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WAVELET DECOMPOSITIONS IN $L^2(\mathbb{R}^2)$

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Abstract

Let $\{V_k\}_{k=-\infty}^{+\infty}$ be a multiresolution analysis generated by a function $\phi(x) \in L^2(\mathbb{R}^2)$. Under this multiresolution framework the key point for studying wavelet decompositions in $L^2(\mathbb{R}^2)$ is to study the properties of W_0 which is the orthogonal complement of V_0 in $V_1 : V_1 = V_0 \oplus W_0$. In this paper the author studies the structure of W_0 and furthermore shows that a box spline of three directions can generate a wavelet decomposition of $L^2(\mathbb{R}^2)$.

§1. Introduction

Recent years the wavelet decompositions in $L^2(\mathbb{R}^s)$ have drawn a lot of attentions. It is not surprising that the literatures about wavelets grow rapidly since the wavelet decompositions show its power both in the field of pure mathematics and in its applications.

It is natural and necessary to consider the general wavelet decompositions in $L^2(\mathbb{R}^s)$ $(s > 1)$. In this paper we study the case $s = 2$. One reason for studying this case is that the case $s = 2$ may be most important in practical applications among the multi-dimensional cases. In addition the wavelet decompositions in higher dimensions (e.g. $s > 4$) should be dealt with in a different way from the case $s = 2$ (as explained in a special case in [9]). We hope that it would be studied in the forthcoming paper.

Here is the outline of this paper. In §2 some notations and lemmas are given. In §3 we study the wavelet decompositions in $L^2(\mathbb{R}^2)$. In §4 we discuss the orthogonal wavelet decompositions in $L^2(\mathbb{R}^2)$. The result which we obtained contains the main theorem in [9]. Finally as a consequence of our theorem in §3 we claim that box splines of three directions generate wavelet decompositions. It is well known that in the case $s = 1$ B-splines generate wavelet decompositions which were widely applied in some practical fields. We believe that the box splines would play an important role in wavelet decompositions in the case $s = 2$ as B-splines do in the case $s = 1$.

§2. Preliminaries

Throughout this paper we use the standard multi-indices. The definitions of the summation over multi-indices and the convergence of multiple series in some metric sense are as usual (for example cf. [7, 10]).

Let $\phi(x) \in L^2(\mathbb{R}^2)$ be a funciton with compact support and

$$\phi_{kj}(x) = \phi(2^k x - j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^2.$$

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We consider a multiresolution analysis generated by $\phi(x)$;

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

$$V_k = \overline{\text{span}}\{\phi_{kj}(x), j \in \mathbf{z}^2\}, \tag{2.1}$$

where the closure is in the sense of $L^2(\mathbb{R}^2)$. For the detail about the multiresolution analysis see [6,8]. For our purpose here we emphasize the following conditions:

(1) $\{\phi(x-j)\}_{j\in\mathbf{z}^2}$ constitute an unconditional basis of V_0,

(2) in the sense of L^2 (see (2.1))

$$\phi(x) = \sum_{\alpha\in\mathbf{z}^2} a_\alpha \phi(2x-\alpha) \tag{2.2}$$

for some coefficients a_α which have exponential decay.

These requirements are not much restrictive for practical purposes[4].

Here a sequence $\{a_\alpha\}_{\alpha\in\mathbf{z}^2}$ is called exponential decay if there are constants $L(>0)$, $\lambda(>0)$ such that

$$|a_\alpha| \le L\exp(-\lambda|\alpha|),$$

where $|\alpha| = |i| + |j|$ for $\alpha = (i,j) \in \mathbf{z}^2$.

Remark. Instead of (2) we can consider (2.2) for some such a_α that $\{a_\alpha\} \in l^2$. Then the relevant proof in the following should be modified slightly without difficulties.

Let W_0 be the orthogonal complement of V_0 in V_1:

$$V_1 = V_0 \oplus W_0. \tag{2.3}$$

To obtain wavelet decompositions in $L^2(\mathbb{R}^2)$ the key point is to study the structure of W_0. In the case $s=1$ it is clear[6,8] that there is a function $\psi(x) \in L^2(\mathbb{R})$ such that

$$W_0 = \overline{\text{span}}\{\psi(x-j) : j \in \mathbf{z}\}.$$

When $s>1$, the situation is different. In our case $(s=2)$ we will claim that there exist functions $\psi^{(i)}(x)$ $(i=1,2,3)$ such that

$$W_{0i} = \overline{\text{span}}\{\psi^{(i)}(x-r) : r \in \mathbf{z}^2\}, i = 1,2,3,$$

$$W_0 = W_{01} + W_{02} + W_{03},$$

which obviously gives the structure of W_0. Then in terms of the multiresolution analysis a dense system in $L^2(\mathbb{R}^2)$ is constructed in a standard way[8]. It provides so-called wavelet decompositions. Hence the main goal of this paper focuses on the study of the structure of W_0.

