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CONVERGENCE TO TRAP ALMOST EVERYWHERE FOR FLOWS GENERATED BY COOPERATIVE AND IRREDUCIBLE VECTOR FIELDS

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Abstract

This paper is concerned with the asymptotic behavior of cooperative systems in $W \subset R^n$. For a C^2 cooperative system whose Jacobian matrices are irreducible, it is proved that the forward orbit converges to an equilibrium for almost every point having compact forward orbit closure and the set of all points which have compact forward orbit closures and do not converge to a semi-asymptotically stable equilibrium is meager in W if the equilibrium set cannot contain a simply ordered curve. The invariant function and the geometry of the stable manifold of an unstable equilibrium are considered.

§1. Introduction

In recent years there has been considerable interest in cooperative systems which can be modelled by the ordinary differential equations:

$$\dot{x} = F(x), \quad x \in W \subset R^n, \tag{1.1}$$

and for which

$$\frac{\partial F_i}{\partial x_j}(x) \geq 0 \tag{1.2}$$

for $i \neq j$, $x \in W$. Much progress has been made in the study of the asymptotic behavior of solutions of (1.1). In a series of papers[1-5], Hirsch has obtained some very important results for general cooperative systems. He showed that limit sets are invariant sets of systems in one dimension lower (see [1, Theorem 3.1]), and that when in addition each Jacobian matrix $DF(x)$ is irreducible then almost all trajectories with compact closures are quasi-convergent, i.e., their ω-limit sets consist of equilibria (see [2, Theorem 4.1]). Under slightly stronger assumptions, in particular, assuming the set of equilibria is countable, he proved that almost all trajectories with compact closures converge to traps (see [2, Theorem 4.4]). Poláčik[6] proved that the set of all points which have bounded nonconvergent trajectories is meager in some fractional power space for the smooth strongly monotone flows defined by semilinear parabolic equations. But his abstract result is not valid for the cooperative and irreducible systems of ordinary differential equations. One reason is that his hypothesis (M_3) ([6, p.93]) does not hold for cooperative and irreducible systems (since $(\mu I + DF(x))^{-1}$ is nonpositive for $\mu < 0$ and $|\mu|$ sufficiently large). The other reason is that we often consider such a system on an open subset of R^n, which is not the Banach space R^n. In earlier work [13], the author

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has given the necessary and sufficient condition that 3-dimensional cooperative system has a globally asymptotically stable equilibrium, which generalizes [3, Theorem 9].

Hirsch[4,p.28] pointed out: "it is difficult to distinguish quasiconvergent trajectories from convergent ones." The purpose of this paper is to improve the above results of Hirsch in case F is C^2. We shall prove that for systems that are cooperative and irreducible, almost all points whose forward orbits are compact in W are convergent (ω-limit set is a singleton). With slightly stronger (but necessary) assumption, in particular, assuming that there cannot exist a simply ordered curve which consists of equilibria, we shall show that the set of all $x \in W$ having compact orbit closures and not converging to traps is rare in W. Under this stronger assumption, we prove that every continuous invariant function is constant. Our final results concern the geometry of the stable manifold of an unstable equilibrium of (1.1) (see Theorems 3.4 and 3.5).

The principal tool is the theory of cooperative systems as developed by Hirsch[1,2]. We employ the same idea used in [6] to study the asymptotic behavior, as $t \to \infty$, of the principal eigenvalue $\lambda(t)$ and the corresponding principal eigenvector $v(t)$ of $DF(y(t))$ associated with a quasiconvergent solution $y(t)$.

§2. Definitions and Preliminaries

In this section, we give some definitions and state some known results which will be useful in subsequent sections.

Definition 2.1. *An $n \times n$ matrix $A = (a_{ij})$ is called cooperative if each off-diagonal term is nonnegative and irreducible if it leaves invariant no non-trivial coordinate subspaces of R^n.*

Define $s(A) = \max \mathrm{Re}\lambda$, where λ runs through the eigenvalues of A. It follows from Theorem 3.1 in [2] that if A is cooperative and irreducible then e^{tA} is positive for $t > 0$. Therefore, by the Perron theorem ([7, p.52]) we obtain the following

Lemma 2.1. *If A is an $n \times n$ matrix which is cooperative and irreducible, then $s(A)$ is an eigenvalue of A (called the principal eigenvalue of A) that is a simple root of the characteristic equation and exceeds the real parts of all other eigenvalues of A. Corresponding to $s(A)$ there is a unit eigenvector (called the principal eigenvector of A) with positive components.*

Definition 2.2. *A C^1 vector field $F : W \to R^n$ (where $W \subset R^n$ is open) is called cooperative (irreducible) if $DF(x)$ is cooperative (irreducible) for all $x \in W$.*

Let $x, y \in R^n$, we write $x \le y (x < y)$ in case the specified inequality holds componentwise. Sometimes we write $x \lneq y$ to signify that $x \le y$ and $x \ne y$.

