{n-1}}(\Omega)^n}+\|\sigma\|_{L^\infty(\Omega;E)}\|\operatorname{grad}\ell(\phi)\|_{L^1(\Omega)^n}\\ \le&C\|\phi\|_{L^1(\partial\Omega)^n}\|\sigma\|_{\Sigma(\Omega)}.\end{aligned}$$

Thus $\gamma_\nu:\Sigma(\Omega)\to L^\infty(\partial\Omega)$ is a bounded linear map.

If σ is a $C^1(\bar\Omega;E)$ function, Green's formula implies that

$$\int_{\partial\Omega}(\sigma.\nu).\phi\,dx=\int_\Omega\operatorname{div}\sigma.u\,dx+\int_\Omega\sigma.\operatorname{grad}u\,dx$$

for a $C^1(\bar\Omega;E)$ function u and a $u|_{\partial\Omega}=\phi$. Since $u_0=u-\ell(\phi)\in W_0^{1,1}(\Omega)$ and

$$\int_\Omega\operatorname{div}\sigma.u_0\,dx+\int_\Omega\sigma.\operatorname{grad}u_0\,dx=0$$

for all $u_0\in W_0^{1,1}(\Omega)$, it follows that

$$\int_{\partial\Omega}(\sigma.\nu).\phi\,dx=\langle\gamma_\nu(\sigma),\phi\rangle.$$

Now the functions ϕ which are restrictions of $C(\bar\Omega;E)$ are dense in $L^1(\partial\Omega)^n$. So it follows that

$$\gamma_\nu(\sigma)=\sigma.\nu\big|_{\partial\Omega}\qquad\text{for all }\sigma\in\Sigma(\Omega)\cap C^1(\bar\Omega;E).$$

By an approximation process, (2.1) follows.

Proposition 2.2. *Let Ω be a bounded set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For any $\sigma \in \Sigma(\Omega)$, there exists a sequence $\{\sigma_m\}$ in $C^\infty(\bar\Omega; E)$ which satisfies*

$$\|\sigma_m\|_{L^\infty(\Omega;E)} \leq \|\sigma\|_{L^\infty(\Omega;E)},$$
$$\|\sigma_m.\nu\|_{L^\infty(\partial\Omega)^n} \leq C\|\gamma_\nu(\sigma)\|_{L^\infty(\partial\Omega)^n}$$

and as $m \to +\infty$,

$$\begin{aligned}
&\mathrm{div}\sigma_m \to \mathrm{div}\sigma \quad && \text{in } L^p(\Omega)^n,\\
&\sigma_m \to \sigma && \text{weakly-star in } L^\infty(\Omega;E),\\
&\sigma_m.\nu \to \gamma_\nu(\sigma) && \text{weakly-star in } L^\infty(\partial\Omega)^n,\\
&\sigma_m \to \sigma && \text{in } L^q(\Omega;E) \text{ for any } 1 \leq q < \infty.
\end{aligned}$$

Proof. The proof is basically the same as in [17].

Remark 2.1. The trace $\gamma_\nu(\sigma)$ is weakly-star continuous in $L^\infty(\partial\Omega)$ with respect to

$$\begin{aligned}
&\mathrm{div}\sigma_m \to \mathrm{div}\sigma \quad && \text{weakly in } L^p(\Omega)^n,\\
&\sigma_m \to \sigma && \text{weakly-star in } L^\infty(\Omega;E).
\end{aligned}$$

2.2. The spaces of $\widetilde{BD}(\Omega)$ and $\widetilde{BD}_0(\Omega)$

2.2.1 Definition

In order to obtain collapse solutions for the strain problem, the space of velocities must be generalized. In fact, in plastic deformation, the velocity may be discontinuous in the interior of Ω as well as along the boundaries. To handle discontinuities in u inside Ω, Temam and Strang[18] have introduced and studied

$$BD(\Omega) = \left\{u \in L^1(\Omega)^n : \epsilon_{ij}(u) \in M(\Omega)\right\},$$

where $M(\Omega)$ is the space of bounded measures on Ω. For a discontinuity in u occurring along Γ, a part of the boundary $\partial\Omega$, they added the term

$$\int_\Gamma \sigma.\nu.(\gamma_0(u) - u_\Gamma)ds$$

in the formula for the strain problem to include the corresponding dissipation work, where u_Γ is a given function on Γ and $\gamma_0(u) \in L^1(\partial\Omega)$ is the trace of a BD function u. But in our strain problem, the boundary conditions are given in inequality forms and the value u_Γ on the part of boundary needs to be determined as a part of the solution. Therefore, the space BD is not convenient for us to study the problem.

We generalize the space BD as the following

$$\widetilde{BD}(\Omega) = \left\{u \in L^1(\Omega)^n : \Lambda_{ij}(u) \in M(\bar\Omega)\right\}$$

with the norm

$$\|u\|_{\widetilde{BD}(\Omega)} = \|u\|_{L^1(\Omega)^n} + \|\Lambda(u)\|_{M(\bar\Omega)}.$$

Instead of $\epsilon_{ij}(u)$ being a measure over Ω, we use $\Lambda_{ij}(u)$, a measure over $\bar\Omega$, the closure of Ω. Note that $M(\bar\Omega) = C(\bar\Omega)^*$ and $M(\Omega) = C_0(\Omega)^*$. For any $\Lambda_{ij} \in M(\bar\Omega)$, which equals ϵ_{ij} (in the distribution sense) in the interior of Ω, the following difference

$$\left\{\langle\Lambda_{ij}, \phi\rangle_{C^*\times C} - \langle\epsilon_{ij}, \phi\rangle_{C_0^*\times C}\right\}, \quad \forall\phi \in C(\bar\Omega)$$

defines a bounded measure on $\partial\Omega$. Hence

$$\widetilde{BD}(\Omega) = BD(\Omega) \times M(\partial\Omega).$$

In fact, for given functions $u \in BD(\Omega)$ and $v \in M(\partial\Omega)$, the corresponding $\Lambda(u)$ is given by

$$\langle \Lambda_{ij}(u), \phi \rangle_{C^* \times C} = \int_\Omega \epsilon_{ij}(u)\,\phi + \int_{\partial\Omega} \mathcal{J}_{ij}(\gamma_0(u) - v)\phi$$

where $\gamma_0(u) \in L^1(\partial\Omega)$ is the trace of the BD function u[17] and

$$\mathcal{J}_{ij}(p) = \frac{1}{2}(p_i\nu_j + p_j\nu_i). \tag{2.2}$$

Conversely, for any $u \in \widetilde{BD}(\Omega)$, we have

$$\langle \epsilon_{ij}, \phi \rangle = \langle \Lambda_{ij}, \phi \rangle \qquad \forall \phi \in C_0(\Omega),$$

and the bounded measure on the boundary is given by

$$\int_{\partial\Omega} \mathcal{J}_{ij}(v)\phi = \int_{\partial\Omega} \mathcal{J}_{ij}(\gamma_0(u))\phi - \int_\Omega \Lambda_{ij}(u)\phi + \int_\Omega \epsilon_{ij}(u)\phi, \qquad \forall \phi \in C(\bar{\Omega}). \tag{2.3}$$

We shall take (2.3) as a definition of the trace $\gamma(u)$ of a $\widetilde{BD}$ function u (replace v by $\gamma(u)$ in (2.3)). Consequently, the norm of a $\widetilde{BD}(\Omega)$ function also can be written as

$$\|u\|_{\widetilde{BD}(\Omega)} = \|u\|_{BD(\Omega)} + \|\mathcal{J}(\gamma_0(u) - \gamma(u))\|_{M(\partial\Omega)}.$$

For further use, we also define two subspaces $\widetilde{BD}_0(\Omega)$ and $LD(\Omega)$ of $\widetilde{BD}(\Omega)$.

$$\widetilde{BD}_0(\Omega) = \left\{ u \in \widetilde{BD}(\Omega), \gamma(u) \in L^1(\partial\Omega) \right\},$$

$$LD(\Omega) = \left\{ u_i \in L^1(\Omega), \epsilon_{ij}(u) \in L^1(\Omega) \right\}.$$

Note that the norm of $LD(\Omega)$

$$\|u\|_{L^1} + \Sigma_{ij}\|\epsilon_{ij}(u)\|_{L^1(\Omega)}$$

is the same as the norm of $BD(\Omega)$ for $u \in LD(\Omega) \subset \widetilde{BD}(\Omega)$. A generalized Green's formula holds in $\Sigma(\Omega) \times \widetilde{BD}_0(\Omega)$ but not in $\Sigma(\Omega) \times \widetilde{BD}(\Omega)$ (§2.3). The disadvantage of using the spaces LD and $\widetilde{BD}_0$ is that a unit ball in $\widetilde{BD}_0(\Omega)$ and $LD(\Omega)$ is not weakly-star compact.

