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ON THE NONLINEAR BOUNDARY STABILIZATION OF THE WAVE EQUATION

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Abstract

The author studies the energy decay rate for the wave equation in a bounded domain under weak growth assumptions on the feedback function and, applying an integral inequality well adapted to this type of problems, improves some earlier results of Zuazua and of Rao ans Conrad.

§1. Introduction and Statement of the Results

Let Ω be a bounded open set in $\mathbb{R}^n$ having a boundary Γ of class C^2. We shall denote by ν the outward unit normal vector to Γ . Fix a point $x_0 \in \mathbb{R}^n$, set

$$m(x) := x - x_0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

and fix an open subset Γ_- of Γ such that setting $\Gamma_+ = \Gamma \backslash \Gamma_-$ we have

$$m \cdot \nu \geq 0 \ \text{ on } \Gamma_+ \ \text{ and } m \cdot \nu \leq 0 \ \text{ on } \Gamma_-. \tag{1.2}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuous function such that $g(0) = 0$, let α be a nonnegative number and consider the following feedback system:

$$u'' - \Delta u = 0 \ \text{ in } \ \Omega \times \mathbb{R}_+, \tag{1.3}$$

$$u = 0 \ \text{ on } \ \Gamma_- \times \mathbb{R}_+, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} + (m \cdot \nu)(\alpha u + g(u')) = 0 \ \text{ on } \ \Gamma_+ \times \mathbb{R}_+, \tag{1.5}$$

$$u(0) = u_0 \ \text{ and } \ u'(0) = u_1 \ \text{ in } \ \Omega. \tag{1.6}$$

This system is well-posd in the following sense (cf. [2, 13]): introducing the Hilbert space V by

$$V = \{v \in H^1(\Omega) : v = 0 \ \text{ on } \ \Gamma_-\},$$

for every $(u_0, u_1) \in V \times L^2(\Omega)$ the system (1.3)-(1.6) has a unique solution u satisfying

$$C(\mathbb{R}_+; V) \cap C^1(\mathbb{R}_+; L^2(\Omega));$$

furthermore its energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$E = \frac{1}{2} \int_\Omega (u')^2 + |\nabla u|^2 \, dx + \frac{\alpha}{2} \int_{\Gamma_+} (m \cdot \nu) u^2 \, d\Gamma \tag{1.7}$$

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is non-increasing.

Let us assume in the sequel that

$$\text{either } \Gamma_- \neq \emptyset \text{ or } \alpha > 0 \tag{1.8}$$

and that

$$\inf_{\Gamma_+}(m \cdot \nu) > 0. \tag{1.9}$$

We shall prove the following result.

Theorem 1.1. *Assume* (1.2), (1.8), (1.9) *and assume that there exist* $p \in [1, +\infty)$ *and two positive constants* c_1, c_2 *such that*

$$c_1|x|^p \leq |g(x)| \leq c_2|x|^{1/p} \quad \text{if } |x| \leq 1 \tag{1.10}$$

and

$$c_1|x| \leq |g(x)| \leq c_2|x| \quad \text{if } |x| \geq 1. \tag{1.11}$$

Then for every $(u_0, u_1) \in V \times L^2(\Omega)$ *the solution of* (1.3)–(1.6) *satisfies the energy estimates*

$$E(t) \leq Ct^{\frac{2}{1-p}}, \quad \forall t > 0 \quad \text{if } p > 1 \tag{1.12}$$

and

$$E(t) \leq E(0) \exp(1 - \frac{t}{C}), \quad \forall t > 0 \quad \text{if } p = 1, \tag{1.13}$$

where in (1.12) *the constant* C *depends only on the initial energy* $E(0)$ *in a continuous way, while in* (1.13) *the constant* C *is independent of the initial data.*

The linear case for $p = 1$ was studied earlier for example by Quinn and Russell[12], Chen[1], Zuazua. Lagness and the author in [6,7,3]. Strong stabilization results were obtained before by Lasiecka in [8]. The first nonlinear uniform stabilization results of this type are due to Zuazua[13]; he needed stronger growth assumptions on g (he needed the exponent 1 instead of $1/p$ in (1.10) and he assumed that α is not too large. His results were improved in different ways at the same time by Conrad and Rao[2] and by Lasiecka and Tataru[9]. Lasiecka and Tataru studied a more general system containing a nonlinear term in the equation (1.3) and also replacing αu by a nonlinear term $f_1(u)$ (at the price of using a compactness argument which did not lead to explicit estimates). Conrad and Rao extended the results of Zuazua in two directions. First, they obtained analogous estimates for the alternative case where g satisfies (1.10) with p replaced by 1 on the left hand side. Furthermore, using a new multiplier they extended all results for large α in both cases (i.e., if either of the two exponents in (1.10) is replaced by 1). Their method is constructive. Our condition (1.10) is even weaker and we conjecture that it is optimal.