For any points x, y in R^n with $x < y$ define

$$\text{the closed order interval } [x, y] = \{u : x \le u \le y\}$$

and

$$\text{the open order interval } [[x, y]] = \{u : x < u < y\}.$$

Definition 2.3. *W is said to be p-convex if it has the following property: whenever $x, y \in W$ and $x \le y$, W contains the entire line segment joining x and y.*

Theorem 2.1. *Let $W \subset R^n$ be p-convex. If the vector field F is cooperative and irreducible then its solution flow $\{\phi_t\}$ is strongly monotone in W, i.e., if $x \lneq y$, then $\phi_t(x) < \phi_t(y)$ for $t > 0$.*

The ω-limit set $\omega(x)$ of a point $x \in W$ is the set of $p \in W$ such that $x(t_k) \to p$ for some sequence $t_k \to \infty$. Let E denote the set of equilibria. x is called quasiconvergent if $\omega(x) \subset E$ and convergent if $\omega(x)$ is a singleton. We use the notations Q and C for the sets of quasiconvergent and convergent points respectively.

Theorem 2.2. (Limit set dichotomy). *Assume that F is a cooperative and irreducible vector field and $x \lneq y$. Let x and y have compact orbit closures in W. Then exactly one of the following holds:*

(a) $\omega(x) < \omega(y)$;

(b) $\omega(x) = \omega(y) \subset E$.

Theorems 2.1 and 2.2 are due to Hirsch and can be found in [2].

Definition 2.4. *An equilibrium $p \in W$ is called a trap if there is some open set N, not necessarily containing p, such that $\phi_t(x)$ converges to p uniformly in $x \in N$ as $t \to \infty$.*

At each point $p \in R^n$ there are positive and negative cones defined by

$$C^+(p) = \{x : p \leq x\} \text{ and } C^-(p) = \{x : p \geq x\}.$$

Definition 2.5. *An equilibrium $p \in W$ of (1.1) is called stable from above (resp. from below) if it is stable in $C^+(p) \cap W$ (resp. $C^-(p) \cap W$). Similarly one defines asymptotical stability from above (resp. from below).*

§3. The Main Results

Our main results may be stated as follows.

Theorem 3.1. *Suppose that F is a C^2 cooperative and irreducible vector field and W is a p-convex open subset of R^n. Then $x(t)$ converges to an equilibrium as $t \to \infty$, for almost all $x \in W^c$, where W^c denotes the set of points whose forward orbits have compact closures in W.*

Theorem 3.2. *Suppose that F satisfies the conditions of Theorem 3.1 and E cannot contain a simply ordered curve. Then the set of all $x \in W$ having compact orbit closures and not converging to a semi-asymptotically stable equilibrium is meager in W.*

Theorem 3.3. *Suppose that F satisfies the conditions of Theorem 3.2. Let $A \subset W$ be a connected open set such that almost every point of A has compact forward orbit closure. Then every continuous invariant function f is constant on A.*

The stable set of an equilibrium p is $S(p) = \{x \in W : \omega(x) = p\}$.

Theorem 3.4. *Let F be a C^1 cooperative and irreducible vector field. $p \in E$ is not a trap if and only if $S(p)$ has Lebesgue measure zero.*

Theorem 3.5. *Let F be a C^1 cooperative vector field. Assume that $p \in E$ is not a trap and $DF(p)$ is irreducible. Then $S(p)$ does not contain distinct points u, v with $u \leq v$.*

Remark. In comparison with the well-known structural Theorem 4.1 in [2], we have imposed the smooth assumption, but our Theorem 3.1 is stronger. It follows from [5, Theorem 9.7] that if E contains a simply ordered curve then there is an order interval $[a, b]$ such that every point in it is convergent to an equilibrium which is not a trap. Therefore, in Theorem

3.2, the condition that E cannot contain a simply ordered curve is necessary. If a cooperative and irreducible vector field possesses an invariant function with positive gradient, then its equilibria consist of a simply ordered curve[14,15]. This implies that in Theorem 3.3, the condition that E cannot contain a simply ordered curve is also necessary. Theorem 3.4 improves Corollary 4.5 in [2] where Hirsch assumed that E is countable; Theorem 3.5 is the same as that of Smith ([16, Theorem 2.10]) except that we eliminate the condition that p is hyperbolic and $s(p) > 0$. Smith's result is a generalization of [9, Theorem 6.3] of Selgrade. But the proof presented here is much simplier than theirs.