2.2.2. Properties

In this subsection, we shall study the trace operator and the compactness in the space $\widetilde{BD}(\Omega)$ under different topologies.

Proposition 2.3. *The trace γ defined by (2.3) is a bounded measure on $\partial\Omega$, satisfying*

$$\gamma(u) = u\big|_{\partial\Omega} \qquad \text{for all } u \in \widetilde{BD}(\Omega) \cap C(\bar{\Omega})^n.$$

The map $\gamma : \widetilde{BD}(\Omega) \to M(\partial\Omega)$ is a continuous linear map. Furthermore, the map γ is weakly-star continuous in $M(\partial\Omega)$, i.e., $\int_{\partial\Omega} \gamma(u)\phi$ is continuous for all $\phi \in C(\partial\Omega)$, with a weak topology in $\widetilde{BD}(\Omega)$, where the weak topology is the one determined by the norm $\|u\|_{L^1(\Omega)^n}$ and the family of seminorms

$$\left| \int_\Omega \phi\Lambda_{ij}(u) \right| \qquad \text{for all } \phi \in C(\bar{\Omega}), \qquad i, j = 1, \cdots, n.$$

Proof. For any $\phi \in C(\partial\Omega)$, there exists an extension function $\Phi \in C(\bar{\Omega})$ satisfying $\Phi\big|_{\partial\Omega} = \phi$ and

$$\|\Phi\|_{C(\bar{\Omega})} \le C\|\phi\|_{C(\partial\Omega)}. \tag{2.4}$$

Thus, using $\|\mathcal{J}_{ij}(\gamma_0(u))\|_{L^1(\partial\Omega)} \leq C\|u\|_{BD(\Omega)}$ (cf.[17]) , we have

$$\left|\int_{\partial\Omega} \mathcal{J}_{ij}(\gamma_0(u))\phi\, dx - \int_{\Omega} \Lambda_{ij}(u)\Phi\right| \leq C(\|u\|_{BD(\Omega)} + \|u\|_{\widetilde{BD}(\Omega)})\|\Phi\|_{C(\bar{\Omega})}. \quad (2.5)$$

Now looking at the third term in (2.3), for any $\delta > 0$, choose $\psi \in C_0(\Omega)$, $\psi = 1$ on Ω_1 where $\bar{\Omega}_1 \subset \Omega$ such that

$$\left|\int_{\Omega} \epsilon_{ij}(u)\Phi(1-\psi)\right| \leq \delta.$$

Then we have

$$\left|\int_{\Omega} \epsilon_{ij}(u)\Phi\right| = \left|\int_{\Omega} \epsilon_{ij}(u)\Phi\psi + \int_{\Omega} \epsilon_{ij}(u)\Phi(1-\psi)\right|$$
$$\leq \|\epsilon_{ij}(u)\|_{M_1(\Omega)}\|\Phi\|_{C(\bar{\Omega})} + \delta.$$

Since δ is arbitrary, combining with (2.4) and (2.5), we obtain

$$\left|\int_{\partial\Omega} \mathcal{J}_{ij}(u)\phi\right| \leq C\|u\|_{\widetilde{BD}(\Omega)}\|\Phi\|_{C(\partial\Omega)}. \quad (2.6)$$

If $u \in C(\bar{\Omega}) \cap \widetilde{BD}(\Omega)$, then $\int_{\Omega} \mu_{ij}\phi = \int_{\Omega} \epsilon_{ij}(u)\phi$. Thus

$$\int_{\partial\Omega} \mathcal{J}_{ij}(\gamma(u))\phi = \int_{\partial\Omega} \mathcal{J}_{ij}(\gamma_0)(u)\phi\, dx.$$

This implies that, for all $u \in \widetilde{BD}(\Omega) \cap C(\bar{\Omega})$,

$$\gamma(u) = \gamma_0(u) = u|_{\partial\Omega}.$$

Next, we prove weakly-star continuity of $\gamma(u)$ in $M(\partial\Omega)$. Suppose that there is a sequence $\{u_m\}$ and a u such that, as $m \to \infty$,

$$\|u_m - u\|_{L^1(\Omega)^n} \to 0,$$
$$\left|\int_{\Omega} \phi[\Lambda_{ij}(u_m) - \Lambda_{ij}(u)]\right| \to 0$$

for all $\phi \in C(\bar{\Omega})$. It is sufficient to prove that, as $m \to \infty$,

$$\int_{\partial\Omega} [\mathcal{J}_{ij}(\gamma_0(u_m)) - \mathcal{J}_{ij}(\gamma_0(u))]\phi\, dx + \int_{\Omega} [\epsilon_{ij}(u_m) - \epsilon_{ij}(u)]\phi \to 0.$$

For any $\delta > 0$, choose $\psi \in C^1(\bar{\Omega})$ such that

$$\|\psi - \phi\|_{C(\bar{\Omega})} \leq \frac{\delta}{5\|\epsilon(u)\|_{BD(\Omega)}}.$$

Thus,

$$\left|\int_{\partial\Omega} [\mathcal{J}_{ij}(\gamma_0(u_m)) - \mathcal{J}_{ij}(\gamma_0(u))](\phi - \psi)\, dx + \int_{\Omega} [\epsilon_{ij}(u_m) - \epsilon_{ij}(u)](\phi - \psi)\right|$$
$$\leq 2\|u_m - u\|_{BD(\Omega)}\|\phi - \psi\|_{C(\bar{\Omega})} \leq \delta.$$

By Green's formula,

$$\left|\int_{\partial\Omega} \mathcal{J}_{ij}(\gamma_0(u_m) - \gamma_0(u))\psi + \int_{\Omega} (\epsilon_{ij}(u_m - u))\psi\right|$$
$$= \left|\frac{1}{2}\int_{\Omega} [(u_j^{(m)} - u_j)\partial_i\psi + (u_i^{(m)} - u_i)\partial_j\psi]dx\right|$$
$$\leq \|u_m - u\|_{L^1(\Omega)^n}\|\psi\|_{C^1(\bar{\Omega})} \leq \delta$$

for large m. Therefore, we have, as $m \to \infty$,

$$\left|\int_{\partial\Omega} \mathcal{J}_{ij}(\gamma(u_m - u))\phi\right| \to 0$$

for all $\phi|_{\partial\Omega} \in C(\partial\Omega)$.

Proposition 2.4. *Assume that Ω is a star-region wrt x_0 and has the outside strong sphere property. Then $C^\infty(\bar\Omega)$ functions are dense in $\widetilde{BD}_0(\Omega)$ with respect to an intermediate topology of $\widetilde{BD}_0(\Omega)$. More precisely, for any $u \in \widetilde{BD}_0(\Omega)$, there exists a sequence of $u_m \in C^\infty(\bar\Omega)$, such that, as $m \to \infty$,*

$$\int_\Omega \Lambda_{ij}(u_m)\phi \to \int_\Omega \Lambda_{ij}(u)\phi \quad \text{for all } \phi \in C(\bar\Omega), \tag{2.7}$$

$$\|u_m - u\|_{L^1(\bar\Omega)^n} \to 0,$$

$$\|\Lambda_{ij}(u_m)\|_{M(\bar\Omega)} \to \|\Lambda_{ij}(u)\|_{M(\bar\Omega)}. \tag{2.8}$$