Now we introduce some notations.

Let $\mathbb{C}^2 = \{z = (z_1, z_2) : z_i \in \mathbb{C}, j = 1,2\}$ be a 2-dimensional complex space. D denotes the open polydisc

$$D = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_j| < 1, j = 1,2\},$$

and ∂D denotes the boundary of D:

$$\partial D = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_j| = 1, j = 1,2\}.$$

When $z = (z_1, z_2) \in \partial D$, we often write

$$z = \left(e^{-i\frac{\omega_1}{2}}, e^{-i\frac{\omega_2}{2}}\right),$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$.

Let $e_0 = (0,0)$, $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (1,1)$, $E = \{e_0, e_1, e_2, e_3\}$. The type of the summation $\sum_{e\in E} F(e)$ always means $\sum_{i=0}^{3} F(e_i)$.

For sequence $\{a_\alpha\}_{\alpha\in\mathbb{Z}^2}$ we formally introduce its symbol

$$a(z) = \sum_{\alpha\in\mathbb{Z}^2} a_\alpha z^\alpha,$$

and its subsymbol ($e \in \mathbb{Z}^2$)

$$a_e(z) = \sum_{\alpha\in\mathbb{Z}^2} a_{e+2\alpha} z^\alpha.$$

Here $z = (z_1, z_2)$.

The symbol of a sequence is useful notion for treating sequences[3]. The relation between the symbol and the subsymbol is described in the following lemma.

Lemma 2.1. *Suppose $\{a_\alpha\}_{\alpha\in\mathbb{Z}^2} \in l^1$, $z \in \partial D$. Then*

$$a(z) = \sum_{e\in E} z^e a_e(z^2). \tag{2.4}$$

Conversely

$$a_e(z^2) = 2^{-2}(z^{-1})^e \sum_{\tilde{e}\in E} (-1)^{e\cdot\tilde{e}} a((-1)^{\tilde{e}} z). \tag{2.5}$$

Proof. (2.4) is obvious. Let $\tilde{e} \in E$. From (2.4) it follows that

$$\begin{aligned}\sum_{\tilde{e}\in E} (-1)^{\hat{e}\cdot\tilde{e}} a((-1)^{\tilde{e}} z) &= \sum_{\tilde{e}\in E} (-1)^{\hat{e}\cdot\tilde{e}} \sum_{e\in E} (-1)^{\tilde{e}\cdot e} z^e a_e(z^2) \\ &= \sum_{e\in E} z^e a_e(z^2) \sum_{\tilde{e}\in E} (-1)^{\hat{e}\cdot\tilde{e}} (-1)^{\tilde{e}\cdot e}.\end{aligned} \tag{2.6}$$

Noting

$$\sum_{\tilde{e}\in E} (-1)^{e\cdot\tilde{e}} = 4\delta_{e_0 e}, \tag{2.7}$$

we obtain (2.5) from (2.6), (2.7).

Let $\alpha \in \mathbb{Z}^2$,

$$\gamma_\alpha = \int_{\mathbb{R}^2} \phi(x)\overline{\phi(x-\alpha)}dx$$

and the symbol of $\{\gamma_\alpha\}$ be $I(z)$. Since $\phi(x)$ has compact support, $\operatorname{supp}(\gamma) := \{\alpha : \gamma_\alpha \neq 0\}$ is a finite set. We have

Lemma 2.2. *For $z \in \partial D$ holds $I(z) > 0$.*

Proof. First we show

$$\sum_{\alpha\in\mathbb{Z}^2} |\hat{\phi}(\omega + 2\pi\alpha)|^2 = I(z^2),$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, $z = \left(e^{-i\frac{\omega_1}{2}}, e^{-i\frac{\omega_2}{2}}\right)$.