It follows from (1.2) and (19) that $\overline{\Gamma_+} \cap \overline{\Gamma_-} \neq \emptyset$; if α is not too large, then (1.9) may be replaced by this last condition (because (1.9) will be used only in the proof of Lamma 2.4. and this lemma is not necessary for small α), and this weaker condition can also be relaxed if $n \leq 3$ (cf. Rmark 2.1 below). We do not know whether the theorem remains valid in the general case without the condition (1.9).

Our next result shows that the condition (1.11) may be weakened if the solution is more regular.

Theorem 1.2. *Assume again (1.2), (1.8) and (1.9), and assume that g is also locally absolutely continuous. Furthermore, assume that there exist three numbers $p_0 \in [1, +\infty)$, $q \in [1, +\infty)$, $r \in [0, 1]$ and two positive constants c_1 and c_2 such that*

$$c_1|x|^{p_0} \leq |g(x)| \leq c_2|x|^{1/p_0} \quad \text{if } |x| \leq 1, \tag{1.14}$$

$$c_1|x|^r \leq |g(x)| \leq c_2|x|^q \quad \text{if } |x| \geq 1, \tag{1.15}$$

$$q \leq \frac{n}{n-2} \quad \text{if } n > 2 \tag{1.16}$$

and

$$q < +\infty \quad \text{if } n = 2. \tag{1.17}$$

Let us choose $p \in [p_0, +\infty)$ such that

$$p - 1 \geq (n-2)(1-r) \quad \text{if } n > 2, \tag{1.18}$$

$$2p \geq n + \frac{4-n}{q} \quad \text{if } q > 1 \tag{1.19}$$

and

$$p > 1 \quad \text{if } n = 2 \text{ and } r < 1. \tag{1.20}$$

Then for every $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ satisfying

$$u_0 = u_1 = 0 \quad \text{on } \Gamma_-$$

and

$$\frac{\partial u_0}{\partial \nu} + (m \cdot \nu)(\alpha u_0 + g(u_1)) = 0 \quad \text{on } \Gamma_+,$$

the solution of (1.3)–(1.6) satisfies the energy estimates (1.12) and (1.13) with some constant C depending on the initial data.

If $r = 0$, then (1.18) is satisfied for a sufficiently large p; hence this theorem permits to obtain decay rate estimates if the feedback is defined with a bounded function g. The possibility of a result of this type was conjectured by F. Conrad (private communication).

The proof of the above theorems will be based on an integral inequality proved in [5]. More precisely, we shall use the following particular case of [4, Theorem 2.1]:

Theorem 1.3. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function, and assume that there exist a nonnegative number α and a positive number A such that*

$$\int_t^\infty E^{\alpha+1} ds \leq AE(t) \quad \text{for all } t \geq 0. \tag{1.21}$$

Then, putting

$$T := AE(0)^{-\alpha}, \tag{1.22}$$

we have

$$E(t) \leq E(0) \left(\frac{T + \alpha T}{T + \alpha t} \right)^{1/\alpha} \quad \text{for all } t \geq T \tag{1.23}$$

if $\alpha > 0$ and

$$E(t) \leq E(0) \exp\left(1 - \frac{t}{T}\right) \quad \text{for all } t \geq T \tag{1.23}$$

if $\alpha = 0$.

We shall also use some of the former ideas of Conrad, Nakao, Rao and Zuazua [2,13].

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§2. Proof of Theorem 1.1

Using a standard density argument as in [13] we may assume without loss of generality that the initial data belong to $(V\cap H^2(\Omega))\times V$ and that

$$\frac{\partial u_0}{\partial \nu}+(m\cdot\nu)(\alpha u_0+g(u_1))=0 \text{ on } \Gamma_+,$$

then the solution of (1.3)–(1.6) satisfies

$$u\in W^{2,\infty}(\mathbb{R}_+;L^2(\Omega))\cap W^{1,\infty}(\mathbb{R}_+;V)\cap L^\infty(\mathbb{R}_+;H^2(\Omega)) \tag{2.1}$$

(we need for this the property $\overline{\Gamma_+}\cap\overline{\Gamma_-}\neq\emptyset$ following from (1.2) and (1.9), cf. [13]). These regularity properties are sufficient to justify all the computations which follow.

