*Chin. Ann. of Math.*
**14B**: 2(1993), 183-188.

# ISOMORPHISMS OF STABLE STEINBERG GROUPS**

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**Abstract**

In [2] the author discussed the isomorphisms between two unstable Steinberg groups $St_m(A)$ and $St_n(R)$ over commutative rings. The aim of the present paper is to determine the isomorphisms between two stable Steinberg groups $St(A)$ and $St(R)$, and the isomorphisms between the corresponding stable linear groups.

## §1. Introduction

Let $R$ be an associative ring with identity, and $V_n = R^{(n)}$ the free (right) $R$-module of rank $n$. Regard $V_n$ as a submodule of $V_{n+1}$ via $V_{n+1} = V_n \oplus R$. Under the standard basis of $V_n$, one has $Aut_R(V_n) \cong GL_n(R)$, and $GL_n(R)$ can be viewed as a subgroup of $GL_{n+1}(R)$ in a natural way. Let $GL(R) = \varinjlim GL_n(R)$ be the direct limit of the $GL_n(R)$, called the stable general linear group over $R$. $GL_n(R)$ can also be viewed as a subgroup of $GL(R)$. Let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by all elementary matrices $e_{ij}(a) = 1 + ae_{ij}(1 \le i, j \le n, i \ne j; a \in R)$. The stable elementary group $E(R)$ is the direct limit of the $E_n(R)$. By the Whitehead lemma, $E(R)$ is just the commutator subgroup of $GL(R)$ (cf. [1]). Define $K_1(R) = GL(R)/E(R)$. Then, one has an exact sequence of groups

$$1 \to E(R) \to GL(R) \to K_1(R) \to 1. \tag{1.1}$$

For $n \ge 3$, the Steinberg group of dimension $n$ over $R$, $St_n(R)$, is the group generated by the symbols $x_{ij}(a)(1 \le i, j \le n, i \ne j, a \in R)$ subject to the following Steinberg relations

$$\left.\begin{aligned} &x_{ij}(a)x_{ij}(b) = x_{ij}(a+b),\\ &[x_{ij}(a), x_{jl}(b)] = x_{il}(ab), i \ne l,\\ &[x_{ij}(a), x_{kl}(b)] = 1, i \ne l \text{ and } j \ne k.\end{aligned}\right\} \tag{1.2}$$

Define the canonical homomorphism

$$\phi_n : St_n(R) \to E_n(R)$$

which sends $x_{ij}(a)$ to $e_{ij}(a)$. On the other hand, one has the canonical homomorphism $St_n(R) \to St_{n+1}(R)$ sending $x_{ij}(a)$ to $x_{ij}(a)$. Let the stable Steinberg group $St(R)$ be the direct limit of the $St_n(R)$ and

$$\phi = \varinjlim \phi_n : St(R) \to E(R).$$

Manuscript received November 6, 1990. Revised January 28, 1992.
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**Project supported by the National Natural Science Foundation of China .

Denote $K_2(R) = \mathrm{Ker}\phi$. Then the following sequence of groups

$$1 \to K_2(R) \to St(R) \to E(R) \to 1 \tag{1.3}$$

is exact. Steinberg[6] and Milnor[3] have proved that $\phi : St(R) \to E(R)$ is a universal central extension, and $K_2(R)$ is just the center of $St(R)$.

For two groups $G$ and $H$, denote by $Iso(G, H)$ the set of isomorphisms from $G$ onto $H$. Let $A$ be another associative ring with identity.

**Theorem 1.1.** *There is a natural* 1-1 *correspondence between* $Iso(St(A), St(R))$ *and* $Iso(E(A), E(R))$.

**Proof.** Let $\Lambda : St(A) \to St(R)$ be a group isomorphism. Then, $\Lambda$ maps $K_2(A)$, the center of $St(A)$, onto $K_2(R)$, the center of $St(R)$. Thus, by the exact sequence (1.3), $\Lambda$ naturally induces an isomorphism from $E(A)$ to $E(R)$. Conversely, assume $\Lambda : E(A) \to E(R)$ is a group isomorphism. Then, $\Lambda$ can be naturally and uniquely lifted to an isomorphism from $St(A)$ to $St(R)$, since $St(A)$ and $St(R)$ are respectively universal central extensions of $E(A)$ and $E(R)$.

## §2. Some Lemmas

By Theorem 1.1, the determination of $Iso(St(A), St(R))$ is equivalent to that of $Iso(E(A), E(R))$. Throughout this section, $A$ and $R$ are commutative rings, and $\Lambda : E(A) \to E(R)$ is a group isomorphism. Denote by $\max(R)$ the set of maximal ideals of $R$.

