Chin. Ann. of Math.
14B: 2(1993), 201-208.

ON THE APPROXIMATION OF (0,2) INTERPOLATION

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Abstract

Let $-1 = x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} = 1$ be the zeros of the polynomial $(1-x^2)P'_{n-1}(x)$, where $P_{n-1}(x)$ is the $(n-1)$th Legendre polynomial and n is even. For $f \in C_{[-1,1]}$, denote by $R_n(f,0,x)$ the polynomial of degree $2n-1$, which satisfies the conditions: $R_n(f,0,x_{k,n}) = f(x_{k,n})$ and $R''_n(f,0,x_{k,n}) = 0$ $(k = 1, 2, \cdots, n)$.

In this paper it is proved that

$$\max_{|x|\le 1} |R_n(f,0,x) - f(x)| = O(nw_2(f, \frac{1}{n})),$$

where $w_2(f,\delta)$ is the continuity modulus of second order of $f(x)$.

§1. Introduction

Let

$$-1 = x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} = 1 \quad (n = 1, 2, \cdots)$$

be the zeros of the polynomial

$$\pi_n(x) = (1-x^2)P'_{n-1}(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)dt,$$

where $P_{n-1}(t)$ is the $(n-1)$th Legendre polynomial with normalization $P_{n-1}(1) = 1$. As in [1], by (0,2) interpolation polynomial of $f \in C_{[-1,1]}$ based on the zeros of polynomial $\pi_n(x)$, we mean those polynomials $R_n(f,\beta,x)$ of degree $\le 2n-1$ whose values and second derivatives at $\{x_{kn}\}_{k=1}^{n}$ are prescribed:

$$R_n(f,\beta,x_{kn}) = f(x_{kn}), \quad R''_n(f,\beta,x_{kn}) = \beta_{kn} \quad (k = 1, 2, \cdots, n), \tag{1.1}$$

where $\beta = \{\beta_{kn}\}_{k=1}^{n}$ is an arbitrary given system of real numbers. In [2] and [3] P. Turan and his collaborators proved that such interpolating polynomials exist if and only if n is even, and the following convergence theorem holds:

Theorem T. *Let $f \in C^1_{[-1,1]}$ with continuity modulus $w(f',\delta)$ of $f'(x)$ such that*

$$\int_0^1 \frac{w(f',\delta)}{\delta} d\delta < \infty \tag{1.2}$$

and let

$$\max_{1\le k\le n} |\beta_{kn}| = o(n). \tag{1.3}$$

Then $R_n(f,\beta,x)$ converges to $f(x)$ uniformly in $[-1,1]$.

Manuscript received December 25, 1990.

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For the sake of brevity let us suppose throughout this paper that $n = 2k$ $(k = 1, 2, \cdots)$. On the importance of the condition (1.2), Turan pointed out that even if all β_{kn} are zeros, the condition (1.2) cannot be replaced by a Lipschitz condition of order $\alpha < 1$. But in [4] G. Freud improved above Theorem T as following

Theorem F. *Let $f \in C_{[-1,1]}$ satisfy the condition*

$$w_2(f,\delta) = o(\delta) \tag{1.4}$$

and let

$$\beta_{kn} = o(\Delta_n^{-1}(x_{kn})) \quad (k = 1, 2, \cdots, n), \tag{1.5}$$

where $w_2(f,\delta)$ is the continuity modulus of second order of $f(x)$, and $\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$. Then $R_n(f,\beta,x)$ converges to $f(x)$ uniformly on $[-1,1]$.

