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A MACHINE PRODUCING NON-$*$LINDELÖF SPACES

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Abstract

The authors construct a machine putting out non-$*$Lindelöf spaces, prove that many interesting topological properties can be preserved after the process, and with the aid of this machine obtain an example of locally compact, ccc, non-$*$Lindelöf space and a consistent example of first countable, locally compact, ccc, non-$*$Lindelöf space.

§1. Introduction

A topological space X is called $*$Lindelöf space (Compactwise $*$Lindelöf, respectively) iff for every open cover $\mathcal{G}$ of X there exists a countable subset $\{x_n : n < \omega\}$ (countable family of compact subsets $\{K_n : n < \omega\}$, respectively) such that $\bigcup_{n=1}^{\infty} \mathrm{st}(x_n, \mathcal{G}) = X(\bigcup_{n=1}^{\infty} \mathrm{st}(K_n, \mathcal{G}) = X)$. It is easy to see that all separable spaces or $\aleph$ compact spaces are $*$Lindelöf, all $*$Lindelöf spaces are compactwise $*$Lindelöf, and a space in which all open subspaces are compactwise $*$Lindelöf is ccc. One question is whether all open subspaces of X are $*$Lindelöf if X is compact Hausdorff ccc space. This question is equivalent to whether every locally compact Hausdorff ccc space is $*$Lindelöf ? To give the negative answer, we construct a machine putting out non-$*$Lindelöf space whenever a non-separable space is put in. The technique used here is similiar to the split point method used by Tall and Watson.

In this paper, all spaces are regular, and N is the set of all positive integers. We refer the reader to [1] and [2] for concepts and terminologies undefined.

§2. Construction of the Machine

Let (X,τ) be a non separable space with $\pi w(X) = \kappa < c$. Choose a π base $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ and a sequence $\{B_{\alpha n} : n \in N\} \in [\mathcal{B}]^\omega$ for each α such that $B_{\alpha 1} = B_\alpha$ and $\bar{B}_{\alpha n+1} \subset B_{\alpha n}$, $(n \in N)$. Let $\{N_\alpha : \alpha < \kappa\} \subset [N]^\omega$ be an almost disjoint family of N with cardinality κ (i.e. for $\alpha, \beta < \kappa$, $\alpha \neq \beta$, $N_\alpha \cap N_\beta$ is finite). Denote $N_{\alpha n} = \{j : j \in N_\alpha, j > n\}$, $X_n = X \times \{n\}$ for $n \in N$. $X \times N = \bigcup_{n=1}^{\infty} X_n$. Let $X_0 = \{p_\alpha : \alpha < \kappa\}$ be any set which disjoint from $X \times N$, $\widetilde{X} = (X \times N) \cup X_0$.

We define a topology of $\widetilde{X}$ as following. (1) If $p =< x, n >$, pick $\{u \times \{n\} : x \in u \in \tau\}$ as a base of neighbourhoods of p. (2) If $p = p_\alpha \in X_0$, pick $\{U_{\alpha mn} : m, n \in N\}$ as a base of neighbourhoods of p, where $u_{\alpha mn} = \{p_\alpha\} \cup (\cup\{B_{\alpha m} \times \{i\} : i \in N_{\alpha n}\})$.

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Thus $\widetilde{X}$ is constructed to be a regular space. (Notice that $Cl_{\widetilde{X}}U_{\alpha\, m+1\, n} \subset U_{\alpha mn}$ for all $m, n \in N$.) The subspaces X_n are clopen in $\widetilde{X}$ and homeomorphic to X for $n \in N$. $X \times N$ is an open dense subspace and X_0 is a closed discrete subset of $\widetilde{X}$.

§3. Behavior of the Machine

Claim 3.1. *X is not $*$Lindelöf.*

Proof. For the open cover $\mathcal{G} = \{X \times N\} \cup \{u_{\alpha 11} : \alpha < \kappa\}$ and any countable subset A of $\widetilde{X}$, denote $A_0 = A \cap X_0$, $A \cap X_n = A_n \times \{n\}$, where $A_n \subset X$. Since the open subspace $X - (\bigcup\limits_{n=1}^{\infty} A_n)^-$ is nonseparable, there are uncountably many $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset X - (\bigcup\limits_{n=1}^{\infty} A_n)^-$. Since A_0 is countable, we can find a $p_\alpha \notin A_0$ such that $B_\alpha \bigcap (\bigcup\limits_{n=1}^{\infty} A_n)^- = \emptyset$. Thus we have $p_\alpha \notin \cup\{\mathrm{st}(p, \mathcal{G}) : p \in A\}$.