§4. Proof of the Theorems

The proof of Theorem 3.1 is conveniently broken into several lemmas. We begin with

Lemma 4.1. *Suppose that A is an $n \times n$ matrix which is cooperative and irreducible. If $s(A) = 0$, then the matrix*

$$\begin{pmatrix} A & -v \\ v^\tau & 0 \end{pmatrix}$$

is invertible, where v is the principal eigenvector of A.

Proof. Let $A^* = (A_{ij})$ denote the adjoint matrix of A. We first prove that A^* is positive or negative. By Lemma 2.1, $s(A)$ is a simple root of the characteristic equation, the associated principal eigenvector v is positive, and other eigenvalues of A hve negative real parts. Since $s(A) = 0$, $A^*A = AA^* = 0$, i.e., every column of A^* is the eigenvector corresponding to $s(A)$. Since the coefficient of one degree term of $D(\lambda) = \det(A - \lambda I)$ is $-\sum\limits_{i=1}^{n} A_{ii}$, there is at least one term, say, $A_{nn} \neq 0$. It is easy to see that A^τ is irreducible. We have $A^\tau(A_{1n}, A_{2n}, \cdots, A_{nn})^\tau = 0$. It follows from $s(A^\tau) = s(A) = 0$ and $A_{nn} \neq 0$ that $A_{in}A_{nn} > 0$ for $i = 1, 2, \cdots, n$. Since the characteristic space corresponding to $s(A)$ is one dimension, there is μ_i such that $(A_{i1}, A_{i2}, \cdots, A_{in})^\tau = \mu_i v$ for $i = 1, 2, \cdots, n$. We deduce $\mu_i \neq 0$ from $A_{in} \neq 0$. Hence, $\mu_i\mu_j > 0$ and $A_{ij}A_{in} > 0$, i.e., $A_{ij}A_{nn} > 0$ for $i, j = 1, 2, \cdots, n$.

By calculation, we can prove that

$$\det\begin{pmatrix} A & -v \\ v^\tau & 0 \end{pmatrix} = v^\tau A^* v = \left(\sum_{i=1}^{n} \mu_i\right)\left(\sum_{i=1}^{n} v_i^2\right) = \sum_{i=1}^{n} \mu_i \neq 0.$$

Lemma 4.2. *Suppose that for any $u \in W$, $A(u)$ is a cooperative and irreducible matrix and that $A(u)$ is continuous on W. If $s(A(u)) \equiv 0$ and the principal eigenvector $v(u) : W \to \mathrm{Int}R^n_+$ is continuous, where $R^n_+ = \{x : x \geq 0\}$, then there exists a basis $\{v(u), v_1(u), \cdots, v_{n-1}(u)\}$ for R^n such that $v_i(u) : W \to R^n$ is continuous for $i = 1, 2, \cdots, n-1$ and $\mathrm{Im}A$ is the linear space spanned by $v_1, v_2, \cdots, v_{n-1}$, where $\mathrm{Im}A$ denotes the image of A.*

Proof. First we shall employs the Gram-Schmidt orthonormalization process. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis for R^n where $e_i = (0, \cdots, 0, {}_i1, 0, \cdots, 0)^\tau$ is the ith vector of the standard basis in R^n. We now define in succession vectors $B_1(u), \cdots, B_n(u)$ by the equations

$$\begin{aligned}
&B_1(u) = v(u),\\
&B_2(u) = e_1 - < e_1, B_1 > B_1,\\
&B_3(u) = e_2 - < e_2, B_1 > B_1 - \frac{< e_2, B_2 >}{< B_2, B_2 >} B_2,\\
&\cdots\cdots\cdots\cdots\\
&B_n(u) = e_{n-1} - \frac{< e_{n-1}, B_{n-1} >}{< B_{n-1}, B_{n-1} >} B_{n-1} - \cdots - < e_{n-1}, B_1 > B_1,
\end{aligned}$$

where $< \cdot, \cdot >$ denotes the usual inner product in R^n. Since $B_1(u) = v(u)$ is continuous on W, it is easy to see that B_i is also continuous on W for $i = 2, 3, \cdots, n$. By induction, we can prove that $\{B_1, B_2, \cdots, B_n\}$ is linearly independent.

