*Chin. Ann. of Math.*
**14B**: 2(1993), 175-182.

# EXISTENCE OF DISCONTINUOUS SOLUTIONS FOR A DOUBLY DEGENERATE ELLIPTIC EQUATIONS ON $I\!R^N$

WANG JUNYU*

**Abstract**

It is demonstrated that under the hypotheses I–III the problem

$$\begin{cases} \operatorname{div}((k(U)+\varepsilon)|DU|^{M-1}DU) = f(|x|,U)+\varepsilon U \text{ in } I\!R^N, N>1, & (1.1)_\varepsilon \\ U(0)>0, U(x)\geq 0 \text{ on } I\!R^N, U(x)\to 0 \text{ as } |x|\to +\infty & (1.2) \end{cases}$$

for each fixed $\varepsilon > 0$ has infinitely many distinct radially symmetric solutions $U_\varepsilon = V_\varepsilon(|x|)$ such that $V_\varepsilon(s)$, $s^{N-1}(k(V_\varepsilon(s))+\varepsilon)|V_\varepsilon'(s)|^{M-1}V_\varepsilon'(s) \in C[0,+\infty)\cap C^1(0,+\infty)$,

$$\begin{cases} (s^{N-1}(k(V_\varepsilon(s))+\varepsilon)|V'(s)|^{M-1}V'(s)) = s^{N-1}(f(s,V_\varepsilon(s))+\varepsilon V_\varepsilon(s)) \text{ for } s>0, & (1.3)_\varepsilon \\ V_\varepsilon(0)=B>0, V_\varepsilon(s)\geq 0 \text{ for } s>0, \text{ and } V_\varepsilon(+\infty)=0, & (1.4) \end{cases}$$

where $B$ is a positive number chosen arbitrarily, which extends the result in [3]. In particular, the author proves that $U_0(x) := V_0(|x|)$ is a weak solution of the problem $(1.1)_0$-(1.2).

## §1. Introduction

In this paper we demonstrate the existence of infinitely many radially symmetric, continuous solutions of the problem

$$\begin{cases} \operatorname{div}((k(U)+\varepsilon)|DU|^{M-1}DU) = f(|x|,U)+\varepsilon U \text{ in } I\!R^N, N>1, & (1.1)_\varepsilon \\ U(0)>0, U(x)\geq 0 \text{ on } I\!R^N, U(x)\to 0 \text{ as } |x|\to +\infty, & (1.2) \end{cases}$$

where $D$ stands for the gradient operator, under the following hypotheses:

I. $\varepsilon > 0$ is a small parameter and $M > N-1 (>0)$ is a given constant.

II. $k(t)$ is a nonnegative continuous function defined on $[0,+\infty)$.

III. $s^{(N-1)(M+1)/M} f(s,t)$ is a nonnegative continuous function defined on $[0,+\infty)\times[0,+\infty)$ such that it is increasing in $s$ for each fixed $t>0$, and $f(s,0)\equiv 0$ for all $s>0$.

Our aim of studying the problem involving a small parameter $\varepsilon$ is to determine solutions for the limiting case $\varepsilon = 0$, in which discontinuities may appear when the function $k(t)$ has intervals of degeneracy, and to ascertain its properties when solutions for the case $\varepsilon>0$ are found. By an interval of degeneracy we mean a closed interval on which $k(t)=0$. It must be pointed out that the equation $(1.1)_0$ is of doubly degenerate elliptic type when $k(t)$ has zeros.

Solutions of the problem $(1.1)_\varepsilon$-(1.2) are sometimes called "ground state", a term borrowed from the physical context (nonlinear field equations) in whcih the problem $(1.1)_\varepsilon$-(1.2)

Manuscript received December 11, 1990. Revised March 19, 1992.

*Department of Mathematics, Jilin University, Changchun 130023, Jilin, China .

arises. In fact, the equation $(1.1)_\varepsilon$ is regarded as a steady state generalized diffusion equation with absorption (see [1, 4]). A similar problem of the form

$$\Delta U = f(U) \text{ in } \mathbb{R}^N, N > 1$$
$$U(x) > 0 \text{ in } \mathbb{R}^N, U(x) \to 0 \text{ as } |x| \to +\infty$$

has been studied by authors under various restrictions on $f(U)$ (see for instance [2] and its references). Moreover, in a recent paper [3] G.Citti proved that the quasilinear degenerate elliptic equation

$$\operatorname{div}(|DU|^{M-1}DU) = f(U) \text{ in } \mathbb{R}^N, 0 < M < N-1$$

has infinitely many distinct radially symmetric solutions under some restrictions on $f(U)$, applying the critical point theory of Ljusternik-Schnirelmann type.