Remark. We say a space X is compactwise separable iff there exist countably many compact subsets of X $\{K_n : n < \omega\}$ such that $(\bigcup\limits_{n=1}^{\infty} K_n)^- = X$. It is obvious that all separable spaces and σ-compact spaces are compactwise separable. Using a method similiar to above, we can prove that $\widetilde{X}$ is not compactwise $*$Lindelöf if X is not compactwise separable.

Claim 3.2. *If X has one of the following properties, then $\widetilde{X}$ has the same one.*

(1) *first countablity,*

(2) *ccc,*

(3) *$Cal(\omega_1, \omega)$,*

(4) *$Cal\ \omega_1$,*

(5) *compactwise separability,*

(6) *Baire,*

(7) *perfect.*

Claim 3.3. (1) *If X is 0-dimensional and the elements of $\mathcal{B}$ are clopen sets, then $\widetilde{X}$ is 0-dimensional.*

(2) *If X is 0-dimensional, locally compact and the elements of $\mathcal{B}$ are clopen compact, then $\widetilde{X}$ is 0-dimensional and locally compact.*

Claim 3.4. (1) *If X is submetacompact, so is $\widetilde{X}$.*

(2) *If X is subparacompact, so is $\widetilde{X}$.*

Proof. (1) Let $\mathcal{G}$ be an open cover of $\widetilde{X}$. $\mathcal{G}|_{X\times N} = \{G \cap (X \times N) : G \in \mathcal{G}\}$ is an open cover of $X \times N$. $X \times N$ is submetacompact because X is so. There exists a sequence of open covers $\{\mathcal{V}_n : n \in N\}$ of $X \times N$ which refines $\mathcal{G}|_{X\times N}$ such that for every $p \in X \times N$, $1 \leq \mathrm{ord}(p, \mathcal{V}_m) < \omega$ for some $m \in N$. Now for every $p_\alpha \in X_0$, $n \in N$, we can find a nbd $u_{\alpha m_\alpha n_\alpha}$ of p_α such that $n_\alpha > n$, and $u_{\alpha m_\alpha n_\alpha} \subset G$ for some $G \in \mathcal{G}$. Let $\mathcal{U}_n = \{u_{\alpha m_\alpha n_\alpha} : \alpha < \kappa\}$, $\mathcal{W}_{mn} = \mathcal{V}_m \cup \mathcal{U}_n$. $\{\mathcal{W}_{mn} : m, n \in N\}$ is a θ-sequence of open covers of $\widetilde{X}$ refining $\mathcal{G}$.

(2) Let $\mathcal{G}$ be an open cover of $\widetilde{X}$. Since clopen subspace X_m is subparacompact, we can find a σ-discrete closed refinement $\bigcup\limits_{n=1}^{\infty} \mathcal{F}_{mn}$ of $\mathcal{G}|_{X_m} = \{G \cap X_m : G \in \mathcal{G}\}$. Then $\mathcal{F} = \{\{p_\alpha\} : \alpha < k\} \cup (\bigcup\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} \mathcal{F}_{mn}\}$ is a σ-discrete closed refinement of $\mathcal{G}$.

Claim 3.5. *(1) If X is a σ space, so is $\widetilde{X}$.*

(2) If X is a Moore space, so is $\widetilde{X}$.

Proof of (2). Let $\{\mathcal{G}_n : n \in N\}$ be a development of X. Let $\widetilde{\mathcal{G}_i} = \bigcup_{j=1}^{\infty} \{G \times \{j\} : G \in \mathcal{G}_i\}$, $\mathcal{U}_{mn} = \{u_{\alpha mn} : \alpha < \kappa\}$, $\mathcal{H}_{imn} = \widetilde{\mathcal{G}_i} \cup \mathcal{U}_{mn}$. Then $\{\mathcal{H}_{imn} : i, m, n \in N\}$, the sequence of open covers of $\widetilde{X}$ is a development of X.

The proof of (1) is similar to (2).

Claim 3.6. *If X has a base $\mathcal{B}$ with one of the following properties, then $\widetilde{X}$ has the same one.*

(1) σ-disjoint base,

(2) σ-point-finite base,

(3) point-countable base,

(4) compact-countable base.

Proof. We prove (1) and (4). The proofs of (2) and (3) are similar to that of (1).