As a result, we also have

$$\|\gamma(u_m) - \gamma(u)\|_{L^1(\partial\Omega)} \to 0. \tag{2.9}$$

Proof. Since the trace $\gamma(u) \in L^1(\partial\Omega)$ for $u \in \widetilde{BD}_0(\Omega)$, we can use the lifting operator $\ell : L^1(\partial\Omega) \to W^{1,1}(\Omega^c)$, where $\Omega^c = I\!R^n\backslash\Omega$, such that, $\ell(\gamma(u)) = w$ in Ω^c and $w|_{\partial\Omega^c} = \gamma(u)$. In this way, the function u is extended as a function $\tilde u$ in $I\!R^n$. It is clear that $\tilde u$ is a $BD(I\!R^n)$ function, and $\Lambda(\tilde u)$ is absolutely continuous with respect to Lebesgue measure in $I\!R^n \setminus \bar\Omega$. Now, we define

$$u_\eta(x) = \int_{I\!R^n} \rho_\eta(x-y)\tilde u(y + \frac{\eta}{M}(y - x_0))\, dy,$$

where ρ_η is a modifier and M is chosen such that

$$|x - x_0| \le M \qquad \text{for all } x \in \Omega. \tag{2.10}$$

It is easy to check that

$$\|u_\eta - u\|_{L^1(\Omega)^n} \to 0 \qquad \text{as } \eta \to 0.$$

Now, we prove (2.7). By differentiating u_η,

$$\int_\Omega \phi(x)\Lambda_{ij}(u_\eta(x)) = \int_\Omega \phi(x)\int_{I\!R^n} \rho_\eta(x-y)\Lambda_{ij}\left(\tilde u(y + \frac{\eta}{M}(y - x_0))\right)(1+\eta)\, dx.$$

Let $z = y + \frac{y}{M}(y - x_0)$ and switch the order of integration.

$$\int_\Omega \phi(x)\Lambda_{ij}(u_\eta) = \int_{\Omega_\eta}\left[\int_\Omega \phi(x)\rho_\eta\left(x - \frac{z + \eta x_0/M}{1+\eta/M}\right) dx\right]\Lambda_{ij}(\tilde u(z)),$$

where

$$\Omega_\eta = \left\{z : \left|x - \frac{z+\eta x_0/M}{1+\eta/M}\right| < \eta, \quad x \in \Omega\right\},$$

which contains the closure of Ω because of (2.10). Write it in two parts

$$\int_\Omega \phi(x)\Lambda_{ij}(u_\eta) = \int_{\Omega_\eta\backslash\bar\Omega} + \int_{\bar\Omega}\left[\int_\Omega \phi(x)\rho_\eta\left(x - \frac{z + \eta x_o/M}{1+\eta/M}\right) dx\right]\Lambda_{ij}(\tilde u(z)).$$

Since $\Lambda(\tilde{u})$ is absolutely continuous wrt Lebesgue measure in $\Omega_\eta \setminus \bar{\Omega}$, the first term tends to zero as $\eta \to 0$. The second term tends to

$$\int_{\bar{\Omega}} \phi(\eta)\, \Lambda_{ij}(u(z)),$$

and (2.7) follows. For (2.8), it follows from the lower semi-continuity of $\|\Lambda(u)\|_{M(\bar{\Omega})}$ in the weak topology of $M(\bar{\Omega})$ that

$$\int_{\bar{\Omega}} |\Lambda(u)| \leq \liminf_{\eta \to 0} \int_{\bar{\Omega}} |\Lambda(u_\eta)|.$$

On the other hand, carrying out the same steps as in the proof of (2.7), we have

$$\int_{\bar{\Omega}} |\Lambda_{ij}(u_\eta(x))|\, dx = \int_{\Omega} \left| \int_{\Omega_\eta} \rho_\eta \left(x - \frac{z + \eta x_0/M}{1 + \eta/M} \right) \Lambda_{ij}(u(z)) \right| dx$$
$$\leq \int_{\bar{\Omega}} |\Lambda_{ij}(u)| + \delta(\eta)$$

where $\delta(\eta) \to 0$ as $\eta \to 0$. This finishes the proof of (2.8).

Finally, let us prove (2.9). Define

$$\Omega_\alpha = \{y : |x - y| < \alpha, \quad x \in \Omega\},$$

and $\Gamma_\alpha = \partial\Omega_\alpha$ ($\Gamma_0 = \partial\Omega$). Since $\|\Lambda_{ij}(u_\eta)\|_{M(\Omega_\alpha)} \to \|\Lambda_{ij}(\tilde{u})\|_{M(\Omega_\alpha)}$, and $\|u_\eta - u\|_{L^1(\Omega_\alpha)} \to 0$, it follows from Fubini's Theorem that

$$\int_{\Gamma_\beta} |u_\eta - \tilde{u}|\, dx \to 0 \quad \text{a.e. } \beta \in (0, \alpha).$$

In $\Omega_\beta \setminus \bar{\Omega}$, we have

$$\int_{\Gamma_0} \mathcal{J}_{ij}(\gamma(u_\eta) - \gamma(\tilde{u})) dx = \int_{\Gamma_\beta} \mathcal{J}_{ij}(\gamma(u_\eta - \tilde{u})) dx + \int_{\Omega_\beta \setminus \bar{\Omega}} \Lambda_{ij}(u_\eta - \tilde{u}) dx.$$

For any $\epsilon > 0$, choose β such that the first term is less than $\frac{\epsilon}{2}$, and then choose η such that the second term is less than $\frac{\epsilon}{2}$. Therefore, as $\eta \to 0$,

$$\int_{\Gamma_0} |\mathcal{J}_{ij}(\gamma(u_\eta - \tilde{u}))|\, dx \to 0.$$

Note that $\gamma(\tilde{u}) = \gamma(u)$, which completes the proof of (2.9).

Remark 2.2. Since $\widetilde{BD}(\Omega)$ is the completion of $\widetilde{BD}_0(\Omega)$ in the weak topology, C^∞ functions are also dense in $\widetilde{BD}(\Omega)$ with the weak topology.

Remark 2.3. The trace $\gamma(u)$ defined by (2.3) is linear and continuous wrt the norm in $\widetilde{BD}_0(\Omega)$. Also, the trace $\gamma(u)$ is weakly-star continuous wrt the intermediate topology. But the trace $\gamma(u)$ is not weakly-star continuous wrt the weak topology in $\widetilde{BD}_0(\Omega)$. The measure defined by the weak limit u of a sequence $\{u_m\} \in \widetilde{BD}_0(\Omega)$ may be concentrated along the boundary. So it may happen that $u \in \widetilde{BD}(\Omega) \setminus \widetilde{BD}_0(\Omega)$.

2.3. The generalized Green formula

For given $u \in \widetilde{BD}(\Omega)$ and $\sigma \in \Sigma(\Omega)$, a product $\sigma \cdot \Lambda(u)$ does not make sense in general, because $\sigma \in L^\infty(\Omega; E)$ and $\Lambda(u) \in M(\bar{\Omega}; E)$. In this subsection, the meaning of $\sigma \cdot \Lambda(u)$ is discussed, and the generalized Green's formula is recovered.

In $\Sigma \times \widetilde{BD}_0$, we define $\sigma \cdot \Lambda(u)$ as

$$\int_\Omega \sigma \cdot \Lambda(u)\phi = \int_\Omega \mathrm{div}\sigma.u\phi\, dx + \int_\Omega \sigma.(u \otimes \mathrm{grad}\phi)\, dx - \int_{\partial\Omega} \gamma_\nu(\sigma).\gamma(u)\phi\, dx \qquad (2.11)$$

for all $\phi \in C^1(\bar{\Omega})$. Since $\mathrm{div}\sigma \in L^p (p \geq n)$, $u \in L^{\frac{n}{n-1}}(\Omega)$[17], $\gamma_\mu(\sigma) \in L^\infty(\partial\Omega)$ and $\gamma(u) \in L^1(\partial\Omega)$, every term on the RHS is meaningful. Furthermore, the product $\sigma \cdot \Lambda(u)$ is a bounded measure on $\bar{\Omega}$ based on the following proposition.