**Lemma 2.1.** $\{E(R) \cap GL(R, M) | M \in \max(R)\}$ *is the set of maximal normal subgroups of* $E(R)$, *where* $GL(R, M)$ *is the principal congruence subgroup of level* $M$, *i.e., the kernel of the canonical homomorphism* $GL(R) \to GL(R/M)$.

**Proof.** Let $M \in \max(R)$. Then, $E(R) \cap GL(R, M)$ is a maximal normal subgroup of $E(R)$, since the quotient group $E(R)/(E(R) \cap GL(R, M))$ is isomorphic to $E(R/M)$ which is a simple group. Conversely, assume $N$ is an arbitrary maximal normal subgroup of $E(R)$. By a theorem of Bass ([1], Chap. V, Theorem (2.1)), there is a unique ideal $I$ of $R$ such that

$$E(R, I) \subseteq N \subseteq GL(R, I),$$

where $E(R, I)$ is the elementary congruence subgroup of level $I$, i.e., the normal subgroup of $E(R)$ generated by all $e_{ij}(b) (i \neq j, b \in I)$. Since $N \neq E(R)$, $I \neq R$. Thus, there is an $M \in \max(R)$ containing $I$. Hence, $N \subseteq E(R) \cap GL(R, M)$. It follows that $N = E(R) \cap GL(R, M)$ by the maximality of $N$.

By Lemma 2.1, the isomorphism $\Lambda : E(A) \to E(R)$ yields a 1-1 correspondence between $\max(A)$ and $\max(R)$, $J \leftrightarrow M$, such that

$$\Lambda(E(A) \cap GL(A, J)) = E(R) \cap GL(R, M).$$

Thus, $\Lambda$ induces an isomorphism of quotient groups, $\bar{\Lambda} : E(A/J) \to E(R/M)$, which has been made clear.

**Lemma 2.2.** *There exist a field isomorphism* $\bar{\sigma} : A/J \to R/M$, *and an infinite invertible matrix* $g$ *over* $R/M$, *such that either*

$$\bar{\Lambda}(x) = g x^{\bar{\sigma}} g^{-1} \text{ for all } x \in E(A/J),$$

*or*

$$\bar{\Lambda}(x) = g\check{x}^{\bar{\sigma}}g^{-1} \text{ for all } x \in E(A/J),$$

*where $\check{x}$ is the transpose inverse of $x$.*

**Proof.** See [4], Theorem 5.3.

Consider now the image of $E_n(A)$ under $\Lambda$. Obviously, there is a smallest integer $N(n)$ such that $\Lambda(E_n(A)) \subseteq E_{N(n)}(R)$, and $N(n) \leq N(m)$ whenever $n \leq m$. For $M \in \max(R)$, let $r_M : E(R) \to E(R_M)$ be the group homomorphism induced by the localization $R \to R_M$.

**Lemma 2.3.** *Let $M \in \max(R)$. Then there exists a set of matrices*

$$\{g(n,M) \in GL_{N(n)}(R_M) | n \geq 4\},$$

*such that either*

$$(r_M \circ \Lambda)e_{ij}(1) = g(n,M)e_{ij}(1)g(n,M)^{-1} \tag{2.1}$$

*for all $n \geq 4$, $1 \leq i,j \leq n$, $i \neq j$, or*

$$(r_M \circ \Lambda)e_{ij}(1) = g(n,M)e_{ji}(-1)g(n,M)^{-1} \tag{2.2}$$

*for all $n \geq 4$, $1 \leq i,j \leq n$, $i \neq j$. Moreover, (2.1) and (2.2) cannot occur simultaneously.*

**Proof.** Use Lemmas 2.1 and 2.2, and proceed as in Sections 3-5 of [5].

Denote $\widetilde{R} = \prod_{M \in \max(R)} R_M$. Then, one has a canonical embedding $R \hookrightarrow \widetilde{R}$, i.e., $\widetilde{R}$ can be regarded as an extension of $R$. Thus, $GL_n(R) \subseteq GL_n(\widetilde{R}) = \prod_{M \in \max(R)} GL_n(R_M)$. For $n \geq 4$, let $g(n) = \prod_{M \in \max(R)} g(n,M) \in GL_{N(n)}(\widetilde{R})$, where $g(n,M)$ is given by Lemma 2.3.