Utilizing only the theory of ordinary differential equations and the Schauder Fixed Point Theorem, we demonstrate in this paper that the problem $(1.1)_\varepsilon$-(1.2) for each fixed $\varepsilon > 0$ has infinitely many distinct radially symmetric solutions of the form

$$U_\varepsilon(x) := V_\varepsilon(s), s = |x| := (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2},$$

such that $V_\varepsilon(s)$, $s^{N-1}(k(V_\varepsilon(s)) + \varepsilon)|V_\varepsilon'(s)|^{M-1}V_\varepsilon'(s) \in C[0,+\infty) \cap C^1(0,+\infty)$, and

$$\begin{cases} [s^{N-1}(k(V_\varepsilon(s)) + \varepsilon)|V_\varepsilon'(s)|^{M-1}V_\varepsilon'(s)]' = s^{N-1}(f(s, V_\varepsilon(s)) + \varepsilon V_\varepsilon(s)) \text{ for } s > 0, & (1.3)_\varepsilon \\ V_\varepsilon(0) = B > 0, V_\varepsilon(s) \geq 0 \text{ on } [0,+\infty), \text{ and } V_\varepsilon(+\infty) = 0, & (1.4) \end{cases}$$

where $B$ is a positive number chosen arbitrarily, under the hypotheses I–III; the function $V_\varepsilon(s)$ pointwise converges to a limit $V_0(s)$ as $\varepsilon \downarrow 0$, the limit $U_0(x) := V_0(|x|)$ is called a solution of the reduced problem $(1.1)_0$-(1.2); the solution $V_0(s)$ has jump points (discontinuity points of the first kind), when the function $k(t)$ possesses at least one interval of degeneracy in $[0, B]$, and there is a one-to-one correspondence between the collection of all intervals of degeneracy in $(0, B)$ and the set of all jump points appearing in $V_0(s)$; $V_0(s)$ satisfies the equation $(1.1)_0$ at all of its continuity points, while at each of its jump points $V_0(s)$ must satisfy certain jump conditions.

## §2. Analysis

A function $U(x)$ is said to be a solution of the problem (1.1)-(1.2) with $\varepsilon > 0$ if

(a) $U_\varepsilon(x) \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \backslash \{0\})$,

(b) $U_\varepsilon(0) > 0$, $U_\varepsilon(x) \geq 0$ on $\mathbb{R}^N$, $U_\varepsilon(x) \to 0$ as $|x| \to +\infty$,

(c) $(k(U_\varepsilon(x)) + \varepsilon)|DU_\varepsilon(x)|^{M-1}DU(x) \in C^1(\mathbb{R}^N\backslash\{0\}) \times \cdots \times C^1(\mathbb{R}^N\backslash\{0\})$, and

(d) $\operatorname{div}((k(U_\varepsilon(x)) + \varepsilon)|DU_\varepsilon(x)|^{M-1}DU_\varepsilon(x)) = f(|x|, U_\varepsilon(x)) + \varepsilon U_\varepsilon(x)$ in $\mathbb{R}^N\backslash\{0\}$.

If the solution $U_\varepsilon(x)$ pointwise converges to a limit $U_0(x)$ as $\varepsilon \downarrow 0$, then the limit is called a solution of the reduced problem $(1.1)_0$-(1.2).

By looking for solutions of the form

$$U(x) = V(s), \quad s = |x|,$$

we arrive at the boundary value problem $(1.3)_\varepsilon$-(1.4).