From Lemma 2.1, we know that the kernel of A is a one dimensional vector space, hence $\mathrm{Im}A$ is an $(n-1)$-dimensional vector space. Since $AB_1 = Av = 0$, $\mathrm{Im}A = L(AB_2, \cdots, AB_n)$ where $L(AB_2, \cdots, AB_n)$ denotes the linear space spanned by $AB_2, \cdots, AB_n$. Obviously, $AB_2, \cdots, AB_n$ is a basis for $\mathrm{Im}A$. When we restrict our attention on $\mathrm{Im}A$,

$$A|_{\mathrm{Im}A} : \mathrm{Im}A \to \mathrm{Im}A$$

is an isomorphism. It follows that $\{A^2B_2, \cdots, A^2B_n\}$ is also a basis for $\mathrm{Im}A$. This implies that $\{v(u), AB_2, \cdots, AB_n\}$ is linearly independent. Let $v_{i-1}(u) = AB_i$ for $i = 2, 3, \cdots, n$. Therefore, $\{v, v_1, \cdots, v_{n-1}\}$ is a desirable basis.

Lemma 4.3. *If u is an equilibrium of F and $s(DF(u)) = 0$, then there exists a neighborhood V of u such that $E \cap V$ is simply ordered.*

This lemma can be easily proved by the centre manifold theorem and the Perron theorem (see [8, Lemmas 4.5 and 4.6] for details).

Let $u \in W$. Denote $\lambda(u) = s(DF(u))$ and write $v(u)$ for the principal eigenvector of $DF(u)$.

Suppose that $y \in W$ has compact orbit closure. The principal eigenvalue and eigenvector of $DF(y(t))$ are denoted, respectively, by $\lambda(t)$ and $v(t)$. We have

Lemma 4.4. *Let $y \in W$ have compact orbit closure. Then $v(t) : (0, \infty) \to \mathrm{Int}R^n_+$ and $\lambda(t) : (0, \infty) \to R$ are C^1-functions and*

(i) if $y \in Q$ then both $\dot{\lambda}(t)$ and $\|\dot{v}(t)\|$ converge to zero, as $t \to \infty$,

(ii) if $y \in Q \backslash C$ or $\omega(y) = u$ with $s(DF(u)) > 0$ then there exist positive numbers m, r, t_1 such that $v(t) > rv(0)$ and $\lambda(t) > m$ for all $t > t_1$.

Proof. Define the mapping $G : W \times \mathrm{Int}R^n_+ \times R \to R^{n+1}$ by

$$G(u, v, \lambda) = \begin{pmatrix} DF(u)v - \lambda v \\ \|v\| \end{pmatrix}.$$

It is easy to see that G is C^1 on $W \times \mathrm{Int}R^n_+ \times R$ and $G(u, v, \lambda) = (0, 1)^\tau$ if and only if $v = v(u)$, $\lambda = \lambda(u)$. Let u_0, v_0, λ_0 with $G(u_0, v_0, \lambda_0) = (0, 1)^\tau$ being fixed. By calculation, we get

$$D_{(v,\lambda)}G(u_0, v_0, \lambda_0) = \begin{pmatrix} DF(u_0) - \lambda_0 I & -v_0 \\ v_0^\tau & 0 \end{pmatrix}.$$

Using Lemma 4.1 we know that $D_{(v,\lambda)}G(u_0, v_0, \lambda_0)$ is invertible. The implicit theorem implies that $v(u)$, $\lambda(u)$ are C^1. Since $y(t) : (0,\infty) \to W$ is C^1, $\lambda(t)$ and $v(t)$ are C^1-functions.