Proposition 2.5. *The product $\sigma \cdot \Lambda(u)$ defined by (2.11)is a bounded measure on $\bar{\Omega}$ and absolutely continuous with respect to $|\Lambda(u)|$, satisfying*

$$\left|\int_\Omega \phi\sigma \cdot \Lambda(u)\right| \leq \|\sigma\|_{L^\infty(\Omega;E)} \int_\Omega |\phi||\Lambda(u)|.$$

Proof. Let σ_m approximate σ as in Proposition 2.2 in §2.1. Substituting σ_m in (2.11), we have, as $m \to \infty$,

$$\int_\Omega \sigma_m.\mu(u)\phi \to \int_\Omega \sigma.\mu(u)\phi$$

for all $\phi \in C^1(\bar{\Omega})$. The inequality

$$\left|\int_\Omega \sigma_m.\mu(u)\phi\right| \leq \|\sigma_m\|_{L^\infty(\Omega;E)} \int_\Omega |\phi||\mu(u)|$$

gives, in the limit,

$$\left|\int_\Omega \sigma.\mu(u)\phi\right| \leq \|\sigma\|_{L^\infty(\Omega;E)} \int_\Omega |\phi||\mu(u)|.$$

The conclusion follows from this by a classical argument[2].

The bounded measure $\sigma \cdot \Lambda(u)$ also satisfies the following

Proposition 2.6. *Suppose that $\sigma \in \Sigma(\Omega)$ and $u \in \widetilde{BD}_0(\Omega)$. Then, we have the following conclusions*

(1) *The measure $\sigma \cdot \Lambda(u)$ depends continuously on σ in the sense that if, as $m \to \infty$,*

$$\mathrm{div}\sigma_m \to \mathrm{div}\sigma \quad \text{weakly in } L^p(\Omega)^n,$$
$$\sigma_m \to \sigma \quad \text{weakly-star in } L^\infty(\Omega; E),$$

then, as $m \to \infty$,

$$\int_\Omega \phi\sigma_m \cdot \Lambda(u) \to \int_\Omega \phi\sigma \cdot \Lambda(u) \qquad \text{for all } \phi \in C(\bar{\Omega}).$$

(2) *The measure $\sigma.\Lambda(u)$ depends continuously on u in the sense that if, as $m \to \infty$,*

$$\Lambda(u_m) \to \Lambda(u) \quad \text{weakly-star in } M(\bar{\Omega})E,$$
$$u_m \to u \quad \text{strongly in } L^{\frac{n}{n-1}}(\Omega)^n,$$
$$\|\Lambda(u_m)\|_{M(\bar{\Omega})} \to \|\Lambda(u)\|_{M(\bar{\Omega})},$$

then, as $m \to \infty$,

$$\int_\Omega \phi\sigma \cdot \Lambda(u_m) \to \int_\Omega \phi\sigma \cdot \Lambda(u) \qquad \text{for all } \phi \in C(\bar{\Omega}).$$

Therefore, one can define $\sigma.\Lambda(u)$ as a continuous functional on $\bar{\Omega}$, such that the generalized Green's formula holds (take $\phi \equiv 1$ in (2.11))

$$\int_\Omega \sigma.\Lambda(u) = \int_\Omega \mathrm{div}\sigma.u\,dx - \int_{\partial\Omega} \gamma_\nu(\sigma).\gamma(u)\,dx,$$

§3. Existence of Solutions

3.1. The statement of the main result

In what follows, we study existence of solutions for both the stress and the strain problems. For the stress problem, existence follows from the theory of convex analysis[6]. For the strain problem, existence comes from weak-compactness in the $\widetilde{BD}(\Omega)$ space. As expected, the solutions for both problems do form a saddle point for the Lagrange function $L(\sigma, u)$ defined on the function spaces $\Sigma \times \widetilde{BD}_0$.

Theorem 3.1. *There exists a pair of solutions $\sigma^* \in \Sigma(\Omega)$ and $u^* \in \widetilde{BD}(\Omega)$ for the variational problems. Moreover, if $\gamma(u^*) \in L^1(\partial\Omega)$, then the triplet $(\sigma^*, \lambda^*, u^*)$ forms a saddle point for $L(\sigma, u)$ on $K_1 \times \Psi \subset \Sigma \times \widetilde{BD}_0$ satisfying*

$$L(\sigma, u^*) \leq L(\sigma^*, u^*) = \lambda^* \leq L(\sigma^*, u),$$

and the extremality relation holds

$$c \cdot \mathrm{div} u^* = \sigma^* \cdot \Lambda(u^*) \quad \text{in } M(\bar{\Omega}),$$
$$\gamma_\nu(\sigma^*) \cdot \gamma(u^*) = 0 \quad \text{a.e. } x \in \partial\Omega.$$

The requirement $\gamma(u^*) \in L^1(\partial\Omega)^n$ guarantees that a generalized Green formula holds (§2.3). So the duality relation can be achieved. In §3.2, we shall study the duality relation and obtain existence of the stress problem. In §3.3, based on a fundamental estimate, we derive existence of the strain problem.

3.2. Minimax theorem and existence for the stress problem

By Proposition 2.6, we know that the Lagrange function

$$L(\sigma, u) = \int_\Omega \sigma . \Lambda(u).$$

is a bilinear continuous function defined on $\Sigma(\Omega) \times \widetilde{BD}_0(\Omega)$. In this subsection, we shall prove

$$\inf_{u \in \Psi} \sup_{\sigma \in K} L(\sigma, u) = \max_{\sigma \in K} \inf_{u \in \Psi} L(\sigma, u), \tag{3.1}$$

where

$$K = \left\{ \sigma \in \Sigma(\Omega);\ \|\sigma^D\| \leq k \left(\frac{\mathrm{tr}\sigma}{n} + c \right) \quad \text{a.e. } x \in \Omega \right\},$$

$$\Psi = \left\{ u \in \widetilde{BD}_0(\Omega);\ \int_\Omega f u \, dx = 1, \begin{array}{l} \gamma(u)_\nu \leq 0,\ \gamma(u)_\tau = 0 \text{ on top} \\ \gamma(u)_\nu + \mu|\gamma(u)_\tau| \leq 0 \text{ on sides} \end{array} \right\}.$$

Using *max* on the RHS of (3.1) means that the supremum is attained for the *sup inf*. In other words, the expression (3.1) gives the existence of the first component σ^* of the saddle point.

To obtain existence for the stress problem, we prove that, for some u_0, the function $\sigma \to L(\sigma, u_0)$ is coercive.

Lemma 3.1. *There is a $u_0 \in \Psi \cap C^1(\bar{\Omega})^n$ such that*

$$\lim_{\sigma \in K, \|\sigma\|_\Sigma \to +\infty} L(\sigma, u_0) = -\infty. \tag{3.2}$$

Proof. In deriving the strain problem, we obtained the dual cone of the cone K in the strain rate space

$$K^* = \left\{ \Lambda(u) \in M(\bar{\Omega}; E);\ -\mathrm{tr}\Lambda(u) \geq k \sup_{f(\sigma^D)=1} \epsilon^D.\sigma^D \quad \text{a.e. } x \in \Omega \right\}. \tag{3.3}$$

We can choose $u_0 \in \overset{\circ}{C}(\bar{\Omega})^n \cap \Psi \cap \overset{\circ}{K^*}$, where $\overset{\circ}{K^*}$ is the interior of K^*. It is obvious that, for any $\sigma \in K$, we have $L(u_0, \sigma) < 0$, and as $\|\sigma\|_\Sigma \to \infty$, the value of $L(u_0, \sigma)$ tends to $-\infty$.

To prove the minimax theorem, weak compactness is required. But the unit ball in both $\widetilde{BD}_0(\Omega)$ and $LD(\Omega)$ is not weakly compact. We introduce a new function space

$$LD_\delta(\Omega) = \{u \in L^1(\Omega)^n, \epsilon_{ij}(u) \in L^{1+\delta}(\Omega; E)\}.$$

Correspondingly, we define

$$\Psi_\delta = \Psi \cap LD_\delta(\Omega).$$

For $\delta > 0$, the unit ball in $LD_\delta(\Omega)$ is weakly compact and $LD(\Omega) = \bigcap_{\delta>0} LD_\delta(\Omega)$.