**Lemma 2.4.** *There is an idempotent $s$ in $R$ such that*

$$\Lambda e_{ij}(1) = g(n)[e_{ij}(1)s + e_{ji}(-1)(1-s)]g(n)^{-1} \tag{2.3}$$

*holds for all $n \geq 4$, $1 \leq i,j \leq n$, $i \neq j$.*

**Proof.** Let $S_1$ (resp. $S_2$) be the product of the $R_M$ where $M$ is so that (2.1) (resp. (2.2)) in Lemma 2.3 holds. Then, $\widetilde{R} = S_1 \times S_2$. Denote by $s$ the identity of $S_1$. Since $\Lambda e_{ij}(1)$ is completely determined by the set

$$\{(r_M \circ \Lambda)e_{ij}(1) | M \in \max(R)\},$$

it follows that (2.3) holds from Lemma 2.3. It can be proved that $s$ belongs to $R$ by using the method in Section 6 of [5].

Write $R = R_1 \times R_2$ where $R_1 = Rs$ and $R_2 = R(1-s)$. Then, $GL(R) = GL(R_1) \times GL(R_2)$. The idempotent $s$ of $R$ determines a generalized contragradient automorphism $k$ of $GL(R)$:

$$k(x_1, x_2) = (x_1, \check{x}_2)$$

for all $(x_1, x_2) \in GL(R_1) \times GL(R_2)$. Clearly, $k(GL_n(R)) = GL_n(R)$ and $k(E_n(R)) = E_n(R)$. On the other hand, for any $M \in \max(R)$, either $R_1$ or $R_2$ is contained in $M$ since $s(1-s) = 0 \in M$. If $s \notin M$, then $1 - s = 0$ in $R_M$, and $R_M = (R_1)_{M_1}$ where $M_1 = M \cap R_1$. So

$$\widetilde{R}_1 = \prod_{M_1 \in \max(R_1)} (R_1)_{M_1} = \prod_{\substack{M \in \max(R) \\ s \notin M}} R_M = S_1.$$

Similarly, $\widetilde{R}_2 = S_2$. Use the same letter $k$ to denote the generalized contragradient automorphism of $GL(\widetilde{R})$ determined by $s$. Evidently, $k^2 = 1$.

Applying $k$ to the two sides of (2.3), one obtains

$$(k \circ \Lambda)e_{ij}(1) = k(g(n))e_{ij}(1)k(g(n)^{-1}).$$

After replacing the original $k(g(n))$ by $g(n)$, we see that the equality

$$(k \circ \Lambda)e_{ij}(1) = g(n)e_{ij}(1)g(n)^{-1} \tag{2.4}$$

holds for all $n \geq 4$, $1 \leq i,j \leq n$, $i \neq j$.

Assume $m > n \geq 4$. One sees, by (2.4), that $g(n)^{-1}g(m)$ commutes with all $e_{ij}(1)(1 \leq i,j \leq n, i \neq j)$. Thus, $g(n)^{-1}g(m)$ is equal to $\begin{pmatrix} c & 0 \\ 0 & * \end{pmatrix}$, where $c$ is a scalar matrix of order $n$. The equality (2.4) will not change if we replace $c^{-1}g(m)$ by $g(m)$. Thus, there is a set of matrices

$$\{g(n) \in GL_{N(n)}(\widetilde{R}) | n \geq 4\}$$

such that (2.4) holds, and $g(n)$ and $g(m)$ have the same first $n$ columns, and $g(n)^{-1}$ and $g(m)^{-1}$ have the same first $n$ rows whenever $m > n$. This means that $\{g(n)|n \geq 4\}$ is compatible in some sense.

An infinite matrix is said to be column-finite (resp. row-finite) if each of its columns (resp. rows) has only a finite number of nonzero entries. Now construct an infinite matrix $g$ over $\widetilde{R}$ whose first $n$ columns are the same as those of $g(n)$ for all $n \geq 4$. By the compatibility in the above sense, $g$ is well-defined, and column-finite. (We say that $g$ is glued by the columns of the $g(n)$.) Similarly, construct an infinite matrix $G$ whose first $n$ rows are the same as those of $g(n)^{-1}$. $G$ is also well-defined, and row-finite. By the construction, $Gg = 1$ where 1 is the infinite identity matrix. Although one does not know, at the moment, whether $gG = 1$ (which will be proved in Theorem 3.1), anyhow, (2.4) can be rewritten as

$$(k \circ \Lambda)e_{ij}(1) = 1 + ge_{ij}G \tag{2.5}$$

for all $i \neq j$.