We begin by establishing two identities. They will be obtained by multiplying the equation (1.3) with u' and $2m\cdot\nabla+(n-1)u$, respectively, and by integrating by parts in $\Omega\times(S,T)$ where (T,S) is an arbitrarily fixed bounded interval in $\mathbb{R}_+$.

Lemma 2.1. *The function $E:\mathbb{R}_+\to\mathbb{R}_+$ is non-increasing, locally absolutely continuous and*

$$E'=\int_{\Gamma_+}(m\cdot\nu)u'g(u')\,d\Gamma \quad a.e.\ \ in\ \mathbb{R}_+. \tag{2.2}$$

Proof. Fixing $0\le S<T<\infty$ arbitrarily, we have the equality

$$\begin{aligned}0&=\int_S^T\int_\Omega u'(u''-\Delta u)\,dxdt\\&=\int_S^T\int_\Omega u'u''+\nabla u\cdot\nabla u'\,dxdt-\int_S^T\int_{\Gamma_+}u'\frac{\partial u}{\partial\nu}\,d\Gamma dt\\&=\int_S^T\int_\Omega u'u''+\nabla u\cdot\nabla u'\,dxdt+\int_S^T\int_{\Gamma_+}(m\cdot\nu)(\alpha u+g(u'))u'\,d\Gamma dt\end{aligned}$$

whence

$$E(S)-E(T)=\int_S^T\int_{\Gamma_+}(m\cdot\nu)u'g(u')\,d\Gamma dt \tag{2.3}$$

Since $(m\cdot\nu)\ge0$ on Γ_+ and since $xg(x)\ge0,\forall x\in\mathbb{R}$, the right hand side of (2.3) is nonnegative; hence E is non-increasing. It follows also from (2.3) that E is locally absolutely continuous and that (2.2) is satisfied.

Lemma 2.2. *Putting for brevity*

$$Mu=2m\cdot\nabla u+(n-1)u \tag{2.4}$$

for all $0 \le S < T < \infty$ *we have*

$$2\int_S^T E^{\frac{p+1}{2}}dt - \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_-}(m\cdot\nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt$$
$$=\left[E^{\frac{p-1}{2}}\int_\Omega u'Mudx\right]_S^T + \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'Mudxdt$$
$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)\{(u')^2 - |\nabla u|^2 + \alpha u^2 - (\alpha u + g(u'))Mu\}d\Gamma dt. \tag{2.5}$$

Proof. We have

$$0=\int_S^T E^{\frac{p-1}{2}}dt - \int_\Omega (Mu)(u''-\Delta u)dxdt$$
$$=\left[E^{\frac{p-1}{2}}\int_\Omega u'Mudx\right]_S^T - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'Mudxdt$$
$$-\int_S^T E^{\frac{p-1}{2}}\int_\Omega u'Mu' + (Mu)(\Delta u)dxdt. \tag{2.6}$$

Integrating by parts and using the relation div$m = n$ we may transform the internal integral in the last term as follows:

$$\int_\Omega u'Mu' + (Mu)(\Delta u)dx$$
$$=\int_\Omega m\cdot\nabla(u')^2 + (n-1)(u')^2 - \nabla u\cdot\nabla(Mu)dx + \int_\Gamma (Mu)\frac{\partial u}{\partial\nu}d\Gamma$$
$$=\int_\Omega m\cdot\nabla(u')^2 + (n-1)(u')^2 - 2|\nabla u|^2 - m\cdot\nabla|\nabla u|^2 - (n-1)|\nabla u|^2dx + \int_\Gamma (Mu)\frac{\partial u}{\partial\nu}d\Gamma$$
$$=-\int_\Omega (u')^2 + |\nabla u|^2dx + \int_\Gamma (m\cdot\nu)((u')^2 - |\nabla u|^2) + (Mu)\frac{\partial u}{\partial\nu}d\Gamma$$
$$=-\int_\Gamma (u')^2 + |\nabla u|^2dx + \int_{\Gamma_-} -(m\cdot\nu)|\nabla u|^2 + (2m\cdot\nabla u)\frac{\partial u}{\partial\nu}d\Gamma$$
$$+\int_{\Gamma_+}(m\cdot\nu)\{(u')^2 - |\nabla u|^2 - (Mu)(\alpha u + g(u'))\}d\Gamma.$$

Since (1.4) implies that $\nabla u = \nu\frac{\partial u}{\partial\nu}$ on Γ_-, we conclude, using also (1.7), that

$$\int_\Omega u'Mu' + (Mu)(\Delta u)dx$$
$$=-2E + \int_{\Gamma_-}(m\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma$$
$$+\int_{\Gamma_+}(m\cdot\nu)\{(u')^2 - |\nabla u|^2 + \alpha u^2 - (Mu)(\alpha u + g(u'))\}d\Gamma.$$

Substituting this expression into (2.6) the lemma follows.