By a solution of the boundary value problem $(1.3)_\varepsilon$-(1.4) with $\varepsilon > 0$, we mean a nonnegative function $V_\varepsilon(s)$ satisfying the following conditions:

(a) $V_\varepsilon(s) \in C[0,+\infty) \cap C^1(0,+\infty)$,

(b) $V_\varepsilon(0) = 0$, $V_\varepsilon(+\infty) = 0$,

(c) $s^{N-1}(k(V_\varepsilon(s)) + \varepsilon)|V'_\varepsilon(s)|^{M-1}V'_\varepsilon(s) \in C[0,+\infty) \cap C^1(0,+\infty)$, and

(d) $(1.3)_\varepsilon$ holds for all $s > 0$.

If the solution $V_\varepsilon(s)$ pointwise converges to a limit $V_0(s)$ as $\varepsilon \downarrow 0$, then $V_0(s)$ is said to be a solution of the reduced boundary value problem $(1.3)_0$-(1.4).

Obviously, if $V_\varepsilon(s)$ is a solution of $(1.3)_\varepsilon$-(1.4) and $V_0(s)$ is a solution of the reduced problem $(1.3)_0$-(1.4), then $U_\varepsilon(x) := V_\varepsilon(|x|)$ is a solution of $(1.1)_\varepsilon$-(1.2) and $U_0(x) := V_0(|x|)$ a solution of the reduced problem $(1.1)_0$-(1.2). So we shall consider only the boundary value problem $(1.3)_\varepsilon$-(1.4) in the sequel.

**Remark.** A solution of the reduced problem $(1.1)_0$-(1.2) is a weak solution of the elliptic problem $(1.1)_0$-(1.2) in the following sense: for any $\phi(x) \in C_0^1(\mathbb{R}^N\backslash\{0\})$

$$\int_{\mathbb{R}^N} (D\phi(x)(k(U_0(x))|DU_0(x)|^{M-1}DU_0(x) + \phi(x)f(|x|, U_0(x)))dx = 0. \qquad (*)$$

In fact, if $U_\varepsilon(x)$ is a solution of $(1.1)_\varepsilon$-(1.2), then for any $\phi(x) \in C_0^1(\mathbb{R}^N\backslash\{0\})$

$$\int_{\mathbb{R}^N} [D\phi(x)(k(U_\varepsilon(x) + \varepsilon)|DU_\varepsilon(x)|^{M-1}DU_\varepsilon(x)) + (f(|x|, U_\varepsilon(x)) + \varepsilon U_\varepsilon(x))\phi(x)]dx = 0.$$

Letting $\varepsilon \downarrow 0$, we get $(*)$. Here we have used the facts that both $U_\varepsilon(x)$ and $Y_\varepsilon(x) := (k(U_\varepsilon(x)) + \varepsilon)|DU(x)|^{M-1}DU_\varepsilon(x)$ are uniformly bounded and $Y_\varepsilon(x)$ pointwise converges to a limit $Y_0(x)$ as $\varepsilon \downarrow 0$, which will be proved in the sequel.

Let $V(s)$ be a solution of $(1.3)_\varepsilon$-(1.4). If it is strictly decreasing, then the function $s = Z(t)$, inverse to $t = V(s)$, exists, $Z(B) = 0$, $V(Z(t)) \equiv t$ in $(0, B]$, and $V'(Z(t)) = 1/Z'(t) < 0$ in $(0, B)$. Inserting $s = Z(t)$ into $(1.3)_\varepsilon$ and then putting

$$W(t) := Z^{N-1}(t)|Z'(t)|^{-M}(k(t) + \varepsilon),$$

we formally obtain a two-point boundary value problem of the form

$$\begin{cases} Z'(t) = -Z^{(N-1)/M}(t)W^{-1/M}(t)(k(t) + \varepsilon)^{1/M} \quad \text{in } (0, B), & (2.1)_\varepsilon \\ W'(t) = -Z'(t)Z^{N-1}(t)(f(Z(t), t) + \varepsilon t) \quad \text{in } (0, B), & (2.2)_\varepsilon \\ W(0) = 0, \quad Z(B) = 0. & (2.3)_0 \end{cases}$$