In order to estimate the order of the difference $f(x) - R_n(f,\beta,x)$, P. O. H. Vertesi introduced a definition: the function $f \in C_{[-1,1]}$ satisfies the condition (A_2) if $w_2(f,\delta) = O(1)w_2(\delta)$, where $w_2(\delta)$ is a non-negative increasing function, $w_2(ct) \le K_c w_2(t)$, and further

$$\sum_{j=1}^{k} \Delta_{2^j}^{-2}(x) w_2(\Delta_{2^j}(x)) = O(1)\Delta_{2^k}^{-2}(x) w_2(\Delta_{2^k}(x)) \tag{1.6}$$

uniformly for $x \in [-1,1]$. Vertesi proved[5]

Theorem V. *If $f \in C_{[-1,1]}$ satisfies the condition (A_2), then*

$$\|f - R_n(f,O)\| = O(1) n w_2\Big(\frac{1}{n}\Big),$$

where $O = \{0\}_{k=1}^n$ and $\|f\| = \max_{|x|\le 1} |f(x)|$.

It is easy to see that the condition (A_2) is very complex. One may raise a problem: does the condition (A_2) is necessary for $\|f - R_n(f,O)\| = O(1) n w_2\Big(\frac{1}{n}\Big)$? The negative answer of this problem will be given in the present paper. We will prove that for any function $f \in C_{[-1,1]}$ the estimation

$$\|f - R_n(f,O)\| = O(1) n w_2\Big(f, \frac{1}{n}\Big)$$

always holds. Furthermore, the pointwise estimation

$$f(x) - R_n(f,O,x) = O(1)\Big\{ n w_2\Big(f,\frac{1}{n}\Big)\frac{|\pi_n(x)|}{\sqrt{n}} + w\Big(f, \frac{|\pi_n(x)|}{n}\Delta_n^{1/2}(x)\Big)\Big\}$$

holds uniformly on $[-1,1]$.

§2. Preliminaries

Denote by

$$l_{\nu n}(x) = \frac{\pi_n(x)}{\pi_n'(x_{\nu n})(x - x_{\nu n})} \quad (\nu = 1, 2, \cdots, n) \tag{2.1}$$

the fundamental polynomials of Lagrange interpolation based on $\{x_{kn}\}_{k=1}^n$. P. Turan proved that the (0,2) interpolation polynomial $R_n(f,\beta,x)$, which is uniquely determined by $f \in C_{[-1,1]}$ and $\beta = \{\beta_{kn}\}_{k=1}^n$, can be represented as follows

$$R_n(f,\beta,x) = \sum_{k=1}^{n} f(x_{kn})\gamma_{kn}(x) + \sum_{k=1}^{n} \beta_{kn}\rho_{kn}(x),$$

where $\gamma_k(x)$ and $\rho_k(x)$[1] are the polynomials of degree $\le 2n-1$ uniquely determined by conditions

$$\gamma_k(x_j) = \begin{cases} 0, & k \ne j, \\ 1, & k = j, \end{cases} \qquad \gamma_k''(x_j) = 0,$$

$$\rho_k''(x_j) = \begin{cases} 0, & k \ne j, \\ 1, & k = j, \end{cases}$$

$$\rho_k(x_j) = 0 \qquad (j,k = 1,2,\cdots,n; \quad n = 1,2,\cdots),$$

respectively.

We need some lemmas.

Lemma 2.1. *On the polynomials $\gamma_k(x)$ and $\rho_k(x)$ $(k = 1,2,\cdots,n)$, the following estimations hold*

$$\gamma_k(x) = O(1)(|l_k(x)| + |\pi_n(x)|n^{-1}\Delta_n^{-1/2}(x_k)), \tag{2.2}$$

$$\rho_k(x) = O(1)(|l_k(x)|n^{-3/2}(1-x_k^2)^{5/4} + |\pi_n(x)|n^{-1}\Delta_n^{3/2}(x_k)). \tag{2.3}$$