Lemma 2.3. *We have*

$$\left|\int_\Omega u'Mudx\right| \le cE. \tag{2.7}$$

(Here and in the sequel we denote by c diverse constants independent of the choice of (u_0, u_1) and of t).

Proof. It follows from (1.8) that the function

$$u \mapsto \left(\int_\Omega |\nabla u|^2 + \alpha \int_{\Gamma_+} (m \cdot \nu) u^2 d\Gamma \right)^{1/2}$$

is a norm on V, equivalent to that defined by $H^1(\Omega)$. Here (2.7) follows by applying the Cauchy-Schwarz inequality.

Now let us first assume that $\alpha = 0$. Using Lemmas 2.2, 2.3 and taking into account that $(m \cdot \nu) \le 0$ on Γ_-, we obtain the inequality

$$\begin{aligned}
&2\int_S^T E^{\frac{p+1}{2}} dt \\
\le & cE^{\frac{p+1}{2}}(S) + cE^{\frac{p+1}{2}}(T) - c\int_S^T E^{\frac{p-1}{2}} E' dt \\
&+ \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 - |\nabla u|^2 - g(u')(2m \cdot \nabla u + (n-1)u)\} d\Gamma dt \\
\le & cE^{\frac{p+1}{2}}(S) + \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 + |m|^2 g(u')^2 + \varepsilon u^2 + \frac{(n-1)^2}{4\varepsilon} g(u')^2\} d\Gamma dt,
\end{aligned}$$

where ε is an arbitrarily fixed positive number (we also used the decreasing property of E). Choosing ε such that

$$\varepsilon \int_{\Gamma_+} (m \cdot \nu) u^2 d\Gamma \le E$$

and taking into account that $|m|^2$ is bounded in Ω (because Ω is bounded), hence we conclude that

$$\int_S^T E^{\frac{p+1}{2}} dt \le cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 + g(u')^2\} d\Gamma dt. \tag{2.8}$$

(In view of (1.7), (1.8) ε may be chosen independently of u). Now we are going to show that (2.8) hold in case $\alpha > 0$, too. Repeating the computations now we have

$$\begin{aligned}
&2\int_S^T E^{\frac{p+1}{2}} dt \\
\le & cE^{\frac{p+1}{2}}(S) + cE^{\frac{p+1}{2}}(T) - c\int_S^T E^{\frac{p-1}{2}} E' dt \\
&+ \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 - |\nabla u|^2 + \alpha u^2 - (\alpha u + g(u'))(2m \cdot \nabla u + (n-1)u)\} d\Gamma dt \\
\le & \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 + \alpha u^2 + |m|^2(\alpha u + g(u'))^2 + (1-n)\alpha u^2 + (1-n) u g(u')\} d\Gamma dt \\
&+ cE^{\frac{p+1}{2}}(S) \\
\le & cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m \cdot \nu)\{(u')^2 + g(u')^2 + u^2\} d\Gamma dt,
\end{aligned}$$

i.e.,

$$2\int_S^T E^{\frac{p+1}{2}}dt \le cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)\{(u')^2+g(u')^2+u^2\}d\Gamma dt. \tag{2.9}$$

In order to deduce (2.8) from (2.9) we need the following lemma:

Lemma 2.4. *There exists a constant c such that for every $\varepsilon\in(0,1)$ we have*

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)u^2\,d\Gamma dt \le cE^{\frac{p+1}{2}}(S)+\varepsilon\int_S^T E^{\frac{p+1}{2}}dt$$
$$+c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)\{|ug(u')|+\varepsilon^{-1}(u')^2\}d\Gamma dt. \tag{2.10}$$

Proof. We apply a method introduced by Conrad and Rao in [2]. we define $\varphi\in H^1(\Omega)$ by

$$\Delta\varphi=0 \ \text{ in } \Omega \ \text{ and } \varphi=u \ \text{ on } \Gamma.$$