Suppose that $y \in Q$. First we shall prove (ii). By [2, Theorem 2.3], there cannot exist u, v in $\omega(y)$ with $u \lneq v$. By Lemma 4.3, if $y \in Q\backslash C$ then each $u \in \omega(y)$ has $\lambda(u) > 0$. If $y \in C$, then this holds by assumption. Since $\omega(y)$ is compact in W and $\lambda(u)$ is continuous on W, there is an $m > 0$ such that $\lambda(u) > m$ for u near $\omega(y)$. From the continuity of $v(u)$ we obtain that the set $v(\omega(y))$ is compact in $\mathrm{Int}R^n_+$. Therefore, there exist two disjoint open subsets U, $V \subset R^n$ such that $0 \in U$, $v(\omega(y)) \subset V$ and $U < V$. Now, there is an $r > 0$ such that $rv(0) \in U$. For u close to $\omega(y), v(u) \in V$. So (ii) holds due to the fact that $y(t)$ approaches $\omega(y)$ in W.

Denote $A(u) = DF(u) - \lambda(u)I$. Then $s(A(u)) \equiv 0$. As proved above, $v(u)$ is C^1. Applying Lemma 4.2, we conclude that there is a basis $\{v(u), v_1(u), \cdots, v_{n-1}(u)\}$ for R^n such that $L(v_1, \cdots, v_{n-1}) = \mathrm{Im}A$ and $v_i(u)$ is continuous on W for $i = 1, 2, 3, \cdots, n-1$. Therefore, for any $w \in R^n$, there exist numbers $k_0, k_1, \cdots, k_{n-1}$ such that

$$w = k_0 v + k_1 v_1 + \cdots + k_{n-1} v_{n-1}. \tag{4.1}$$

Define the mapping $P : W \times R^n \to R^n$ by

$$P(u, w) = k_0(u, w) v(u).$$

Let $B(u)$ denote the matrix $(v(u), v_1(u), \cdots, v_{n-1}(u))$. Then $(k_0, k_1, \cdots, k_{n-1}) = B^{-1}(u)w$. Hence $k_i(u, w)$ is continuous on W for $i = 0, 1, \cdots, n-1$. It follows that $P(u, w)$ is continuous on $W \times R^n$. From (4.1), for fixed u, P is a linear mapping about w.

Now differentiating the identity

$$DF(y(t))v(t) = \lambda(t)v(t)$$

with respect to t we obtain

$$D^2F(y(t))\dot{y}(t)v(t) + DF(y(t))\dot{v}(t) = \dot{\lambda}(t)v(t) + \lambda(t)\dot{v}(t) \tag{4.2}$$

and hence

$$P(y(t), D^2F(y(t))\dot{y}(t)v(t)) = \dot{\lambda}(t)v(t). \tag{4.3}$$

Since $y(t)$ is quasiconvergent, $\lim\limits_{t\to\infty} \dot{y}(t) = 0$ (see [4, p.31]). Due to the fact that F is C^2 it follows from (4.3) and the continuity of P that

$$\lim_{t\to\infty} |\dot{\lambda}(t)| = \lim_{t\to\infty} |P(y(t), D^2F(y(t))\dot{y}(t)v(t))| = 0.$$

We claim that $\lim\limits_{t\to\infty} \dot{v}(t) = 0$. Suppose not, there exists a sequence $t_n \to \infty$ such that $\lim\limits_{n\to\infty} \dot{v}(t_n) = v_0 \neq 0$. Without loss of generality, we may assume that $\lim\limits_{n\to\infty} y(t_n) = y_0$ and $\lim\limits_{n\to\infty} \lambda(t_n) = \lambda_0$. From (4.2), $DF(y_0)v_0 = \lambda_0 v_0$. Therefore, either $v_0 > 0$, or $v_0 < 0$. We assume the former case holds. $v(y_0) = v_0/\|v_0\|$. Since $< v(t), v(t) >\equiv 1$, $< v(t), \dot{v}(t) >\equiv 0$. It follows that $< v(t_n), \dot{v}(t_n) >= 0$ for $n = 1, 2, \cdots$. Letting $n \to \infty$, we obtain $< v(y_0), \|v_0\| v(y_0) >= 0$, i.e., $\|v_0\| = 0$, a contradiction, which implies our claim holds.