**Lemma 2.5.** *There is an injective homomorphism of rings, $\sigma : A \to R$, such that*

$$(k \circ \Lambda)e_{ij}(a) = g(n)e_{ij}(\sigma(a))g(n)^{-1} \tag{2.6}$$

*holds for all $n > 4$, $1 \leq i,j \leq n$, $i \neq j$, and $a \in A$, or in other words,*

$$(k \circ \Lambda)e_{ij}(a) = 1 + \sigma(a)ge_{ij}G \tag{2.7}$$

*for all $i \neq j$, and $a \in A$.*

**Proof.** Let $a \in A$. Since $e_{ij}(a)$ commutes with all $e_{pq}(1)$ $(p \neq j$ and $q \neq i)$, applying the isomorphism $k \circ \Lambda$, one sees, by (2.5), that $(k \circ \Lambda)e_{ij}(a)$ commutes with all $1 + ge_{pq}G$ $(p \neq j$ and $q \neq i)$. It follows that $(k \circ \Lambda)e_{ij}(a) = g(n)e_{ij}(\sigma(a))g(n)^{-1}$ for some $\sigma(a) \in \widetilde{R}$. But, $\sigma(a)$ belongs to $R$ just as the element $s$ in Lemma 2.4 does. By using the Steinberg relations and the fact that $k \circ \Lambda$ is an isomorphism, one derives that $\sigma : A \to R$ is an injective homomorphism of rings.

**Remark.** It follows from Lemma 2.2 that the homomorphism $\sigma$ given by Lemma 2.5 is compatible with the 1-1 correspondence between $\max(A)$ and $\max(R)$, i.e., if $J \in \max(A)$ corresponds to $M \in \max(R)$, then $\sigma(J) \subseteq M$, and $\sigma(A - J) \subseteq R - M$. Thus, $\sigma$ induces a homomorphism $A_J \to R_M$. Their product

$$\widetilde{A} = \prod_{J \in \max(A)} A_J \to \prod_{M \in \max(R)} R_M = \widetilde{R}$$

is also denoted by $\sigma$.

## §3. Main Results

**Theorem 3.1.** *Let $A$ and $R$ be commutative rings, and let $\Lambda : E(A) \to E(R)$ be a group isomorphism. Then, there exist*

(i) *a ring isomorphism $\sigma : A \to R$,*

(ii) *a generalized contragradient automorphism $k$ of $E(R)$ determined by an idempotent in $R$, and*

(iii) *an infinite invertible matrix $g$ over $\widetilde{R}$ which is both column-finite and row-finite, such that*

$$(k \circ \Lambda)e_{ij}(a) = ge_{ij}(\sigma(a))g^{-1} \tag{3.1}$$

*holds for all $i \neq j$, and $a \in A$, or in other words,*

$$(k \circ \Lambda)x = gx^{\sigma}g^{-1} \tag{3.2}$$

*for all $x \in E(A)$.*

**Proof.** Section 2 has given an injective homomorphism of rings $\sigma : A \to R$, and a generalized contragradient automorphism $k$ of $E(R)$ determined by an idempotent $s$ in $R$, and two infinite matrices $g$ and $G$ over $\widetilde{R}$, such that (2.6) and (2.7) hold. Here, $g$ is column-finite, and $G$ is row-finite, and $Gg = 1$.

Apply Lemmas 2.1-2.5 to the isomorphism $\Lambda^{-1} : E(R) \to E(A)$. One obtains an injective homomorphism $\tau : R \to A$, a generalized contragradient automorphism $k'$ of $E(A)$ determined by an idempotent $t$ in $A$, and infinite matrices $h$ and $H$ over $\widetilde{A}$, such that

$$(k' \circ \Lambda^{-1})e_{ij}(r) = h(n)e_{ij}(\tau(r))h(n)^{-1}$$

holds for all $n \geq 4$, $1 \leq i, j \leq n$, $i \neq j$ and $r \in R$, i.e., $(k' \circ \Lambda^{-1})e_{ij}(r) = 1 + \tau(r)he_{ij}H$ for all $i \neq j$ and $r \in R$. Here $h$ is glued by the columns of the $h(n)$, and $H$ is glued by the rows of the $h(n)^{-1}$, and $Hh = 1$.