Let

$$C(y) = \int_{\mathbb{R}^2} \phi(x)\overline{\phi(x-y)}dx, \quad f(y) = C(y)e^{-i\omega y}.$$

we have

$$\hat{f}(u) = |\hat{\phi}(\omega + u)|^2. \tag{2.8}$$

Now by the Poisson Summation formula

$$\sum_{\alpha \in z^2} |\hat{\phi}(\omega + 2\pi\alpha)|^2 = I(z^2). \tag{2.9}$$

Suppose

$$C = \{C_\alpha\}_{\alpha \in z^2} \in l^2, \quad S(\omega) = \sum_{\alpha \in z^2} C_\alpha e^{-i\omega\alpha}.$$

Note the periodicity of $S(\omega)$. Then

$$\left\| \sum_{\alpha \in z^2} C_\alpha \phi(x - \alpha) \right\|_{L^2}^2 = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |S(\omega)|^2 \sum_{\alpha \in z^2} |\hat{\phi}(\omega + 2\pi\alpha)|^2 d\omega. \tag{2.10}$$

It is obvious that

$$\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |S(\omega)|^2 d\omega = \|C\|_{l^2}^2.$$

From (2.9) we know $I(z) > 0$ $(z \in \partial D)$ because $\{\phi(x - j), j \in z^2\}$ is an unconditional basis of V_0.

§3. Main Result

In order to describe our main result, set

$$\overline{b^{(1)}(z)} = z^{-e_1} z^{e_2} I((-1)^{e_1} z) a((-1)^{e_1} z),$$
$$\overline{b^{(2)}(z)} = z^{-e_2} I((-1)^{e_2} z) a((-1)^{e_2} z),$$
$$\overline{b^{(3)}(z)} = z^{e_1} I((-1)^{e_3} z) a((-1)^{e_3} z).$$

It is obvious that we can write $b^{(i)}z$ as follows

$$b^{(i)}(z) = \sum_{\alpha \in z^2} b_\alpha^{(i)} z^\alpha, \quad z \in \partial D, i = 1, 2, 3,$$

where the coefficients $b_\alpha^{(i)}$ have exponential decay. Then we define

$$\psi^{(i)}(x) = \sum_{\alpha \in z^2} b_\alpha^{(i)} \phi(2x - \alpha), \quad i = 1, 2, 3.$$

Now we are in a position to prove the following

Theorem 3.1. *Suppose*

$$\sum_{e \in E} (I((-1)^e z) a((-1)^e z))^2 \neq 0 \tag{3.1}$$

for $z \in \partial D$. Then

(1) $W_0 = W_{01} + W_{02} + W_{03}$, *where*

$$W_{0i} = \overline{\text{span}}\{\psi^{(i)}(x - \alpha) : \alpha \in z^2\};$$

(2) *for $C = \{C_\alpha\}_{\alpha \in z^2} \in l^2$,*

$$A\|C\|_{l^2} \leq \left\| \sum_{\alpha \in z^2} C_\alpha \psi^{(i)}(x - \alpha) \right\|_{L^2} \leq B\|C\|_{l^2}, i = 1, 2, 3,$$

where A, B $(0 < A \leq B)$ are constants.

Proof. First we claim $\psi^{(i)}(x) \in W_0$. In view of orthogonal decomposition (2.3) this is equivalent to

$$\langle \phi(x-\alpha), \psi^{(i)}(x) \rangle = 0, \quad \forall \alpha \in \mathbb{Z}^2, \tag{3.2}$$

$$\psi^{(i)}(x) \in V_1. \tag{3.3}$$

From the definitions of $\psi^{(i)}(x)$, (3.3) is obvious.

By (2.2) and the definitions of $\psi^{(i)}(x)$ we know that (3.2) is equivalent to

$$H_\alpha := \sum_{\beta \in \mathbb{Z}^2} \sum_{\theta \in \mathbb{Z}^2} a_\beta \overline{b_\theta^{(i)}} \gamma_{\theta-2\alpha-\beta} = 0, \quad \alpha \in \mathbb{Z}^2.$$

Taking the symbol of $\{H_\alpha\}$ equivalently we should show

$$\sum_{e \in E} \sum_{\tilde{e} \in E} a_{\tilde{e}}(z) \overline{b_e^{(i)}(z)} I_{e-\tilde{e}}(z) = 0, \quad z \in \partial D. \tag{3.4}$$