Lemma 4.5. *Let $y \in Q\backslash C$ or $\omega(y) = u_0$ which is not a trap and isolated in E. Assume that there is a neighborhood $[[u, v]]$ of y such that every point in it has compact orbit closure. Then there are two equilibria p, q and two points u_1, v_1 such that $p < u < u_1 < y < v_1 < v < q$ and $\lim\limits_{t\to\infty} z(t) = p$ for all $z \in [[u_1, y]]$ and $\lim\limits_{t\to\infty} w(t) = q$ for all $w \in [[y, v_1]]$.*

Proof. Let $z \in [[y,v]]$. We first prove that $\omega(z) > \omega(y)$. As in Lemma 4.4, denote $v(t) = v(y(t))$, $\lambda(t) = \lambda(y(t))$. Let $\lambda(u) > 0$ for $u \in \omega(y)$.

$$\begin{aligned} &\frac{d}{dt}(y(t)+\varepsilon v(t)) - F(y(t)+\varepsilon v(t)) \\ =&F(y(t)) + \varepsilon\dot{v}(t) - F(y(t)+\varepsilon v(t)) \\ =&\varepsilon\dot{v}(t) - \varepsilon\int_0^l DF(y(t)+\varepsilon sv(t))v(t)ds \\ =&\varepsilon\dot{v}(t) + \varepsilon\int_0^l [DF(y(t)) - DF(y(t)+\varepsilon sv(t))]v(t)ds - \varepsilon\lambda(t)v(t). \end{aligned} \tag{4.4}$$

Since F is C^2, for $\varepsilon > 0$ sufficiently small and $t > t_1$

$$\int_0^t [DF(y(t)) - DF(y(t)+\varepsilon sv(t))]v(t)ds \leq 1/2mrv(0) < 1/2mv(t). \tag{4.5}$$

By Lemma 4.4, $\dot{v}(t) \to 0$. Therefore, there is a $t_2 > t_1$ such that

$$\dot{v}(t) < 1/2mrv(0) < 1/2mv(t), \quad \text{for } t > t_2. \tag{4.6}$$

Using (4.4)-(4.6) we see that for small $\varepsilon > 0$ and $t > t_2$ the function $\widetilde{y}(t) = y(t) + \varepsilon v(t)$ satisfies

$$\frac{d}{dt}\widetilde{y}(t) - F(\widetilde{y}(t)) < (1/2m + 1/2m - m)v(t) = 0. \tag{4.7}$$

Since $z > y$, $z(t_2) > y(t_2)$. Make $\varepsilon > 0$ so small that $z(t_2) > \widetilde{y}(t_2)$. Then by the well-known Kamke theorem $z(t) > \widetilde{y}(t)$ for all $t > t_2$, i.e.,

$$z(t) - y(t) \geq \varepsilon v(t) > \varepsilon rv(0). \tag{4.8}$$

By Theorem 2.3 and (4.8), $\omega(z) > \omega(y)$. If $\omega(y) = u$ which is not a trap, then it easily follows from Theorem 2.3 and Definition 2.4 that $\omega(z) > \omega(y)$ for all $z \in [[y,v]]$.

Applying Lemma 4.3 we know that if $y \in Q\backslash C$ then $\lambda(u) > 0$ for each $u \in \omega(y)$. Let $u \in \omega(y)$. If $\lambda(u) > 0$, then u has a 1-dimensional strong unstable manifold tangent to the principal eigenvector $v(u)$ at u. If $\lambda(u) = 0$, by assumption, u is not a trap, then [12, Theorem 3] implies that u also has a 1-dimensional unstable manifold tangent to the principal eigenvector $v(u)$ at u. Fix $u \in \omega(y)$ and choose a point $u_1 > y(t_3)$ (for some t_3) such that it belongs to the unstable manifold of u. Since the solution of (1.1) is continuous with respect to initial conditions, there is a point $v_1 \in [[y,v]]$ such that $z \in [[y,v_1]]$ implies $z(t_3) < u_1$. Applying Theorem 2.2, we get $\omega(z) \leq \omega(u_1)$. It follows from Lemma 2.1 and [9, Lemma 2.3] that $u_1(t)$ is increasing for $t \in (-\infty,\infty)$, i.e., if $t_1 < t_2$, then $u_1(t_1) < u_1(t_2)$. As proved above $\omega(z) > u$. Therefore, $\omega(z) > u_1$, which implies $\omega(z) \geq \omega(u_1)$. So far we can conclude that $\omega(z) = \omega(u_1)$. This shows that $\omega(z) = \omega(u_1) = q$ for any $z \in [[y,v_1]]$. It is easy to see that q is asymptotically stable from below.

The proof of the case $z < y$ is analogous.