Proposition 3.1. *The perturbed Lagrange function*

$$L_\delta(\sigma, u) = L(\sigma, u) + \delta\|\epsilon_1(u)\|_{1+\delta}$$

possesses at least one saddle point $(\sigma_\delta^*, u_\delta^*)$ on $K \times \Psi_\delta \subset \Sigma(\Omega) \times LD_\delta(\Omega)$, and

$$L_\delta(\sigma_\delta^*, u_\delta^*) = \min_{u\in\Psi_\delta} \sup_{\sigma\in K} L_\delta(\sigma, u) = \max_{\sigma\in K} \inf_{u\in\Psi_\delta} L_\delta(\sigma, u).$$

Proof. The function $u \to L_\delta(\sigma, u)$ is also coercive in the following sense: there is a $\sigma_0 \in K$ such that

$$\lim_{u\in\Psi_\delta, \|u\|_{LD_\delta(\Omega)}\to\infty} L(\sigma_0, u) = +\infty.$$

For each $u \in \Psi_\delta$, the function $\sigma \to L_\delta(\sigma, u)$ is concave and upper semi-continuous; for each $\sigma \in K$, the function $u \to L_\delta(\sigma, u)$ is convex and lower semi-continuous. The conclusion easily follows from game theory[6].

Proposition 3.2. *The stress problem has at least one solution σ^*. On $K \times \Psi \subset \Sigma(\Omega) \times LD(\Omega)$, we have*

$$\inf_{u\in\Psi} \sup_{\sigma\in K} L(\sigma, u) = \max_{\sigma\in K} \inf_{u\in\Psi} L(\sigma, u). \tag{3.4}$$

Proof. For fixed $\delta > 0$, we have from Proposition 3.1

$$\begin{aligned} L(\sigma, u_\delta^*) + \delta\|\Lambda(u_\delta^*)\|_{1+\delta} &\leq L(\sigma_\delta^*, u_\delta^*) + \delta\|\Lambda(u_\delta^*)\|_{1+\delta} \\ &\leq L(\sigma_\delta^*, u) + \delta\|\Lambda(u)\|_{1+\delta} \qquad \forall u \in \Psi_\delta,\ \sigma \in K. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\begin{gathered} L(\sigma_\delta^*, u_\delta^*) \geq L(\sigma, u_\delta^*) \qquad \forall \sigma \in K, \\ L(\sigma_\delta^*, u_\delta^*) \geq \sup_{\sigma\in K} L(\sigma, u_\delta^*) \geq \inf_{u\in\Psi} \sup_{\sigma\in K} L(\sigma, u) = \gamma. \end{gathered} \tag{3.6}$$

Setting $u = u_0$ in (3.5), we have

$$L(\sigma_\delta^*, u_0) + \delta\|\Lambda(u_0)\|_{1+\delta} \geq L(\sigma_\delta^*, u_\delta^*) + \delta\|\Lambda(u_\delta^*)\|_{1+\delta}. \tag{3.7}$$

Combining with (3.6), the above (3.7) gives

$$L(\sigma_\delta^*, u_0) \geq L(\sigma_\delta^*, u_\delta^*) - \delta\|\Lambda(u_0)\|_{1+\delta} \geq \gamma - \delta\|\Lambda(u_0)\|_{1+\delta}. \tag{3.8}$$

Assume $\gamma > -\infty$ (when $\gamma = -\infty$, the equality (3.4) is obviously true). Thus, $L(\sigma_\delta^*; u_0)$ is bounded from below when $\delta \to 0$, and from (3.2),

$$\sigma_\delta^* \text{ is bounded using the norm of } \Sigma(\Omega) \text{ for } \delta \to 0.$$

There then exists a sequence $\delta_j \to 0$ and $\sigma^* \in K$ such that

$$\sigma_{\delta_j}^* \to \sigma^* \text{weakly in } K.$$

Let $\delta_j \to 0$ in (3.5), by (3.6),

$$L(\sigma^*, u) \geq \limsup_{\delta_j \to 0} L(\sigma_\delta^*, u) \geq \limsup_{\delta_j \to 0} L(\sigma_\delta^*, u_\delta^*) \geq \gamma, \qquad \forall u \in \Psi.$$

This gives

$$\gamma \leq \inf_{u \in \Psi} L(\sigma^*, u) \leq \sup_{\sigma \in K} \inf_{u \in \Psi} L(\sigma, u),$$

and the conclusion follows.

Note that $LD(\Omega)$ and $\widetilde{BD}_0(\Omega)$ have the same norm, and $LD(\Omega)$ functions are dense in $\widetilde{BD}_0(\Omega)$ in the weak topology. As a result, we have the following conclusion.

Corollary 3.1. *The stress problem has at least one solution σ^*. On $K \times \Psi \subset \Sigma(\Omega) \times BD_0(\Omega)$, we have*

$$\inf_{u \in \Psi} \sup_{\sigma \in K} L(\sigma, u) = \max_{\sigma \in K} \inf_{u \in \Psi} L(\sigma, u).$$

3.3. Existence for the strain problem and the extremality relation

We recall the strain problem

$$\inf_u \left\{ c \int_\Omega \operatorname{div} u \, dx : u \in K^* \cap \Psi \right\}, \tag{3.9}$$

where K^* is the dual cone of K, defined by (3.3). Suppose that $\{u_k\}$ is the minimizing sequence of (3.9) in the space $LD(\Omega)$. We shall prove that $\{u_m\}$ is uniformly bounded in the norm topology of $LD(\Omega)$. But we cannot expect to obtain solutions in $LD(\Omega)$ in general, since a bounded set in $LD(\Omega)$ is not weakly compact. We can take $\{u_m\}$ as elements in $\widetilde{BD}(\Omega)$ in which a bounded set is weakly compact. Then existence of the strain problem in $\widetilde{BD}(\Omega)$ follows. When the solution u^* belongs to $\widetilde{BD}_0(\Omega)$, the duality relation is proved.

Lemma 3.2. *Suppose that $u \in LD(\Omega)$, and $\gamma(u)$ is the trace on the boundary. Then*

$$q(u) = \int_{\text{top}} |\gamma(u)_t| \, ds$$

is a continuous seminorm in $LD(\Omega)$, which is a norm on the set of rigid motion

$$\Re = \{u : \ |\epsilon(u)|_{L^1(\Omega)} = 0\}.$$

In other words, the equation $q(u) = 0$ and $u \in \Re$ imply $u \equiv 0$. In fact, $\Re$ is a 6-dimensional space in which only rotation and translation are allowed. The equation $q(u) = 0$ restricts the rigid motion and $u \equiv 0$ follows.

When $q(u) = 0$, we can obtain a Poincare inequality

$$\|u\|_{L^1(\Omega)^n} \leq C \sum_{ij} \|\epsilon_{ij}(u)\|_{L^1}.$$

In general, the norm in the space $LD(\Omega)$ is equivalent to

$$q(u) + \sum_{ij} \|\epsilon_{ij}(u)\|_{L^1}.$$

Together with Proposition 1.1, we obtain the following result.

Proposition 3.3. *Suppose that $\{u_k\} \in LD(\Omega)$ is the minimizing sequence of (3.9). Then, there exists a constant $N > 0$ such that*

$$\|u_k\|_{LD(\Omega)} \le N.$$

Moreover there exists at least one solution of the strain problem in $\widetilde{BD}(\Omega)$.

For the second part of Proposition 3.6, we take u_n as elements in $\widetilde{BD}(\Omega)$, and then extract a subsequence, still denoted by u_n, such that

$$\begin{aligned} &u_m \to u^* \quad \text{strongly in } L^1(\Omega),\\ &\mathrm{div} u_m \to \mathrm{div} u^* \quad \text{weakly-star in } M(\bar{\Omega}),\\ &|\Lambda^D(u_m)| \to |\Lambda^D(u^*)| \quad \text{weakly-star in } M(\bar{\Omega}),\\ &\gamma(u_m) \to \gamma(u^*) \quad \text{weakly-star in } M(\partial\Omega). \end{aligned} \tag{3.10}$$

The convergence of $\gamma(u_m)$ follows from Proposition 2.3 in §2.2.2. Since $c\int_\Omega \mathrm{div} u\, dx$ is continuous with respect to (3.10), function u^* is the solution of the strain problem.

When $u^* \in \widetilde{BD}_0(\Omega)$, the generalized Green formula holds. In this case, the duality relation can be proved.