## §3. Two-Point Boundary Value Problem $(2.1)_\varepsilon$–$(2.2)_\varepsilon$– $(2.3)_0$

As the endpoint $t = 0$ is singular for the two-point boundary value problem $(2.1)_\varepsilon$-$(2.2)_\varepsilon$-$(2.3)_0$, we need to consider the two-point boundary value problem without singularity

$$\begin{cases} Z'(t) = -Z^{(N-1)/M}(t)W^{-1/M}(t)(k(t) + \varepsilon)^{1/M} \quad \text{in } (0, B], & (2.1)_\varepsilon \\ W'(t) = -Z'(t)Z^{N-1}(t)(f(Z(t), t) + \varepsilon t) \quad \text{in } (0, B], & (2.2)_\varepsilon \\ W(0) = h > 0, \quad Z(B) = 0. & (2.3)_h \end{cases}$$

It is clear that a pair $(Z(t), W(t))$ is a solution of the two-point boundary value problem

$(2.1)_\varepsilon$-$(2.2)_\varepsilon$-$(2.3)_h$ with $h \geq 0$, if and only if it is a solution of the system

$$\begin{cases} Z(t) = (TW)(t) := \left(\int_t^B PW^{-1/M}(s)(k(s)+\varepsilon)^{1/M} ds\right)^{1/p}, \ p := \dfrac{M-N+1}{M} > 0, & (3.1) \\ W(t) = (\phi W)(t) := \int_0^t (TW)^{(N-1)(M+1)/M}(s) W^{-1/M}(s)(k(s)+\varepsilon)^{1/M} & \\ \qquad (f((TW)(s),s)+\varepsilon s)ds + h \text{ for } t \in [0,B]. & (3.2)_h \end{cases}$$

Moreover, for any subinterval $[a,b]$ of $[0,B]$, we have

$$Z(t) = (TW)(t) := \left(\int_t^b PW^{-1/M}(s)(k(s)+\varepsilon)^{1/M} ds + Z^p(b)\right)^{1/p}, Z(b) = (TW)(b), \quad (3.1)_b$$

$$W(t) = (\phi_a W)(t) := \int_a^t (TW)^{(N-1)(M+1)/M}(s) W^{-1/M}(s)(k(s)+\varepsilon)^{1/M}$$

$$(f((TW)(s),s)+\varepsilon s)ds + W(a) \text{ for } t \in [a,b]. \quad (3.2)_{ab}$$

**Lemma 3.1.** *The equation $(3.2)_h$, $h > 0$, has at least one solution $W(t;\varepsilon,h) \geq h$.*

**Proof.** Define a mapping $\phi : X \to X$ by the right hand side of $(3.2)_h$, where

$$X := \{w(t) \in C[0,B]; 0 < h \leq w(t) \leq (\phi h)(t)\}.$$

By the hypotheses I, II and III, it is readily verified that the mapping $\phi$ is a compactly continuous mapping from $X$ into $X$. The Schauder Fixed Point Theorem tells us that in the set $X$ the mapping $\phi$ has at least one fixed point, denoted by $W(t;\varepsilon,h)$, which is clearly a solution of the equation $(3.2)_h$ with $h > 0$.

**Lemma 3.2.** *If $h_1 > h_2 > 0$, then*

$$0 \leq W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) \leq h_1 - h_2 \text{ on } [0,B].$$

**Proof.** We first prove that $W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) \geq 0$ on $[0,B]$. If not, then there will be a point $t = A \in (0,B]$ at which $W(A;\varepsilon,h_1) - W(A;\varepsilon,h_2) < 0$. There are two cases.

Case (i). $W(B;\varepsilon,h_1) - W(B;\varepsilon,h_2) < 0$. We can elect the endpoint $t = B$ as the point $t = A$. Because $W(0;\varepsilon,h_1) - W(0;\varepsilon,h_2) = h_1 - h_2 > 0$, there is a point $t = a \in (0,B)$ such that

$$W(a;\varepsilon,h_1) - W(a;\varepsilon,h_2) = 0 \text{ and } W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) < 0 \text{ in } (a,B].$$

Whence it follows by $(3.2)_{ab}$ that

$$\begin{aligned} 0 &> W(B;\varepsilon,h_1) - W(B;\varepsilon,h_2) \\ &= (\phi_a W(\cdot;\varepsilon,h_1))(B) - (\phi_a W(\cdot;\varepsilon,h_2))(B) \geq 0, \end{aligned}$$

which is a contradiction.