It follows from the standard variational theory of elliptic equations (using also (1.9)) that

$$\int_\Omega \varphi^2 \le c\int_{\Gamma_+}(m\cdot\nu)u^2 d\Gamma \le cE \tag{2.11}$$

and applying the Green formula we also have

$$\int_\Omega \nabla\varphi\cdot\nabla(u-\varphi)dx = -\int_\Omega(\Delta\varphi)(u-\varphi)dx+\int_\Gamma\frac{\partial\varphi}{\partial\nu}(u-\varphi)d\Gamma=0$$

whence

$$\int_\Omega \nabla\varphi\cdot\nabla u dx=\int_\Omega|\nabla\varphi|^2dx\ge 0. \tag{2.12}$$

Since φ' satisfies a system analogous to that of defining φ, we have also the estimate

$$\int_\Omega(\varphi')^2\le c\int_{\Gamma_+}(m\cdot\nu)(u')^2d\Gamma. \tag{2.13}$$

Now multiplying (1.3) with $E^{\frac{p-1}{2}}\varphi$ and integrating by part we obtain the equality

$$\begin{aligned}
0 =&2\int_S^T E^{\frac{p-1}{2}}\int_\Omega\varphi(u''-\Delta u)dxdt\\
=&\left[E^{\frac{p-1}{2}}\int_\Omega\varphi u'dx\right]_S^T-\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega\varphi u'dxdt\\
&-\int_S^T E^{\frac{p-1}{2}}\int_\Omega(\varphi'u'-\nabla\varphi\cdot\nabla u)\,dxdt-\int_S^T E^{\frac{p-1}{2}}\int_\Gamma\varphi\frac{\partial u}{\partial\nu}d\Gamma dt\\
=&\left[E^{\frac{p-1}{2}}\int_\Omega\varphi u'dx\right]_S^T-\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega\varphi u'dxdt\\
&-\int_S^T E^{\frac{p-1}{2}}\int_\Omega(\varphi'u'-\nabla\varphi\cdot\nabla u)\,dxdt+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)u(\alpha u+g(u'))d\Gamma dt.
\end{aligned}$$

Using (2.11)-(2.13) hence we conclude that

$$\begin{aligned}
&\alpha\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)u^2\,d\Gamma dt\\
\le&-\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)ug(u')\,d\Gamma dt+\int_S^T E^{\frac{p-1}{2}}\int_\Omega \varphi' u'\,dxdt\\
&+\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega \varphi u'\,dxdt+\left[E^{\frac{p-1}{2}}\int_\Omega \varphi u'\,dx\right]_S^T\\
\le&\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)|ug(u')|d\Gamma dt+cE^{\frac{p+1}{2}}(S)+cE^{\frac{p+1}{2}}(T)\\
&+\int_S^T E^{\frac{p-1}{2}}\int_\Omega\left(\frac{\alpha\varepsilon}{2}(u')^2+\frac{1}{2\alpha\varepsilon}(\varphi')^2\right)dxdt-c\int_S^T E^{\frac{p-1}{2}}E'\,dt\\
\le&\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)|ug(u')|d\Gamma dt+cE^{\frac{p+1}{2}}(S)\\
&+\alpha\varepsilon\int_S^T E^{\frac{p+1}{2}}dt+\frac{c}{\alpha\varepsilon}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)(u')^2 d\Gamma dt.
\end{aligned}$$

Hence (2.10) follows.

Using the inequality

$$c|ug(u')|\le\frac{1}{2}u^2+\frac{c^2}{2}g(u')^2$$

we deduce from (2.10) the estimate

$$\begin{aligned}
&\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)u^2\,d\Gamma dt\\
\le& cE^{\frac{p+1}{2}}(S)+2\varepsilon\int_S^T E^{\frac{p+1}{2}}dt+c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)\left(\frac{1}{\varepsilon}(u')^2+g(u')^2\right)d\Gamma dt.
\end{aligned}$$

Choosing ε sufficiently small and combining with (2.9) hence (2.8) follows.