Proposition 3.4. *If $\gamma(u^*) \in L^1(\partial\Omega)$, then the triplet $(\sigma^*, \lambda^*, u^*)$ forms a saddle point for $L(\sigma, u)$ on $K \times \Psi \subset \Sigma \times \widetilde{BD}_0$ satisfying*

$$L(\sigma, u^*) \le L(\sigma^*, u^*) = \lambda^* \le L(\sigma^*, u),$$

and the extremality relation holds

$$\begin{aligned} &c\mathrm{div} u^* = \sigma^* \cdot \Lambda(u^*) \quad \text{in } M(\bar{\Omega}),\\ &\lambda_\nu(\sigma^*) \cdot \lambda(u^*) = 0 \quad \text{a.e. } x \in \partial\Omega. \end{aligned}$$

Proof. To prove the extremality relation, we need to check that, for any $u \in \Psi$,

$$\sup_{\sigma\in K} L(\sigma, u) = \begin{cases} c\int_\Omega \mathrm{div} u & \text{if } u \in K^*,\\ +\infty & \text{if } u \notin K^*, \end{cases}$$

and for any $\sigma \in K$,

$$\inf_{u\in\Psi} L(\sigma, u) = \begin{cases} \lambda & \text{if } \sigma \in \Psi = \cup_{\lambda>0}\Psi_\lambda,\\ -\infty & \text{if } \sigma \notin \Psi. \end{cases}$$

When u and σ are nice functions and $\partial\Omega$ is smooth, we derived these formula in §1.4 and §1.5. Working on the generalized function spaces $\Sigma(\Omega)$ and $\widetilde{BD}_0(\Omega)$, we need an approximation procedure to reproduce the results[17, 14]. We do not carry out the calculation here. After proving these results, we have, on the saddle point (σ^*, u^*),

$$c\int_{\bar{\Omega}} \mathrm{div} u^* = L(\sigma^*, u^*) = \lambda^*.$$

This gives the extremality relations. The first part of the proposition follows from Corollary 3.1.

3.4. Discontinuities of the velocity field

Both the stresses and velocity in our problems can be discontinuous. It was known[11] that a jump discontinuity of the stresses does not make a contribution to the energy dissipation rate but a jump discontinuity of the velocities does. So when we study the strain problem, it is an important issue to know how to handle discontinuities of velocity inside Ω as well as along the boundary. This has been done in the previous sections. Here, we just want to observe a different feature between velocity discontinuities for pressure-insensitive and pressure-sensitive materials.

For pressure-insensitive materials, it follows from the flow rule that the material is incompressible. As a result, the normal component $u.\nu$ of the velocity is continuous across the surface of velocity discontinuity. But pressure-sensitive materials are compressible, so the tangential discontinuity is accompanied by a normal discontinuity in general. We elaborate on this point in the following.

Before proceeding, we mention the basic fact that, on a surface of a velocity discontinuity, the stress components are continuous[11]. In fact, on a surface of velocity discontinuity, the relations that the stress components satisfy are determined by the flow rule. By contrast, the relation that the stress components satisfy across a stress discontinuity is determined by the equilibrium equations, $\operatorname{div}\sigma = \lambda g$.

Suppose that u is the collapse solution of the strain problem and σ is the corresponding stress tensor. Also assume that the discontinuity of u happens on a surface $\mathcal{L}$ which separates Ω into Ω_1 and Ω_2, and σ is continuous on $\mathcal{L}$. Then, by Proposition 3.4,

$$\begin{aligned} c\int_\Omega \operatorname{div} u &= \int_\Omega \sigma.\Lambda(u) = \int_\Omega \sigma.\operatorname{div}\sigma\, dx \\ &= \int_{\Omega_1\cup\Omega_2} \sigma.\Lambda(u)\, dx + \int_{\mathcal{L}} \sigma.\mathcal{J}([u])ds, \end{aligned}$$

where ν is the unit vector normal to $\mathcal{L}$ from Ω_1 to Ω_2, the jump $[u] = (u_{\Omega_1} - u_{\Omega_2})\big|_{\mathcal{L}}$ and $\mathcal{J}$ is given by (2.2). By the extremality relation, we have

$$c\int \operatorname{div} u = c\int_{\Omega_1\cup\Omega_2} \operatorname{div} u\, dx - c\int_{\mathcal{L}} [u].\nu\, ds,$$

and $\epsilon(u)$, $\mathcal{J}([u]) \in K^*$, that is

$$\operatorname{div} u \geq k \sup_{f(\sigma^D)=1} \sigma^D.\epsilon^D(u), \qquad [u].\nu \geq k \sup_{f(\sigma^D)=1} \sigma^D.\mathcal{J}^D([u]).$$

So the normal component of the velocity suffers discontinuity.

In our case, since $[u].\nu \geq 0$, abrupt thickening could happen along the slip surface. The "thickness" of the slip surface depends on the magnitude of the discontinuity. This observation could be helpful in studying the formation of shear bands in the region of plastic deformation.

§4. Explicit Solutions and Asymptotic Analysis

4.1. The case of radial gravity

Consider radial gravity in two dimensions. Assume that the wall of the hopper is smooth (see (1.1)) and the yield condition is given by (1.4). In this event, solutions of both the stress and the strain problems can be obtained explicitly, and the largest multiplier λ_r^* (subscript r is mnemonic for "radial") can be expressed in terms of the top and bottom radius and the parameters of the material.

For the stress problem, we rewrite the equilibrium equations in polar coordinates. On the region $\Omega = [r_0, R] \times [-\theta_0, \theta_0]$,

$$\begin{aligned} &\text{(a)} \quad \partial_r \sigma_{rr} + r^{-1}\partial_\theta \sigma_{r\theta} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = -\lambda, \\ &\text{(b)} \quad \partial_r \sigma_{r\theta} + r^{-1}\partial_\theta \sigma_{\theta\theta} + 2r^{-1}\sigma_{r\theta} = 0 \end{aligned} \tag{4.1}$$

and the boundary conditions become

$$\begin{aligned} &\sigma_{rr}\big|_{r=r_0} = \sigma_{r\theta}\big|_{r=r_0} = \sigma_{r\theta}\big|_{\theta=\pm\theta_0} = 0, \\ &\sigma_{rr}\big|_{r=R} \geq 0, \quad \sigma_{\theta\theta}\big|_{\theta=\pm\theta_0} \geq 0. \end{aligned} \tag{4.2}$$

The yield condition is

$$\sqrt{\frac{1}{2}(\sigma_{rr} - \sigma_{\theta\theta})^2 + 2\sigma_{r\theta}^2} \leq k\left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} + c\right). \tag{4.3}$$

Suppose that $\sigma_{r\theta} \equiv 0$ and $\sigma_{\theta\theta} \geq \sigma_{rr}$ (we will check later that the solution we find satisfies these conditions). From the extreme case of the yield condition, we have

$$\sigma_{\theta\theta} = \frac{\sqrt{2}+k}{\sqrt{2}-k}\sigma_{rr} + \frac{2kc}{\sqrt{2}-k}. \tag{4.4}$$

Substituting (4.4) and $\sigma_{r\theta} \equiv 0$ in (4.1(a)), we obtain

$$r\partial_r \sigma_{rr} - \frac{2K}{\sqrt{2}-k}\sigma_{rr} + \frac{2kc}{\sqrt{2}-k} = -\lambda r.$$

Solving this equation with the initial data $\sigma_{rr}\big|_{r=r_0} = 0$, we have

$$\sigma_{rr} = \begin{cases} c\left(\left(\frac{r}{r_0}\right)^{\frac{2k}{\sqrt{2}-k}} - 1\right) - \lambda r \frac{\sqrt{2}-k}{\sqrt{2}-3k}\left(1 - \left(\frac{r}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\right), & k \neq \frac{\sqrt{2}}{3}, \\ c\left(\frac{r}{r_0} - 1\right) - \lambda r \ln \frac{r}{r_0}, & k = \frac{\sqrt{2}}{3}. \end{cases} \tag{4.5}$$

The condition $\sigma_{rr}\big|_{r=R} \geq 0$ gives

$$\hat{\lambda} = \begin{cases} c\frac{\sqrt{2}-3k}{\sqrt{2}-k} \cdot \frac{\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\left(1-\left(\frac{R}{r_0}\right)^{-\frac{2k}{\sqrt{2}-k}}\right)}{r_0\left(1-\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\right)}, & k \neq \frac{\sqrt{2}}{3}, \\ c \cdot \frac{r_0^{-1} - R^{-1}}{\ln \frac{R}{r_0}}, & k = \frac{\sqrt{2}}{3}. \end{cases} \tag{4.6}$$

The fact that $\sigma_{rr} \geq 0$ (easy to check) and (4.4) give $\sigma_{\theta\theta} \geq \sigma_{rr}$. So $\hat{\lambda}$ in (4.6) provides a lower bound of the limit load.