Case (ii). $W(B;\varepsilon,h_1) - W(B;\varepsilon,h_2) \geq 0$. We may without loss of generality assume that there is an interval $(a,b) \subset (0,B)$, which contains the point $t = A$, such that

$$W(a;\varepsilon,h_1) - W(a;\varepsilon,h_2) = W(b;\varepsilon,h_1) - W(b;\varepsilon,h_2) = 0,$$

$$W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) < 0 \text{ in } (a,b), \text{ and } W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) \geq 0 \text{ on } [b,B].$$

Hence $W'(b;\varepsilon,h_1) \geq W'(b;\varepsilon,h_2)$, i.e., $Z(b;\varepsilon,h_1) \geq Z(b;\varepsilon,h_2)$, because

$$W'(b;\varepsilon;h_j) = Z^{(N-1)(M+1)/M}(b;\varepsilon,h_j)(f(Z(b;\varepsilon,h_j),b)+\varepsilon b)W^{-1/M}(b;\varepsilon,h_j)(k(b)+\varepsilon)^{1/M},$$

where $Z(t;\varepsilon,h_j) = (TW(\cdot;\varepsilon,h_j))(t)$, $j = 1,2$. Here it has been used that the function $s^{(N-1)(M+1)/M}(f(s,b)+\varepsilon b)$ is strictly increasing in $s > 0$. Whence it follows by $(3.2)_{ab}$ that

$$0 > W(A;\varepsilon,h_1) - W(A;\varepsilon,h_2) = (\phi_a W(\cdot;\varepsilon,h_1))(A) - (\phi_a W(\cdot;\varepsilon,h_2))(A) \geq 0,$$

which is also a contradiction.

This shows that $W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) \geq 0$ on $[0,B]$. From the assertion proved above, it follows by $(3.2)_h$ that $W(t;\varepsilon,h_1) - W(t;\varepsilon,h_2) \leq h_1 - h_2$ on $[0,B]$.

In very much the same way, we can prove the following two lemmas.

**Lemma 3.3.** *For each fixed $h > 0$ the solution $W(t;\varepsilon,h)$ is increasing in $\varepsilon \geq 0$.*

**Lemma 3.4.** *The equation $(3.2)_h$, $h \geq 0$, has at most one solution $W(t;\varepsilon,h)$.*

Lemmas 3.1, 3.2 and 3.4 assert that the equation $(3.2)_h$, $h > 0$, has a unique solution $W(t;\varepsilon,h) \geq h$ and the solution $W(t;\varepsilon,h)$ converges to a limit, denoted by $W(t;\varepsilon,0)$, uniformly on $[0,B]$ as $h \downarrow 0$. Inserting the solution $W(t;\varepsilon,h)$ into the equation $(3.2)_h$ and then letting $h \downarrow 0$ yields the equality $(3.2)_0$, which shows that the uniform limit $W(t;\varepsilon,0) \geq 0$ is a solution of the equation $(3.2)_0$, by the Monotone Convergence Theorem.

Put $W_\varepsilon(t) = W(t;\varepsilon,0)$ and $Z_\varepsilon(t) = (TW_\varepsilon)(t)$. It is easy to check that the pair $(Z_\varepsilon(t), W_\varepsilon(t))$ is a unique solution of the two-point boundary value problem $(2.1)_\varepsilon$–$(2.2)_\varepsilon$–$(2.3)_0$.

Lemma 3.2 implies that for any $h > 0$

$$W(t;0,0) \leq W(t;0,h) \leq W(t;0,0) + h \quad \text{on } [0,B].$$

Lemma 3.3 tells us that for fixed $h > 0$ there exists an $\varepsilon_h > 0$ such that

$$W(t;\varepsilon,h) \leq W(t,\varepsilon_h,h) \leq W(t;0,h) + h \quad \text{on } [0,B]$$

whenever $\varepsilon \in (0,\varepsilon_h)$, and hence

$$W(t;0,0) \leq W(t;\varepsilon,0) \leq W(t;\varepsilon_h,h) \leq W(t;0,0) + 2h \quad \text{on } [0,B]$$

whenever $\varepsilon \in (0,\varepsilon_h)$. This shows that as $\varepsilon \downarrow 0$ the function $W_\varepsilon(t)$ converges to the function $W_0(t)$ uniformly on $[0,B]$. Whence it follows by $(3.2)_0$ that as $\varepsilon \downarrow 0$ the function $Z_\varepsilon(t)$ converges to the function $Z_0(t)$ uniformly on $[\delta,B]$ for any $0 < \delta < B$.