Now we are going to majorize the last integral in (2.8). We set

$$\Gamma_1=\{x\in\Gamma_+:|u'(x)|\le 1\}\ \text{ and } \Gamma_2=\{x\in\Gamma_+:|u'(x)|>1\}. \tag{2.14}$$

Using (1.11) and (2.2) we obtain that

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)\{(u')^2+g(u')^2\}\,d\Gamma dt\le -c\int_S^T E^{\frac{p-1}{2}}E'\,dt\le cE^{\frac{p+1}{2}}(S). \tag{2.15}$$

Furthermore, using (1.10) we have

$$\begin{aligned}
&\int_{\Gamma_1}(m\cdot\nu)\{(u')^2+g(u')^2\}\,d\Gamma\le c\int_{\Gamma_1}(m\cdot\nu)(u'g(u'))^{\frac{2}{p+1}}d\Gamma\\
\le& c\left(\int_{\Gamma_1}(m\cdot\nu)u'g(u')d\Gamma\right)^{\frac{2}{p+1}}\le c(-E')^{\frac{2}{p+1}}
\end{aligned}$$

and hence, applying the Young inequality, for any $\varepsilon>0$ we have

$$\begin{aligned}
&\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2+g(u')^2\}\,d\Gamma dt\le -c\int_S^T E^{\frac{p-1}{2}}(-E')^{\frac{2}{p+1}}dt\\
\le&\int_S^T\left(\varepsilon E^{\frac{p+1}{2}}-c(\varepsilon)E'\right)dt\le\varepsilon\int_S^T E^{\frac{p+1}{2}}dt+c(\varepsilon)E(S).
\end{aligned} \tag{2.16}$$

Combining (2.8),(2.15) and (2.16) we conclude that

$$\int_S^T E^{\frac{p+1}{2}} dt \le cE^{\frac{p+1}{2}}(S) + \varepsilon c \int_S^T E^{\frac{p+1}{2}} dt + c(\varepsilon)E(S).$$

Choosing ε sufficiently small (such that $\varepsilon c < 1$) it follows that

$$\int_S^T E^{\frac{p+1}{2}} dt \le c(E^{\frac{p+1}{2}}(S) + E(S)). \tag{2.17}$$

Since E is non-incerasing and since $\frac{p+1}{2} \ge 1$, we deduce from (2.17) that

$$\int_S^T E^{\alpha+1} dt \le AE(S) \quad \text{for all } 0 \le S < T < +\infty$$

with $\alpha := \frac{p-1}{2}$ and $A := c(1 + E(0)^\alpha)$. Letting $T \to +\infty$ we obtain (1.21), and Theorem 1.1 follows by applying Theorem 1.3.

Remark 2.1. The conditions (1.2) and (1.9) are satisfied if Ω is star-shaped with respect to x_0 or more generally, if it is the set-theoretical difference of two such domains having disjoint boundaries, but otherwise they represent a strict geometric assumption on Ω. For $n \le 3$ and for small α Theorem 1.1 remains valid without the assumption (1.9): this can be shown by the methods of [3, 6] or [13], applying an inequality due to Grisvard. It is an open question whether Theorem 1.1 remains always valid without the assumption (1.9).

§3. Proof of Theorem 1.2

It follows from our assumptions that u satisfies the regularity property (2.1); this will be sufficient to justify the computations of this section. (We remark that contrary to the preceeding section some constants below will not depend only on the initial energy of u and therefore our result will not extend by density to more general initial data).

Consider first the case $\alpha = 0$. introducing Γ_1 and Γ_2 by (2.14), we deduce from Lemmas 2.2 and 2.3 the following inquality:

$$\begin{aligned} 2\int_S^T E^{\frac{p+1}{2}} dt \le & cE^{\frac{p+1}{2}}(S) + \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_+} (m\cdot\nu)\{(u')^2 - |\nabla u|^2 - g(u')Mu\} d\Gamma dt \\ \le & cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} (m\cdot\nu)\{(u')^2 + g(u')^2\} d\Gamma dt \\ & + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)\{(u')^2 + |Mu||g(u')|\} d\Gamma dt. \end{aligned}$$

Majorizing the first integral on the right hand side in the same way as in section 2, i.e., applying (2.16) with a sufficiently small ε, we obtain that

$$\int_S^T E^{\frac{p+1}{2}} dt \le cE^{\frac{p+1}{2}}(S) + cE(S) + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)\{(u')^2 + |Mu||g(u')|\} d\Gamma dt. \tag{3.1}$$

Now consider the case $\alpha > 0$. We have

$$
\begin{aligned}
&2\int_S^T E^{\frac{p+1}{2}}dt\\
\le& cE^{\frac{p+1}{2}}(S) + \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_+}(m\cdot\nu)\{(u')^2 - |\nabla u|^2 + \alpha u^2 - (\alpha u + g(u'))Mu\}\,d\Gamma dt\\
\le& cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2 + g(u')^2 + u^2\}d\Gamma dt\\
&+ c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)\{(u')^2 + u^2 + |g(u')||Mu|\}d\Gamma dt.
\end{aligned}
$$