For the strain problem, we rewrite it in the following equivalent form.

$$\inf\left\{\frac{c\int \mathrm{div} u\, dx}{\int fu\, dx} : \begin{array}{l} u \in K^*,\ u_\theta \leq 0 \text{ on the sides} \\ u_r \leq 0,\ u_\theta = 0 \text{ on the top} \end{array}\right\}.$$

Let $(\bar{u}_r, \bar{u}_\theta) = \left(-r^{-\frac{\sqrt{2}+k}{\sqrt{2}-k}}, 0\right)$. It is not difficult to check that

$$\hat{\lambda} = \frac{c\int_\Omega \mathrm{div}\bar{u}\, dx}{\int_\Omega f\bar{u}\, dx}.$$

The upper bound found from the strain problem is equal to the lower bound found from the stress problem. Therefore, a pair of solutions of our variational problems are found and the limit load λ_r^* is given by (4.6).

In terms of the formula (4.6), we can obtain the following asymptotic behavior of λ_r^* with respect to R, as $R \to \infty$,

$$\lambda_r^* \sim \frac{c}{r_0}\frac{\sqrt{2}-3k}{\sqrt{2}-k}\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}, \qquad k < \frac{\sqrt{2}}{3},$$

$$\lambda_r^* \sim \frac{c}{r_0}\frac{1}{\ln\frac{R}{r_0}}, \qquad k = \frac{\sqrt{2}}{3},$$

$$\lambda_r^* \to \frac{c}{r_0}\frac{3k-\sqrt{2}}{\sqrt{2}-k}, \qquad k > \frac{\sqrt{2}}{3}.$$

Also it is easy to check that, as $R \to \infty$, the limit load λ_r^* is monotone increasing and converges uniformly with respect to k in the interval $[0, \sqrt{2}-\delta]$ $(\delta > 0)$ to the function

$$h(k) = \begin{cases} 0, & k \le \frac{\sqrt{2}}{3}, \\ \frac{c}{r_0}\cdot\frac{3k-\sqrt{2}}{\sqrt{2}-k}, & k > \frac{\sqrt{2}}{3}. \end{cases}$$

Note that k is the internal friction of the material. Physically, in this specific model, only when this value is greater than $k > \frac{\sqrt{2}}{3}$, can the hopper support material with infinite height.

Now we calculate the maximal stress attained inside the hopper. In fact, the relation between the maximal stress in the material and the height R of the hopper gives us information about how strong we should make the hopper. Using the explicit solution found above, we obtain, as $R \to \infty$,

$$(\sigma_{\theta\theta})_{max} \sim c\left(\frac{\sqrt{2}-k}{2k}\right)^{\frac{2k}{3k-\sqrt{2}}}\frac{(\sqrt{2}-k)(\sqrt{2}+k)}{(\sqrt{2}-k)^2}\left(\frac{R}{r_0}\right)^{\frac{2k}{\sqrt{2}-k}} \qquad \text{if } k < \frac{\sqrt{2}}{3},$$

$$(\sigma_{\theta\theta})_{max} \sim \frac{c}{e}\frac{R}{r_0\ln\frac{R}{r_0}}\frac{\sqrt{2}+k}{\sqrt{2}-k} \qquad \text{if } k = \frac{\sqrt{2}}{3},$$

$$(\sigma_{\theta\theta})_{max} \sim c\left(\frac{\sqrt{2}-k}{2k}\right)^{\frac{2k}{3k-\sqrt{2}}}\frac{(3k-\sqrt{2})(\sqrt{2}+k)}{(\sqrt{2}-k)^2}\left(\frac{R}{r_0}\right) \qquad \text{if } k > \frac{\sqrt{2}}{3}.$$

4.2. The case of vertical gravity

When the gravity is vertical, explicit solutions cannot be found. But we still can obtain a lower bound from the stress problem and a upper bound from the strain problem. Asymptotically, the limit load in the vertical case possesses the same properties as that in the radial case.

In the vertical case, the forcing term in the equilibrium equations (4.1) is replaced by $(-\lambda\cos\theta, \lambda\sin\theta)$. When choosing $\sigma_{r\theta} \equiv 0$ and solving the equations with the condition $\sigma_{rr}\big|_{r=r_0} = 0$, we obtain

$$\begin{aligned} \sigma_{\theta\theta} &= f(r) - \lambda r\cos\theta, \\ \sigma_{rr} &= \frac{1}{r}\int_{r_0}^{r} f(r)\,dr - \lambda\left(r^2 - r_0^2\right)\cos\theta, \end{aligned} \tag{4.7}$$

where $f(r)$ is an arbitrary function of r. Suppose $\sigma_{\theta\theta} \ge \sigma_{rr}$, then the yield condition (4.3) implies

$$f(r)-\lambda r\cos\theta-\frac{\sqrt{2}+k}{\sqrt{2}-k}\cdot\frac{1}{r}\left[\int_{r_0}^{r}f(r)\,dr-\lambda\left(r^2-r_0^2\right)\cos\theta\right]\leq\frac{2kc}{\sqrt{2}-k}.$$

To get a lower bound, it is sufficient to choose $f(r)$ such that $f(r)$ satisfies the equation

$$rf(r)-\frac{\sqrt{2}+k}{\sqrt{2}-k}\int_{r_0}^{r}f(r)\,dr+\lambda\frac{2k}{\sqrt{2}-k}\left(r^2-r_0^2\right)-\lambda r_0^2\cos\theta_0=\frac{2kc}{\sqrt{2}-k}r,$$

where θ_0 is the open angle of the hopper. Solving this equation and then substituting $f(r)$ in (4.7), we obtain expressions for σ_{rr} and $\sigma_{\theta\theta}$. Using $\sigma_{rr}\big|_{r=R}\geq 0$, we get a lower bound of λ_v^* (the subscript v is mnemonic for "vertical"); for $k\neq\frac{\sqrt{2}}{3}$,

$$\lambda_v^*\geq\lambda=\frac{cr_0\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\left(1-\left(\frac{R}{r_0}\right)^{-\frac{2k}{\sqrt{2}-k}}\right)}{\frac{\sqrt{2}-k}{\sqrt{2}-3k}\left(1-\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\right)+\frac{\sqrt{2}-k}{\sqrt{2}+k}(1-\cos\theta_0)\left(\frac{R}{r_0}\right)^{-2}\left(\left(\frac{R}{r_0}\right)^{\frac{\sqrt{2}+k}{\sqrt{2}-k}}-1\right)},$$

and for $k=\frac{\sqrt{2}}{3}$,

$$\lambda_v^*\geq\lambda=\frac{c\left(\frac{R}{r_0}-1\right)}{R\ln\frac{R}{r_0}+\frac{r_0^2}{2R}(1-\cos\theta_0)\left(\left(\frac{R}{r_0}\right)^2-1\right)}.$$

To complete our analysis, the reader can check that $\sigma_{\theta\theta}\geq\sigma_{rr}$, and that σ satisfies the remainder of the boundary conditions.

To derive an upper bound, we use the same u as was used in the radial case,

$$\lambda_v^*\leq\frac{c\int_\Omega \operatorname{div}u\,dx}{\int_\Omega u.f\,dx}=\lambda_r^*\frac{\theta_0}{\sin\theta_0},$$

where λ_r^* (4.6) is the limit load obtained in the radial case. From the lower and upper bound of λ_v^*, we conclude that

$$\lambda_v^*\sim O\left(\left(\frac{R}{r_0}\right)^{\frac{3k-\sqrt{2}}{\sqrt{2}-k}}\right)\qquad\text{if } k<\frac{\sqrt{2}}{3},$$

$$\lambda_v^*\sim O\left(\frac{1}{\ln\frac{R}{r_0}}\right)\qquad\text{if } k=\frac{\sqrt{2}}{3},$$

$$\lambda_v^*\sim O(1)\qquad\text{if } k>\frac{\sqrt{2}}{3}.$$

So, the limit load λ_v^* in the vertical case has the same asymptotic behavior as that in the radial case.