We summarize the results above in the following statement.

**Theorem 3.1.** *Under the hypotheses* I, II *and* III, *the two-point boundary value problem $(2.1)_\varepsilon$–$(2.2)_\varepsilon$–$(2.3)_0$ for each fixed $\varepsilon \geq 0$ has a unique solution $(Z_\varepsilon(t), W_\varepsilon(t))$, where both $Z_\varepsilon(t)$ and $-W_\varepsilon(t)$ are decreasing and continuously differentiable in $(0,B]$. Moreover, as $\varepsilon$ tends to zero, the function $W_\varepsilon(t)$ converges to the function $W_0(t)$ uniformly on $[0,B]$ and the function $Z_\varepsilon(t)$ converges to the function $Z_0(t)$ uniformly on $[\delta,B]$ for any $\delta \in (0,B)$.*

## §4. Radially Symmetric Solutions

Let us begin with the following two definitions.

**Definition 4.1.** *Let $k(t)$ be a nonnegative continuous function defined on $[0,B]$. Put*

$$E_0 = \{t \in [0,B]; k(t) = 0\} \quad \text{and} \quad E_+ = \{t \in [0,B]; k(t) > 0\}.$$

*Each connected component of the set $E_0$ which is not an isolated point set is called an interval of degeneracy in $[0,B]$ and each connected component of the set $E_+$ an interval of non-degeneracy in $[0,B]$.*

**Definition 4.2.** *Let $Z(t)$ be a decreasing continuous function defined in $(0, B]$ with $Z(B) = 0$. A function $t = V(s)$ is said to be a generalized inverse function of the function $s = Z(t)$, if it satisfies the following conditions:*

*(i) $V(s)$ is a decreasing, possibly multiple-valued function defined on $[0, +\infty)$ with $V(0) = B$ and $V(+\infty) = 0$, and*

*(ii) the restriction of $V(s)$ to $[0, Z(0+))$ is strictly decreasing so that its inverse function is exactly the function $Z(t)$, where $Z(0+) = \lim_{t \downarrow 0} Z(t)$; when $Z(0+)$ is finite, $V(s)$ is defined to be zero for $s \geq Z(0+)$.*

In the $s$-$t$ plane, as far as graphs of such two functions $s = Z(t)$ and $t = V(s)$ are concerned, the continuous curve $s = Z(t)$ is exactly the part of the continuous curve $t = V(s)$ in the region $[0, Z(0+)) \times (0, B]$. For example, if $Z(t) = 0$ in $(0, B]$, then $V(s) = B - BH(s)$, where $H(s)$ is a multiple-valued Heaviside function, that is, $H(s) = 0$ for $s < 0$, $H(s) = 1$ for $s > 0$, and $H(0) = [0, 1]$. The fact is the foundation for all the arguments in the ensuing paragraphs.

We now prove that the boundary value problem $(1.3)_\varepsilon$–$(1.4)$ has a solution $V_\varepsilon(s)$.

Theorem 3.1 shows that the two-point boundary value problem $(2.1)_\varepsilon$-$(2.2)_\varepsilon$-$(2.3)_0$ for each fixed $\varepsilon \geq 0$ has a unique solution $(Z_\varepsilon(t), W_\varepsilon(t))$ and as $\varepsilon \downarrow 0$ the function $Z_\varepsilon(t)$ converges to the function $Z_0(t)$ uniformly on $[\delta, B]$ for any $\delta \in (0, B)$. As the function $Z_\varepsilon(t)$ is a decreasing continuous function defined in $(0, B]$ with $Z(B) = 0$, its generalized inverse function $V(s)$ exists and automatically satisfies the conditions in (1.4).

**Lemma 4.1.** *As $\varepsilon \downarrow 0$ the function $V_\varepsilon(s)$ pointwise converges to the function $V_0(s)$.*

**Proof.** The lemma follows from the fact that as $\varepsilon \downarrow 0$ the function $Z_\varepsilon(t)$ converges to the function $Z_0(t)$ uniformly on $[\delta, B]$ for any $\delta \in (0, B)$.