Proof of Theorem 3.1. Define

$$N = \{x \in W^c : \omega(x) \text{ is not a singleton}\}.$$

To prove that N has measure zero it suffices by Fubini's theorem or [5, Lemma 7.7] to prove that $N \cap L$ is countable for every line segment L joining a and b with $a < b$. By Lemma 4.5, the set $N \cap L$ is discrete, therefore countable. This completes the proof of Theorem 3.1.

Recall that $A \subset W$ is called nowhere dense in W if its closure $\bar{A}$ has empty interior; A is called meager (of first category) in W if A is the union of a countable set of nowhere dense subsets of W. From [5, Theorem 7.5] and the proof above we conclude that N is meager in W.

Let $L \subset R^n$ be a one-dimensional linear subspace spanned by a positive vector. Then L has a complementary closed linear subspace $M \subset R^n$ so that $R^n = M \oplus L$. Define $Y = \{x \in W^c \backslash N : \omega(x) \text{ is not a trap}\}$. Due to compactness of the semiflow and the Kuratowski-Ulam Theorem[18], simple modifications of Hirsch's argument in Lemma 7.4 of [5] yield Lemma 4.6.

Lemma 4.6. *Assume that for any $u \in M$, $(u+L) \cap Y$ is meager in $u+L$. Then Y is meager in W.*

Proof of Theorem 3.2. In the proof of Theorem 3.1, we have proved that N is meager in W. Therefore, we only have to prove that the set Y is meager in W. By Lemma 4.6, it suffices to examine that $L \cap Y$ is meager in L for every straight line spanned by a positive vector. Since the union of any countable family of meager sets is meager, we only have to prove that $J \cap Y$ is meager in J for any line segment J which joins points a and b with $a < b$. Without loss of generality, we can assume that every point in J has compact orbit closure. We claim that Y is nowhere dense in J. Suppose the contrary. Then there is an open line segment $J_1 \subset J$ such that $J_1 \subset \text{clos}(Y)$. We shall prove that $J_1 \subset Y$. For any $x \in J_1$, there exist two sequences $\{x_n^i\} \subset Y$ such that $x_n^i \to x$ as $n \to \infty$ for $i = 1, 2$, and

$$x_1^1 > x_2^1 > \cdots > x_n^1 > x_{n+1}^1 > \cdots > x,$$

$$x_1^2 < x_2^2 < \cdots < x_n^2 < x_{n+1}^2 < \cdots < x.$$

By assumption, $p_n = \omega(x_n^1)$ is not a trap, neither is $q_n = \omega(x_n^2)$. Therefore,

$$p_1 > p_2 > \cdots > p_n > p_{n+1} > \cdots,$$

$$q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots.$$

Let $p = \lim\limits_{n\to\infty} p_n$ and $q = \lim\limits_{n\to\infty} q_n$. Obviously, $p, q \in E$. Hence, $q \le \omega(x) \le p$. We assert that $q = \omega(x) = p$. Otherwise, for example, $\omega(x) < p$, then there is a number t_0 such that $x(t_0) < p$. Since the solutions of (1.1) continuously depend on initial conditions, there is a neighborhood U of x such that $z(t_0) < p$ for any $z \in U$. It follows from $\lim\limits_{n\to\infty} x_n^1 = x$ that there is an N such that $x_n^1 \in U$ for $n \ge N$, so $\omega(x_n^1) = \omega(x_n^1(t_0)) \le p$, i.e., $p_n < p$ for $n > N$, a contradiction. This shows that $\omega(x) = p = q$. Therefore, p is not a trap. By definition, $x \in Y$, which shows $J_1 \subset Y$.

Let $x \in J_1$ and $\omega(x) = p$. Then there is a p-convex neighborhood V of p in W and a C^1 locally centre manifold C of V of dimension one, passing through p and tangent to $v(p)$ at p such that every invariant set of (1.1) in V belongs to C (see [17, p.322]). Since $v(p) > 0$, C can be chosen to be simply ordered. Choose $y \in J_1$ with $x < y$ such that $\omega(y) = q \in C$. Therefore, for any $z \in [x, y] \cap J_1$, $\omega(z) \in C$. Replace $[x, y] \cap J_1$ by J_2. We assert that $C_1 = [p, q] \cap C \subset E$. Suppose not, there is an $r \in C_1$ such that $r \notin E$. So there is an open arc C_2^1 of C such that $C_2^1 \cap E = \emptyset$. Suppose that C_2 is such an arc which is maximal in the sense that $E \cap \text{clos} C_2 \ne \emptyset$. Therefore, $E \cap \text{clos} C_2 = \{w, z\}$. Since p is not a trap and is not isolated in $E, p < w$. For $u \in J_2$ close to $x, \omega(u) < w$. Define

$$u = \sup\{a \in J_2 : \omega(v) \leq w \text{ for any } v \in [x, a] \cap J_2\}.$$