By Lemma 2.1

$$\begin{aligned} &\sum_{e \in E} \sum_{\tilde{e} \in E} a_{\tilde{e}}(z^2) \overline{b_e^{(i)}(z^2)} I_{e-\tilde{e}}(z^2) \\ =&\sum_{e \in E} \sum_{\tilde{e} \in E} \left(\sum_{e^* \in E} 2^{-2} (z^{-1})^{\tilde{e}} (-1)^{e^* \cdot \tilde{e}} a((-1)^{e^*} z) \right) \overline{b_e^{(i)}(z^2)} I_{e-\tilde{e}}(z^2) \\ =&2^{-2} \sum_{e^* \in E} \overline{b^{(i)}((-1)^{e^*} z)} I((-1)^{e^*} z) a((-1)^{e^*} z). \end{aligned} \tag{3.5}$$

By the definitions of $b^{(i)}(z)$ one can prove that (3.4) follows from (3.5). Thus $\psi^{(i)}(x) \in W_0 (i = 1, 2, 3)$.

Now we show that for $\psi^{(i)}(x) (i = 1, 2, 3)$ we can find sequences $\{C_\alpha^{(i)}\} (i = 0, 1, 2, 3)$ with exponential decay, which satisfy

$$\psi(2x - e) = \sum_{\alpha \in \mathbb{Z}^2} C_{e-2\alpha}^{(0)} \phi(x-\alpha) + \sum_{i=1}^{3} \sum_{\alpha \in \mathbb{Z}^2} C_{e-2\alpha}^{(i)} \psi^{(i)}(x-\alpha), \tag{3.6}$$

where $e \in E$.

We know that (3.6) is equivalent to

$$\hat{\phi}(\frac{\omega}{2}) z^e = \left[\sum_{\alpha \in \mathbb{Z}^2} C_{e-2\alpha}^{(0)} z^{2\alpha} a(z) + \sum_{i=1}^{3} \sum_{\alpha \in \mathbb{Z}^2} C_{e-2\alpha}^{(i)} z^{2\alpha} b^{(i)}(z) \right] \hat{\phi}(\frac{\omega}{2}),$$

where $z = \left(e^{-i\frac{\omega_1}{2}}, e^{-i\frac{\omega_2}{2}}\right)$. It follows that

$$z^e = C_e^{(0)}(z^{-2}) a(z) + \sum_{i=1}^{3} C_e^{(i)}(z^{-2}) b^{(i)}(z). \tag{3.7}$$

Take $\hat{e} \in E$, then from (3.7)

$$((-1)^{\hat{e}} z)^e = C_e^{(0)}(z^{-2}) a((-1)^{\hat{e}} z) + \sum_{i=1}^{3} C_e^{(i)}(z^{-2}) b^{(i)}((-1)^{\hat{e}} z),$$

namely

$$(-1)^{\hat{e} \cdot e} = z^{-e} C_e^{(0)}(z^{-2}) a((-1)^{\hat{e}} z) + \sum_{i=1}^{3} z^{-e} C_e^{(i)}(z^{-2}) b^{(i)}((-1)^{\hat{e}} z).$$

By Lemma 2.1 and (2.7) we obtain the equations which are equivalent to (3.7):

$$C^{(0)}(z^{-1})a((-1)^{\hat{e}}z)+\sum_{i=1}^{3}C^{(i)}(z^{-1})b^{(i)}((-1)^{\hat{e}}z)=4\delta_{e_0\hat{e}},\quad \hat{e}=e_0,e_1,e_2,e_3. \tag{3.8}$$

Let $\Delta(z)$ denote the coefficient determinant of (3.8). If

$$\Delta(z)\neq 0 \tag{3.9}$$

in a neighborhood of ∂D, the equations (3.8) (about unknown $C^{(i)}(z^{-1})$, $i=0,1,2,3$) are solvable; furthermore the solutions $C^{(i)}(z^{-1})(i=0,1,2,3)$ are analytic in a neighborhood of ∂D. Hence there exist sequences $\{C_\alpha^{(i)}\}_{\alpha\in z^2}(i=0,1,2,3)$ with exponential decay such that

$$C^{(i)}(z^{-1})=\sum_{\alpha\in z^2}C_\alpha^{(i)}z^\alpha$$

for $z\in\partial D$. From the above discussion we know that the sequences $\{C_\alpha^{(i)}\}_{\alpha\in z^2}$ are what we want to find. Thus it remains to verify (3.9). By the definitions o f $b^{(i)}(z)$ the direct calculation shows

$$\overline{\Delta(z)}=-I(z^2)\sum_{e\in E}(I((-1)^e z)a((-1)^e z))^2.$$

From Lemma 2.2 and the hypothesis (3.1) we know $\overline{\Delta(z)}\neq 0$, $z\in\partial D$, namely $\Delta(z)\neq 0$ in a neighborhood of ∂D by the continuity as required.