**Lemma 4.2.** *For each fixed $\varepsilon \geq 0$ the function $V_\varepsilon(s)$ is a solution of the boundary value problem $(1.3)_\varepsilon$–$(1.4)$.*

**Proof.** When $\varepsilon > 0$, $W_\varepsilon(t) > 0$ in $(0, B]$, $Z'_\varepsilon(t) < 0$ in $(0, B)$, and $Z'_\varepsilon(0+) = -\infty$. Whence it follows that the inverse function of $Z_\varepsilon(t)$ exists and is exactly the restriction of $V_\varepsilon(s)$ to $[0, Z_\varepsilon(0+))$. Hence,

$$Z_\varepsilon(V_\varepsilon(s)) \equiv s \text{ in } [0, Z_\varepsilon(0+)),$$

$$V'_\varepsilon(s) = 1/Z'_\varepsilon(V_\varepsilon(s)) < 0 \text{ in } (0, Z_\varepsilon(0+)), \quad \text{and} \quad \lim_{s \uparrow Z_\varepsilon(0+)} V'_\varepsilon(s) = 0;$$

if $Z_\varepsilon(0+)$ is finite,

$$V'_\varepsilon(s) \equiv 0 \text{ on } [Z_\varepsilon(0+), +\infty).$$

This shows that $V_\varepsilon(s) \in C[0, +\infty) \cap C^1(0, +\infty)$.

The equality $(2.1)_\varepsilon$ can be rewritten as

$$Z_\varepsilon^{N-1}(t)(k(t) + \varepsilon)|Z'_\varepsilon(t)|^{-M} = W_\varepsilon(t) \text{ for all } t \in [0, B]. \tag{$(2.1)'_\varepsilon$}$$

Substituting the function $t = V_\varepsilon(s)$ into the equalities $(2.1)'_\varepsilon$ and $(2.2)_\varepsilon$ yields

$$s^{N-1}(k(V_\varepsilon(s)) + \varepsilon)|V'_\varepsilon(s)|^{M-1}V'_\varepsilon(s) = -W_\varepsilon(V_\varepsilon(s)) \quad \text{in } [0, Z_\varepsilon(0+)),$$

$$W'_\varepsilon(V_\varepsilon(s))V'_\varepsilon(s) = -s^{N-1}(f(s, V_\varepsilon(s)) + \varepsilon V_\varepsilon(s)) \quad \text{in } (0, Z_\varepsilon(0+)),$$

and hence

$$(s^{N-1}(k(V_\varepsilon(s))+\varepsilon)|V_\varepsilon'(s)|^{M-1}V_\varepsilon'(s))' = s^{N-1}(f(s,V_\varepsilon(s))+\varepsilon V_\varepsilon(s)) \text{ in } (0,Z_\varepsilon(0+));$$

when $Z_\varepsilon(0+)$ is finite, all the above equalities read $0=0$ on $[Z_\varepsilon(0+),+\infty)$, and as $s\uparrow Z_\varepsilon(0+)$ the right hand sides of the above equalities approach to zero without exception. This shows that the function $s^{N-1}(k(V_\varepsilon(s))+\varepsilon)|V_\varepsilon'(s)|^{M-1}V_\varepsilon'(s)$ is continuous on $[0,+\infty)$ and continuously differentiable in $(0,+\infty)$ and the equality $(1.3)_\varepsilon$ holds everywhere in $(0,+\infty)$. In one word, the function $V_\varepsilon(s)$ is a solution of the boundary value problem $(1.3)_\varepsilon$–(1.4).

Lemmas 4.1 and 4.2 point out that the function $V_0(s)$ is a solution of the reduced boundary value problem $(1.3)_0$-(1.4). We now investigate some properties of the solution $V_0(s)$.