Applying Lemma 2.4 with a sufficiently small $\varepsilon > 0$ hence we deduce the inequality

$$
\begin{aligned}
2\int_S^T E^{\frac{p+1}{2}}dt \le& cE^{\frac{p+1}{2}}(S) + c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2 + g(u')^2 + |ug(u')|\}\,d\Gamma dt\\
&+ c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)\{(u')^2 + |g(u')|(|u| + |\nabla u|)\}d\Gamma dt.
\end{aligned}
$$

Using (1.7) and (2.13), for any fixed $\varepsilon > 0$ the first integral on the right hand side is majorized as follows:

$$
\begin{aligned}
&\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2 + g(u')^2 + |ug(u')|\}\,d\Gamma dt\\
\le& \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2 + \varepsilon u^2 + c(\varepsilon)g(u')^2\}d\Gamma dt\\
\le& \frac{2\varepsilon}{\alpha}\int_S^T E^{\frac{p+1}{2}}dt + \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{(u')^2 + c(\varepsilon)g(u')^2\}\,d\Gamma dt\\
\le& \frac{3\varepsilon}{\alpha}\int_S^T E^{\frac{p+1}{2}}dt + c(\varepsilon)E(S);
\end{aligned}
$$

choosing ε such that $\frac{3\varepsilon}{\alpha} < 2$, hence we conclude that

$$
\begin{aligned}
\int_S^T E^{\frac{p+1}{2}}dt \le& cE^{\frac{p+1}{2}}(S) + cE(S)\\
&+ c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)\{(u')^2 + |g(u')|(|u| + |\nabla u|)\}\,d\Gamma dt. \qquad (3.2)
\end{aligned}
$$

Let us observe that (3.1) implies (3.2), too; hence (3.2) is satisfied for any $\alpha \ge 0$. Now we are going to prove for any $\varepsilon > 0$ the estimates

$$
E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)(u')^2\,d\Gamma \le \varepsilon E^{\frac{p+1}{2}} - c(\varepsilon)E' \qquad (3.3)
$$

and

$$
E^{\frac{p-1}{2}}\int_{\Gamma_2}(m\cdot\nu)|g(u')|(|u| + |\nabla u|)\,d\Gamma \le \varepsilon E^{\frac{p+1}{2}} - c(\varepsilon)E'. \qquad (3.4)
$$

Then the theorem will follow. Indeed, choosing ε sufficiently small, we deduce from (3.2)-(3.4) the estimate (2.14) and then the proof may be completed in the same way as that of Theorem 1.1.

In the sequel we write for brevity $\|\ \|_S$ instead of $\|\ \|_{L^S(\Gamma)}$.

First we prove (3.3) and (3.4) for $n=1$. Observe that (1.14) and the increasing property of g imply that $\inf\{|g(x)| : |x| \ge 1\} > 0$. Hence, using also (2.1), (2.2) and applying the trace theorem $(V \subset)H^1(\Omega) \hookrightarrow L^\infty(\Gamma)$ we have

$$\begin{aligned} E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)(u')^2 d\Gamma \le & cE^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)|u'|u'g(u')d\Gamma \\ \le & cE^{\frac{p-1}{2}} \|u'\|_\infty(-E') \le -cE' \end{aligned}$$

and

$$\begin{aligned} & E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)|g(u')|(|u|+|\nabla u|)d\Gamma \\ \le & cE^{\frac{p-1}{2}} \int_{\Gamma_2} |g(u')|d\Gamma \|u\|_{H^2(\Omega)} \le c\int_{\Gamma_2} u'g(u')d\Gamma \le -cE'. \end{aligned}$$

In what follows we assume that $n \ge 2$. First we prove (3.3). We begin by showing for any given $s \in (0,1)$ and $\varepsilon \in (0,1)$ the inequality

$$E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)(u')^2 d\Gamma \le \varepsilon E^{\frac{p-1}{2(1-s)}} \|u'\|_{\frac{2-(r+1)s}{1-s}}^{\frac{2-(r+1)s}{1-s}} - c(\varepsilon)E'. \tag{3.5}$$