Next we show that if sequences $d^{(i)}=\{d_\alpha^{(i)}\}\in l^2(i=0,1,2,3)$ satisfy

$$\sum_{\alpha\in z^2}d_\alpha^{(0)}\phi(x-\alpha)+\sum_{i=1}^{3}\sum_{\alpha\in z^2}d_\alpha^{(i)}\psi^{(i)}(x-\alpha)=0, \tag{3.10}$$

then $d^{(i)}=0$, $i=0,1,2,3$.

In fact, using Fourier transform, from (3.10) we have

$$d^{(0)}(z^2)a(z)+\sum_{i=1}^{3}d_\alpha^{(i)}(z^2)b^{(i)}(z)=0,\quad z\in\partial D, \tag{3.11}$$

where $d^{(i)}(z)$ is the symbol of the sequence $\{d_\alpha^{(i)}\}$.

Respectively taking $(-1)^{e_0}z$, $(-1)^{e_1}z$, $(-1)^{e_2}z$, $(-1)^{e_3}z$ in (3.11), we obtain the equations about the unknown $d^{(i)}(z^2)$:

$$\begin{aligned}
&d^{(0)}(z^2)a(z)+\sum_{i=1}^{3}d^{(i)}(z^2)b^{(i)}(z)=0,\\
&d^{(0)}(z^2)a((-1)^{e_1}z)+\sum_{i=1}^{3}d^{(i)}(z^2)b^{(i)}((-1)^{e_1}z)=0,\\
&d^{(0)}(z^2)a((-1)^{e_2}z)+\sum_{i=1}^{3}d^{(i)}(z^2)b^{(i)}((-1)^{e_2}z)=0,\\
&d^{(0)}(z^2)a((-1)^{e_3}z)+\sum_{i=1}^{3}d^{(i)}(z^2)b^{(i)}((-1)^{e_3}z)=0\quad(z\in\partial D).
\end{aligned} \tag{3.12}$$

The determinant of (3.12) is also $\Delta(z)$. Since $\Delta(z)\neq 0$, $z\in\partial D$, it follows that

$$d^{(i)}(z^2)=0,\quad i=0,1,2,3,z\in\partial D.$$

Hence $d^{(i)} = 0$, $i = 0, 1, 2, 3$.

Thus the conclusion (1) has been proved.

Finally we will show that $\{\psi^{(i)}(x-\alpha)\}_{\alpha\in\mathbb{Z}^2}$ satisfy the conclusion (2) of the theorem. Let $C = \{C_\alpha\}_{\alpha\in\mathbb{Z}^2} \in l^2$. We should show that there exist constants A and B $(0 < A \le B)$ such that

$$A\|C\|_{l^2} \le \left\| \sum_{\alpha\in\mathbb{Z}^2} C_\alpha \psi^{(i)}(x-\alpha) \right\|_{L^2} \le B\|C\|_{l^2}. \tag{3.13}$$

It is clear that (3.13) is equivalent to

$$0 < A \le \sum_{\alpha\in\mathbb{Z}^2} |\hat{\psi}^{(i)}(\omega + 2\pi\alpha)|^2 \le B, \tag{3.14}$$

(see the proof of Lemma 2.2).

Noting the constructions of $\psi^{(i)}(x)$, we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^2} |\hat{\psi}^{(i)}(\omega + 2\pi\alpha)|^2 &= 2^{-4} \sum_{e\in E} |b^{(i)}((-1)^e z)|^2 \left(\sum_{\alpha\in\mathbb{Z}^2} |\hat{\psi}(\frac{\omega}{2} + \pi e + 2\pi\alpha)|^2 \right) \\ &= 2^{-4} \sum_{e\in E} |b^{(i)}((-1)^e z)|^2 I((-1)^e z) \\ &= 2^{-4} \sum_{e\in E} |I((-1)^e z) a((-1)^e z)|^2 I((-1)^e z), \quad z \in \partial D.\end{aligned}$$

In view of the condition (3.1), Lemma 2.2 and the compactness of ∂D we know that there exist constants A $(A > 0)$ and B $(B \ge A)$ which satisfy (3.13). Thus we complete the proof of the theorem.