4.3. A linearly ill-posed free boundary problem

In the previous sections, we thought of the bottom boundary being fixed. Now we take the bottom boundary as a free boundary and thus it should be determined as a part of solution. Physically, we want to find maximal stable arches in the hopper.

Suppose that the material deforms plastically everywhere. Using the yield condition to replace one variable, the equilibrium equations are reduced to a 2×2 hyperbolic system. Take the traction-free condition on the bottom as initial data, and tangential traction-free (the wall is smooth) on the sides as a boundary condition. One extra condition on the top, normal traction-free, is used to determine the largest multiplier as well as the shape of the free boundary.

When gravity is radial, we find such a solution (§4.1). When gravity is perturbed away from the radial direction, we try to use the Nash-Moser Implicit Function Theorem to solve this problem. The linearization of the equation near the radial solution can be written, on $[0,T] \times [0,L]$, as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & \frac{2s}{1-s} \\ \frac{1+s}{2s} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \frac{1}{1-s} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\lambda - \lambda_r^*) f(t) \\ 0 \end{pmatrix},$$

subject to a boundary condition $v\big|_{x=0,L} = 0$ and initial conditions

$$u(x,0) = -\alpha h(x), \qquad v(x) = h'(x),$$

where $0 < s < 1$ ($s = \frac{k}{\sqrt{2}}$), the number α is positive, the function $f(t)$ is known and related to the radial solution, and $h(x)$ is essentially the shape of the bottom boundary. In order to use the Nash-Moser Theorem, we need to investigate the mapping

$$M : h(x) \to u(x,T) = g(x).$$

Using Fourier Series, we can solve the linearized problem,

$$u(x,t) = \int_0^t (\lambda - \lambda_r^*) e^{\frac{1}{1-s}(t-\tau)} f(\tau)\, d\tau - \frac{2}{L} \int_0^L \Bigg\{ \alpha e^{\frac{1}{1-s}t} + \sum_{n=1}^{\infty} \Bigg[\alpha \cosh(\beta_n t)$$

$$+ \left(\alpha - 2s \left(\frac{n\pi}{L} \right)^2 \right) \frac{1}{(1-s)\beta_n} \sinh(\beta_n t) \Bigg] \cos \frac{n\pi x}{L} \cos \frac{n\pi \xi}{L} \Bigg\} h(\xi)\, d\xi, \tag{4.8}$$

where

$$\beta_n = \sqrt{\frac{1}{(1-s)^2} - \frac{1+s}{1-s} \left(\frac{n\pi}{L} \right)^2}.$$

For the mapping M, it is easy to see that the following estimate holds

$$\|g\|_{H^k[0,L]} \le C \left(\|h\|_{H^{k+2}[0,L]} + \|f\|_{H^0[0,T]} \right).$$

Now, we look at the inverse map M^{-1}. Substituting

$$h(\xi) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi \xi}{L}$$

in (4.8), and calculating the coefficients, we obtain

$$h(x) = \frac{\lambda - \lambda_r^*}{2\alpha} \int_0^t e^{-\frac{1}{1-s}\tau} f(\tau)\, d\tau - \frac{1}{2L\alpha} e^{-\frac{1}{1-s}T} \int_0^L g(\xi)\, d\xi$$

$$+ \sum_{n=1}^{\infty} \frac{2}{L \left[\alpha \cosh(\beta_n t) + \dfrac{L^2 \alpha - 2s(n\pi)^2}{(1-s) L^2 \beta_n} \sinh(\beta_n t) \right]} \int_0^L g(\xi) \cos \frac{n\pi \xi}{L}\, d\xi \cos \frac{n\pi x}{L}.$$

Conjecture. *For any $\gamma > 0$, the set*

$$\left\{t : \liminf_{n\to\infty} n^\gamma \left|\alpha\cosh(\beta_n t) + \frac{L^2\alpha - 2s(n\pi)^2}{(1-s)L^2\beta_n}\sinh(\beta_n t)\right| = 0\right\}$$

is dense in $I\!R$.

This is fundamentally a conjecture in Number Theory.

Note that term in brackets appears in the denominator. If this conjecture is true, then, generically, the function h is unbounded no matter how smooth g is. In fact, the coefficients of high mode terms in the Fourier Series cannot be controlled unless $g(x)$ only contains finite modes. So the inverse map M^{-1} is not continuous at $g \equiv 0$ in any reasonable topology. In this sense, this problem is not linearly well-posed.

Acknowledgements. This work was done when the author studied at Duke University, and worked at McMaster University as a postdoctoral fellow supported by National Sciences and Engineering Research Council of Canada. The author is grateful to Profs. Chadam and Peirce for their sponsoring this research program at McMaster. The author is grateful to Prof. Schaeffer for his patient discussion regarding this problem and format of this paper. Thanks also go to Profs. Allard and Venakides for their help on some parts of this problem.

References

[1] Anzellotti, G. & Giaquinta, M., On the existence of the fields of stresses and displacements for an elasto-perfectly plastic body in static equilibrium, *J. Math. Pures et Appl.*, **61** (1982), 219–244.

[2] Bourbaki, N., Eléments de Marhématiques, Livre vi, Hermann, Paris, 1950-51 (2nd edition, 1957).

[3] Christiansen, E., Limit analysis in plasticity as a mathematical programming problem, *Calcolo*, **17** (1980), 41–65.

[4] Christiansen, E., On the collapse solution in limit analysis, *Arch. Rat. Mech. Anal.*, **91** (1985), 119–135.

[5] Drucker, D. C. & Prager, W., Soil Mechanics and plastic analysis or limit design, *Q. Appl. Math.*, **10** (1954), 157–165.

[6] Ekeland, I. & Temam. R., Convex analysis and variational problems, North-Holland, Amsterdam, New York, 1976.

[7] Gagliardo, E., Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rend. Sem. Mat. Padova*, **27** (1957), 284–305.

[8] Hardt, R. & Kinderlehrer, D., Elastic plastic deformation, *Appl. Math. Optim.*, **10** (1983), 203–246.

[9] Hill, R., The mathematical theory of plasticity, Oxford at the Clarenden Press, 1950.

[10] Jackson, R., Some mathematical and physical aspects of continuum models for the motion of granular materials, Theory of dispersed multiphase flow, (R. Meyer, Ed., Proc. of Adv. Sem. Math. Research Center, Univ. Wisconsin, May, 1982), Academic Press New York/London, 1982, 291–337.

[11] Kachanov, L., Foundations of the theory of plasticity, American Elsevier, 1971.

[12] Kohn, R. & Temam, R., Dual spaces of stresses and strains with applications to Hencky plasticity, *Appl. Math. Optim.*, **10** (1983), 1–35.

[13] Matthies, H., Strang, G. & Christiansen, E., The saddle point of a differential program, Energy methods in finite element analysis, (R. Glowinski, E. Rodin & D. C. Zienkiewicz Ed.), John Wiley, New-York, 1979, 309–318.

[14] Nečas, J., Sur les normes équivalentes dans $W_p^k(\Omega)$ et sur la coercivité der formes formellement positives, Equations aux dérivées partielles, Presses de l'Université de Montréal, 1966, 102–128.

[15] Roscoe, K. H.,Schofield, A. N. & Wroth, C. P., On the yielding of soils, *Géotech.*, **8** (1958), 22–53.

[16] Schofield, A. N. & Wroth, C. P., Critical state soil mechanics, McGraw-Hill Publishing Company, 1968.

[17] Temam, R., Problèmes mathématiques en plasticité, Dunod, Paris, 1983; or English translation, Dunod USA, 1985.

[18] Temam, R. & Strang, G., Duality and relaxation in the variational problems of plasticity, *J. Mécanique*, **19** (1980), 493–527.

[19] Strang, G., A minimax problem in plasticity theory, Functional analysis methods in numerical analysis, (Proc. Special Sessio, Annual Meeting, Amer. Math. Soc., St Louis, Mo., 1977) Lecture Notes in Math. 701, Springer, Berlin-New York, 1979, 319–333.