Let $\{[a_j,b_j];j=1,2,\cdots\}$ be the collection of all intervals of degeneracy possessed by the function $k(t)$ in $[0,B]$. It follows from $(2.1)_0$ that the function $Z_0'(t)$ has in $[0,B]$ the same intervals of degeneracy as the function $k(t)$. Clearly, $Z_0(t)=s_j=$ constant on $[a_j,b_j]$, $Z_0(a_j-0)=s_j$, and $Z_0(b_j+0)=s_j$, $j=1,2,\cdots$. This shows that the point $s=s_j$, $j=1,2,\cdots$, is a jump point of the function $V_0(s)$, where

$$V_0(s_j)=[a_j,b_j], V_0(s_j-0)=b_j, \text{ and } V_0(s_j+0)=a_j, j=1,2,\cdots. \tag{4.1}$$

Note that

$$(0,Z_0(0+))=Z_0(E_0)\cup Z_0(E_+),$$

where $Z_0(E_0)$ is a null set, which contains all jump points of $V_0(s)$ and on which $V_0'(s)=-\infty$, and $Z_0(E_+)$ is a set in which $V_0'(s)<0$. Repeating the arguments in the proof of Lemma 4.2, we can conclude that in each connected component of the set $Z_0(E_+)$, $V_0(s)$ is continuously differentiable and $s^{N-1}k(V_0(s))|V_0'(s)|^{M-1}V_0'(s)$ is also continuously differentiable, and the equality $(1.3)_0$ holds everywhere. This shows that $V_0(s)$ satisfies, in the classical sense, the equation $(1.3)_0$ in the open set $(0,+\infty)\setminus Z_0(E_0)$. Integrating the equation $(2.2)_0$ over $[a_j,b_j]$ gives the equality $W_0(a_j)=W_0(b_j)$, namely

$$s^{N-1}k(V_0(s))|V_0'(s)|^{M-1}V_0'(s)\Big|_{s=s_j-0}^{s=s_j+0}=0, j=1,2,\cdots. \tag{4.2}$$

Moverover, the solution $V_0(s)$ can be represented by

$$V_0(s)=B+\sum_j(a_j-b_j)H(s-s_j)+\int_0^s V_0'(t)dt \text{ for all } s\in[0,+\infty). \tag{4.3}$$

We summarize the results above in the following statement.

**Theorem 4.1.** *The boundary value problem $(1.3)_\varepsilon$–(1.4) for each fixed $\varepsilon>0$ has a solution $V_\varepsilon(s)$ and the reduced boundary value problem $(1.3)_0$–(1.4) has a solution $V_0(s)$. Moreover, the solution $V_0(s)$ can be represented by the formula (4.3),where $\{[a_j,b_j];j=1,2,\cdots\}$ is the collection of all intervals of degeneracy possessed by the function $k(t)$ in $[0,B]$, $\cup\{s=s_j\}$ is the set of all jump points of $V_0(s)$, and $V_0(s)$ satisfies the jump conditions (4.1) and (4.2) at each of its jump points. From the formula (4.3), we conclude that if and only if the function $k(t)$ has at least one interval of degeneracy in $[0,B]$ discontinuities appear in $V_0(s)$ and there exists a one-to-one correspondence between the collection of all intervals of degeneracy possessed by $k(t)$ in $[0,B]$ and the set of all jump points appearing in $V_0(s)$.*

Clearly, $U_\varepsilon(x):=V_\varepsilon(|x|)$ is a radially symmetric solution of the problem $(1.1)_\varepsilon$–(1.2) and $U_0(x):=V_0(|x|)$ is a radially symmetric solution of the reduced problem $(1.1)_0$–(1.2), in which discontinuities appear if and only if $k(t)$ has at least one interval of degeneracy in $[0, B]$. Without doubt, there are infinitely many such solutions.

## References

[1] Atkinson, F. V. & Peletier, L. A., Ground states and Dirichlet problems for $-\Delta u = f(u)$ in $I\!R^2$, *Arch. Rational Mech. Anal.*, **96** (1986), 147-166.

[2] Atkinson, C. & Bouillet, J. E., Some qualitative properties of solutions of a generalised diffusion equation, *Math. Proc. Camb. Phil. Soc.*, **86** (1979), 495-510.

[3] Citti, G., Existence of infinitely many solutions for a quasilinear degenerate elliptic equation in $I\!R^n$, *Nonlinear Anal. TMA*, **12** (1988), 871-879.

[4] Philip, J. R., $n$-diffusion, *Austral. J. Phys.*, **14** (1961), 1-13.