Proof of (1). Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ be a σ-disjoint base of X. Let $\widetilde{\mathcal{B}_{ij}} = \{B \times \{j\} : B \in \mathcal{B}_i\}$, $\mathcal{U}_{imn} = \{u_{\alpha mn} : \alpha < \kappa, B_{\alpha 1} \in \mathcal{B}_i\}$ for $i, j, m, n \in N$. Then $\cup\{\widetilde{\mathcal{B}_{ij}} : i, j \in N\} \cup (\cup\{\mathcal{U}_{imn} : i, m, n \in N\})$ is a σ-disjoint base of $\widetilde{X}$.

Proof of (4). Let $\mathcal{B}$ be a compact-countable base of X. Let $\widetilde{\mathcal{B}_j} = \{B \times \{j\} : B \in \mathcal{B}\}$, $\mathcal{U}_{mn} = \{u_{\alpha mn} : \alpha < \kappa\}$ for $i, j, m, n \in N$. Then $\widetilde{\mathcal{B}} = (\bigcup_{j=1}^{\infty} \widetilde{\mathcal{B}_j}) \cup (\cup\{\mathcal{U}_{mn} : m, n \in N\})$ is a base of $\widetilde{X}$. If K is a compact subset of $\widetilde{X}$, then $K_j = K \cap X_j$ is a compact subset of X_j. Since X_j is homeomorphic to X, we have $\mathrm{ord}(K_j, \widetilde{\mathcal{B}_i}) = 0$ for $j \neq i$ and $\mathrm{ord}(K_j, \widetilde{\mathcal{B}_j}) \leq \omega$, $\mathrm{ord}(K_j, \mathcal{U}_{mn}) = 0$ for $j \leq n$ and $\mathrm{ord}(K_j, \mathcal{U}_{mn}) \leq \omega$ for $j > n$. Since $K_0 = K \cap X_0$ is finite, we have $\mathrm{ord}(K_0, \widetilde{\mathcal{B}}) \leq \omega$ and $\mathrm{ord}(K, \widetilde{\mathcal{B}}) = \mathrm{ord}(\bigcup_{j=1}^{\infty} K_j, \widetilde{\mathcal{B}}) = \sum_{j=1}^{\infty} \mathrm{ord}(K_j, \widetilde{\mathcal{B}}) \leq \omega$. Thus $\widetilde{\mathcal{B}}$ is a compact countable base of $\widetilde{X}$.

Claim 3.7. *If the π base $\mathcal{B}$ of X is not point countable, then $\widetilde{X}$ is not metalidelöf.*

Proof. Choose an open cover $\mathcal{G} = \{X \times N\} \cup \{u_{\alpha 11} : \alpha < \kappa\}$ of $\widetilde{X}$. By the principle of pigeonhole, for every open refinement $\mathcal{V}$ of $\mathcal{G}$, there exists $p \in X \times N$ such that $\mathrm{ord}(p, \mathcal{V}) > \omega$.

§4. Existence of Locally Compact 0-Dimensional ccc Non-*Lindelöf Space

Theorem 4.1. *There exists a locally compact, 0-dimensional, ccc space which is not *Lindelöf .*

Proof. Bell has given a compactification $\gamma\omega$ of ω in [3] such that $X = \gamma\omega - \omega$ is a locally compact, 0-dimensional, ccc, nonseparable space with $w(x) = c$. Put it in the machine described in §1. We obtain the space $\widetilde{X}$ that we need. Noting $X \times N$ is a σ-compact open dense set in $\widetilde{X}$, we see that the space $\widetilde{X}$ is compactwise separable, hence compactwise *Lindelöf .

It is known that the Suslin line T with the order topology is a locally compact, 0-dimensional first countable ccc nonseparable space with $w(T) = \omega_1$ (cf. [4]). Putting T

into the machine, we obtain a consistent example of a locally compact, 0-dimensional, first countable, ccc, non-$*$Lindelöf space.

Theorem 4.2. *If there exists a compact, 0-dimensional, ccc, nonseparable space X with $\pi w(x) \leq c$ such that X has a P point p, then there exists a locally compact, 0-dimensional ccc space which is not compactwise $*$Lindelöf.*

Proof. Let $Y = X - \{p\}$, $\{K_n : n \in N\}$ be a sequence of compact subsets in Y. The interior of $X - \bigcup_{n=1}^{\infty} K_n$ is nonempty because p is a P point of X. From the remark of Claim 1 in [3], we can see that $\widetilde{Y}$ is not compactwise $*$Lindelöf.

Theorem 4.3. *There exists a first countable, 0-dimensional, compactwise separable ccc space which is not $*$Lindelöf.*

Proof. Bell has given a first countable, 0-dimensional, ccc, nonseparable space X in [5]. The output $\widetilde{X}$ of X is what we need.

References

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