Fix an $n \geq 4$. (2.6) implies that $(k \circ \Lambda)x = g(n)x^{\sigma}g(n)^{-1}$ holds for all $x \in E_n(A)$, i.e.,

$$\Lambda(x) = k(g(n))(x^{\sigma}s + \check{x}^{\sigma}(1-s))k(g(n)^{-1}).$$

For any $x \in E_n(A)$, and any $m$ large enough,

$$\begin{aligned} xt + \check{x}(1-t) =& k'(x) = (k' \circ \Lambda^{-1} \circ \Lambda)(x) = (k' \circ \Lambda^{-1})[k(g(n))(x^{\sigma}s + \check{x}^{\sigma}(1-s))k(g(n)^{-1})] \\ =& h(m)[k(g(n))]^{\tau}[x^{\sigma}s + \check{x}^{\sigma}(1-s)]^{\tau}[k(g(n)^{-1})]^{\tau}h(m)^{-1}. \end{aligned}$$

It follows that $\tau(s) = t$, and $\sigma$ and $\tau$ are both isomorphisms, and $\tau = \sigma^{-1}$. Moreover, $h(m)[k(g(n))]^{\tau}$ is equal to $\begin{pmatrix} c & 0 \\ 0 & * \end{pmatrix}$, where $c$ is a scalar matrix of order $n$. This shows that, for all $m$ large enough, $h(m)$ and $c[k(g(n)^{-1})]^{\tau}$ have the same first $n$ rows, and $h(m)^{-1}$ and $c^{-1}[k(g(n))]^{\tau}$ have the same first $n$ columns. Thus, the infinite matrix $h$ glued by the columns of the $h(n)$ is not only column-finite but also row-finite, so is $H$. Moreover,

$Hh = hH = 1$. Similarly, $g$ and $G$ are both column-finite and row-finite, and $Gg = gG = 1$; i.e., $G = g^{-1}$. Hence, (2.7) can be rewritten as (3.1).

Since there is a natural 1-1 correspondence between $Iso(St(A), St(R))$ and $Iso(E(A), E(R))$, every isomorphism from $St(A)$ to $St(R)$ has also been made clear. In particular, for the $g$ in Theorem 3.1, $gx_{ij}(r)g^{-1}$ makes sense, and

$$k(x_{ij}(r)) = (x_{ij}(rs), x_{ji}(-r(1-s))) \in St(R_1) \times St(R_2) = St(R).$$

By using Theorem 3.1, every isomorphism from $GL(A)$ to $GL(R)$ can also be determined.

**Theorem 3.2.** *Let $A$ and $R$ be commutative rings, and let $\Lambda : GL(A) \to GL(R)$ be a group isomorphism. Then there exist*

(i) *a ring isomorphism $\sigma : A \to R$,*

(ii) *a generalized contragradient automorphism $k$ of $GL(R)$ determined by an idempotent in $R$, and*

(iii) *an infinite invertible matrix $g$ over $\widetilde{R}$ which is both column-finite and row-finite, such that $(k \circ \Lambda)x = gx^\sigma g^{-1}$ for all $x \in GL(A)$.*

**Proof.** Since $E(A)$ and $E(R)$ are the commutator subgroups of $GL(A)$ and $GL(R)$ respectively, $\Lambda$ induces an isomorphism from $E(A)$ to $E(R)$. By Theorem 3.1, one obtains suitable $\sigma$, $k$ and $g$, such that $(k \circ \Lambda)y = gy^\sigma g^{-1}$ for all $y \in E(A)$. Now, $xyx^{-1} \in E(A)$ for any $x \in GL(A)$ and $y \in E(A)$. Therefore,

$$g(xyx^{-1})^\sigma g^{-1} = (k \circ \Lambda)(xyx^{-1}) = (k \circ \Lambda)(x)gy^\sigma g^{-1}(k \circ \Lambda)(x^{-1}).$$

Thus, $(x^{-1})^\sigma g^{-1}(k \circ \Lambda)(x)g$ commutes with all $y^\sigma \in E(R)$. This implies that $(x^{-1})^\sigma g^{-1}(k \circ \Lambda)(x)g = 1$, i.e., $(k \circ \Lambda)x = gx^\sigma g^{-1}$.

**Corollary 3.1.** *Let $A$ and $R$ be commutative rings. Then,*

$$Iso(St(A), St(R)) \cong Iso(E(A), E(R)) \cong Iso(GL(A), GL(R)).$$

*In particular, $Aut(St(R)) \cong Aut(E(R)) \cong Aut(GL(R))$.*

**Proof.** It follows from Theorems 1.1, 3.1 and 3.2.

**Corollary 3.2.** *Let $A$ and $R$ be commutative rings. Then the following statements are equivalent.*

(i) $A \cong R$. (ii) $St(A) \cong St(R)$.

(iii) $E(A) \cong E(R)$. (iv) $GL(A) \cong GL(R)$.

Here, (i) *is an isomorphism of rings, and* (ii)–(iv) *are isomorphisms of groups.*

**Proof.** It follows from Theorems 1.1, 3.1 and 3.2.

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