By the definition of u and the continuity of solutions of (1.1) with respect to initial conditions, $\omega(u) = w$. By similar way, we can prove that there is a point $v \in J_2$ such that $\omega(v) = z$. Obviously, $u < v$. For any $r \in [[u, v]] \cap J_2$, $\omega(u) < \omega(r) < \omega(v)$, i.e., $w < \omega(r) < z$, which implies that the equilibrium $\omega(r) \in C_2$, a contradiciton. This shows that (1.1) has a simply ordered curve which consists of equilibria, contradicting our assumption. Therefore, Y is nowhere dense in W.

It remains to prove that if $p \in E$ is a trap then p is either asymptotically stable from above or asymptotically stable from below. It is easy to see that $\lambda(p) \leq 0$. If $\lambda(p) < 0$, then p is asymptotically stable. Suppose that $\lambda(p) = 0$. Then there is a change of coordinates in a neighborhood of p which transforms (1.1) to

$$\begin{aligned} \dot{y} &= u(y, z), \\ \dot{z} &= Bz + v(y, z) \end{aligned} \tag{4.9}$$

where $y \in R^1$, $z \in R^{n-1}$ and B is a constant matrix such that all eigenvalues of B have negative real parts. It follows from the centre manifolds theorem that there exists a centre manifold for (4.9) $z = h(y)$, $|y| < \delta$, where h is C^2 (see [10, p.4]). The flow on the centre manifold is governed by the one-dimensional system

$$\dot{w} = u(w, h(w)). \tag{4.10}$$

In [12], we have proved that the zero solution of (4.9) is a trap if and only if the zero solution of (4.10) is a trap. Applying this result, we know that the zero solution of (4.10) is a trap, that is, there is a nonzero solution $w(t)$ of (4.10) such that $\lim\limits_{t\to\infty} w(t) = 0$, correspondingly (1.1) has a solution $x(t)$ such that $\lim\limits_{t\to\infty} x(t) = p$ and it is tangent to the principal eigenvector $v(p)$ at p. Therefore, for t sufficiently large, $x(t)$ is strictly monotone with respect to the relation $<$. This implies that p is either asymptotically stable from above or asymptotically stable from below. The proof of Theorem 3.2 is complete.

Proof of Theorem 3.3. Define

$$T = \{x \in E : x \text{ is a trap}\}.$$

Then $S(x) \cap S(y) = \emptyset$ for distinct points $x, y \in T$. Since the stable set of a trap p contains an open subset of W, $S(p)$ contains a point r_p every component of which is a rational nubmer. Define the map $e : T \to R^n$ by

$$e(p) = r_p.$$

In fact, e is injective, which proves that T is countable.

The rest of the proof is similar to [2, Theorem 4.7].

Proof of Theorem 3.4. By [5, Lemma 7.7], it suffices to prove that every simply ordered subset of $S(p)$ is countable. Since p is not a trap, there cannot exist distinct points $x, y \in S(p)$ with $x < y$. Therefore, every simply ordered subset of $S(p)$ is a single point. This proves Theorem 3.4.

Proof of Theorem 3.5. It follows from the irreducibility of $DF(p)$ and the continuity of $DF(x)$ that there is a neighborhood U of p such that F is cooperative and irreducible on U. If there are two points u, v in $S(p)$ with $u \lesssim v$, then there exists a $T > 0$ such that $u(t), v(t) \in U$ for $t \geq T$. From the Kamke theorem, we have $u(T) \lesssim v(T)$. It follows from the irreducibility of $DF(x)$ on U that $u(t) < v(t)$ for $t > T$. Choose $t_0 > T$. Then $\lim_{t\to\infty} z(t) = p$ for any $z \in [u(t_0), v(t_0)]$, i.e., $S(p)$ contains an open set $[[u(t_0), v(t_0)]]$. Therefore, p is a trap, contradicting our assumption. This contradiction completes the proof of Theorem 3.5.

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