Indeed, we have

$$\begin{aligned} & E^{\frac{p-1}{2}} \int_{\Gamma_2} (u')^2 d\Gamma \le cE^{\frac{p-1}{2}} \int_{\Gamma_2} |u'|^{2-(r+1)s}|u'g(u')|^s d\Gamma \\ \le & cE^{\frac{p-1}{2}} \||u'|^{2-(r+1)s}\|_{1/(1-s)} \||u'g(u')|^s\|_{1/s} = -cE^{\frac{p-1}{2}} \|u'\|_{\frac{2-(r+1)s}{1-s}}^{2-(r+1)s} \|u'g(u')\|_1^s \\ = & cE^{\frac{p-1}{2}} \|u'\|_{\frac{2-(r+1)s}{1-s}}^{2-(r+1)s} (-E')^s \le \varepsilon E^{\frac{p-1}{2(1-s)}} \|u'\|_{\frac{2-(r+1)s}{1-s}}^{\frac{2-(r+1)s}{1-s}} - c(\varepsilon)E'. \end{aligned}$$

If $r<1$, then we choose $s=\frac{2}{p+1}$ (it follows from (1.18) and (1.20) that $p>1$). Using (1.18) and applying the trace theorem we have

$$H^1(\Omega) \hookrightarrow L^{\frac{2(p-r)}{p-1}}(\Gamma) = L^{\frac{2-(r+1)s}{1-s}}(\Gamma).$$

Using also (2.1) hence we conclude from (3.5) that

$$E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)(u')^2 d\Gamma \le c\varepsilon E^{\frac{p+1}{2}} - c(\varepsilon)E'$$

which proves (3.3) (with another ε).

If $r=1$, then we have simply

$$E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)(u')^2 d\Gamma \le cE^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)u'g(u')d\Gamma \le -cE'.$$

Turning to the proof of (3.4) we start with the inequality

$$\begin{aligned} & E^{\frac{p-1}{2}} \int_{\Gamma_2} (m\cdot\nu)|g(u')|(|(u')|+|\nabla u|)d\Gamma \\ \le & cE^{\frac{p-1}{2}} \|g(u')\|_{1+q^{-1}} (\|u\|_{1+q} + \|\nabla u\|_{1+q}) \\ \le & cE^{\frac{p-1}{2}} (\|u\|_{1+q} + \|\nabla u\|_{1+q}) \|(u'g(u'))^{\frac{1}{1+q^{-1}}}\|_{1+q^{-1}} \\ \le & cE^{\frac{p+1}{2}} (\|u\|_{1+q} + \|\nabla u\|_{1+q}) (-E')^{\frac{1}{1+q^{-1}}} \\ \le & \varepsilon E^{\frac{(p-1)(1+q)}{2}} (\|u\|_{1+q} + \|\nabla u\|_{1+q})^{1+q} - c(\varepsilon)E'. \end{aligned}$$

Hence (3.4) will be proved if we show that

$$E^{\frac{(p-1)(1+q)}{2}}\left(\|u\|_{1+q}+\|\nabla u\|_{1+q}\right)^{1+q}\le cE^{\frac{p+1}{2}}. \tag{3.6}$$

By the trace theorem we have

$$\|u\|_{1+q}+\|\nabla u\|_{1+q}\le c\|u\|_{H^\beta(\Omega)}, \tag{3.7}$$

where β is defined by

$$\beta=1+\frac{n}{2}-\frac{n-1}{1+q}. \tag{3.8}$$

(It is easy to see that $1\le\beta\le 2$). Set

$$t:=\max\left\{\frac{p+1}{q+1}+1-p,\ 0\right\},$$

then we have $t\in[0,1]$. It follows from (1.6), (1.7), (1.9) and (3.8) that

$$\beta\le 2-t=t+2(1-t);$$

therefore we have the interpolation inequality

$$\|u\|_{H^\beta(\Omega)}\le c\|u\|^t_{H^1(\Omega)}\|u\|^{1-t}_{H^2(\Omega)}.$$

Using (3.7), (1.7) and (2.1) hence we conclude that

$$\|u\|_{1+q}+\|\nabla u\|_{1+q}\le cE^{t/2}$$

and therefore

$$E^{\frac{(p-1)(1+q)}{2}}\left(\|u\|_{1+q}+\|\nabla u\|_{1+q}\right)^{1+q}\le cE^{\frac{(p-1+t)(1+q)}{2}}.$$

Since $(p-1)(1+q)\ge p+1$ by (3.9), hence (3.6) follows.

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