§4. Further Results

Suppose that $\phi(x)$ generate the multiresolution analysis studied in the above sections. If in addition

$$\int_{\mathbb{R}^2} \phi(x)\overline{\phi(x-\alpha)}dx = \delta_{0\alpha}, \quad \alpha \in \mathbb{Z}^2,$$

we can obtain more about this important case in wavelet decompositions.

Theorem 4.1. *Suppose $\phi(x)$ generate the multiresolution analysis described above. In addition suppose*

$$\int_{\mathbb{R}^2} \phi(x)\overline{\phi(x-\alpha)}dx = \delta_{0\alpha}, \quad \alpha \in Z^2 \tag{4.1}$$

and $a(z)$ is a real function $(z \in \partial D)$. Then we have

(1) $I(z) = 1$;

(2) $\langle \psi^{(i)}(x), \psi^{(i)}(x-\alpha) \rangle = \delta_{0\alpha}$, $\alpha \in \mathbb{Z}^2$, $i = 1, 2, 3$;

(3) $\langle \psi^{(i)}(x), \psi^{(j)}(x-\alpha) \rangle = 0$, $\alpha \in \mathbb{Z}^2$, $i \ne j$, $i, j = 1, 2, 3$;

(4) $\sum_{e\in E} (a((-1)^e z))^2 \ne 0$.

Here $\psi^{(i)}(x)$ are defined as in Theorem 3.1. Thus we have

$$W_0 = W_{01} \oplus W_{02} \oplus W_{03}, \quad W_{0i} = \overline{\operatorname{span}}\{\psi^{(i)}(x-\alpha), \alpha \in \mathbb{Z}^2\}.$$

Proof. By Theorem 3.1. The details are omitted.

Finally as a consequence of Theorem 3.1 we show that in the case $s = 2$ box splines of three directions generate wavelet decompositions. It is well known that in the case $s = 1$ the wavelet decompositions which are obtained with use of the B-splines were widely applied in practical fields. We believe that in the case $s = 2$ the wavelet decompositions generated by box splines will play an important role as B-splines do in the case $s = 1$.

A detailed knowledge of box spline theory can be gained from [2,5]. Let $x^1 = (0,1)$, $x^2 = (1,0)$, $x^3 = (1,1)$. Box splines of three directions x^1, x^2, x^3 are

$$B(x) = B(x|x^1, \cdots, x^1; x^2, \cdots, x^2; x^3, \cdots, x^3).$$

It is known that $B(x)$ generates a multiresolusion analysis[3]. By Theorem 3.1 it is clear that the wavelet decompositions can be obtained if $B(x)$ satisfy the condition (3.1). Now we will verify it. It is obvious that we only need to consider the simplest case $B_1(x) = B(x|x^1, x^2, x^3)$.

For the box spline $B_1(x)$ there exists a sequence $\{a_\alpha\}_{\alpha \in \mathbb{Z}^2}$ satisfying

$$B_1(x) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha B_1(2x - \alpha).$$

Let $z = (z_1, z_2) = (e^{-i\frac{\omega_1}{2}}, e^{-i\frac{\omega_2}{2}})$, $\omega = (\omega_1, \omega_2)$.

$$\hat{B}_1(\omega) = \left(\frac{1 - e^{-i\omega \cdot x^1}}{i\omega \cdot x^1}\right) \cdot \left(\frac{1 - e^{-i\omega \cdot x^2}}{i\omega \cdot x^2}\right) \cdot \left(\frac{1 - e^{-i\omega \cdot x^3}}{i\omega \cdot x^3}\right),$$

$$a(z) = \frac{\hat{B}_1(\omega)}{\frac{1}{2^2}\hat{B}_1(\frac{\omega}{2})} = \frac{1}{2}(1 + z_1)(1 + z_2)(1 + z_1 z_2).$$

By some calculation we can obtain $\sum_{e \in E} (I((-1)^e z) a((-1)^e z))^2 \neq 0$. Hence the condition (3.1) is satisfied by $B_1(x)$.

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