Furthermore, for $x \in (x_{\nu+1}, x_{\nu-1})$ $(\nu = 2,3,\cdots,n-1)$, we have

$$\rho_\nu(x) = O(|\pi_n(x)|n^{-2}(1-x_\nu^2)). \tag{2.4}$$

Proof. The explicit form of the polynomials $\gamma_k(x)$ and $\rho_k(x)$ is the following (see [2, 3]): for $k = 1, n$

$$\begin{aligned}\gamma_k(x) =& \frac{3+x_kx}{4}l_k^2(x) - x_k\frac{1-x^2}{4}l_k(x)l_k'(x) \\ &+ \Big(\frac{5}{16} + \frac{1}{8n(n-1)}\Big)\pi_n(x)\Big(1 + \frac{x_k}{3}P_{n-1}(x)\Big).\end{aligned} \tag{2.5}$$

and for $2 \le k \le n-1$,

$$\begin{aligned}\gamma_k(x) =& (x-x_k)l_k(x)\int_{\frac{x-x_k}{|x-x_k|}}^{x} \frac{l_k(t)}{(t-x_k)^2}dt \\ &+ \frac{1}{n(n-1)}\Big(\frac{x_k-x}{2|x-x_k|} + \frac{P_{n-1}(x)}{6}\Big)\frac{\pi_n(x)}{(1-x_k^2)P_{n-1}^3(x_k)}.\end{aligned} \tag{2.6}$$

Further, for $k = 1,\ n$

$$\rho_k(x) = \frac{\pi_n(x)}{n^2(n-1)^2}\Big\{x_k + \frac{1}{3}P_{n-1}(x)\Big\}, \tag{2.7}$$

and for $k = 2,3,\cdots,n-1$

$$\begin{aligned}\rho_k(x) =& \frac{\pi_n(x)}{2\pi_n'(x_k)P_{n-1}''(x_k)} \\ &\times\Big\{\int_{\frac{x-x_k}{|x-x_k|}}^{x} \frac{P_{n-1}'(t)}{t-x_k}dt + P_{n-1}(x)\Big(-\frac{x_k}{1-x_k^2} + \frac{1}{3}\frac{1}{(1-x_k^2)P_{n-1}(x_k)}\Big) \\ &+ \frac{x_k-x}{|x_k-x|}\Big(-\frac{3}{1-x_k^2} + \frac{1}{(1-x_k^2)P_{n-1}(x_k)}\Big)\Big\}.\end{aligned} \tag{2.8}$$

It is well known that

$$|l_k(x)| \le 1 \qquad (|x| \le 1),$$

[1]Here and in many cases we denote x_{kn}, γ_{kn}, β_{kn} and ρ_{kn} by x_k, γ_k, β_k and ρ_k respectively.

$$|P_{n-1}(x_k)| \geq \frac{1}{100} n^{-1} \Delta_n^{-1/2}(x_k) \quad (k = 1, 2, \cdots, n). \tag{2.9}$$

Hence, for $k = 1$, n, using the Bernstein inequality we obtain

$$(1 - x^2) l_k(x) l_k'(x) = O(|\pi_n(x)|),$$

and from (2.5) we have

$$\gamma_k(x) = O(1)(|l_k(x)| + |\pi_n(x)| n^{-1} \Delta_n^{-1/2}(x_k)). \tag{2.10}$$

For $k = 2, 3, \cdots, n-1$, from (2.6) we have also

$$\gamma_k(x) = O(1)(|l_k(x)| + |\pi_n(x)| n^{-1} \Delta_n^{-1/2}(x_k)), \tag{2.11}$$

(2.10) and (2.11) complete the proof of inequality (2.2).

Now, we are going to prove (2.3) and (2.4). For $k = 1$, n, (2.3) and (2.4) are valid obviously. In order to prove (2.3) and (2.4) for $k = 2, 3, \cdots, n-1$, first let $x < x_k \leq 1$. Since for $x < x_k$

$$\int_{-1}^{x} \frac{P_{n-1}'(t)}{t - x_k} dt = O(1) \Big(\frac{1}{|x - x_k|} + \frac{1}{1 - x_k^2} \Big)$$

and for $x \in (x_{k+1}, x_k)$

$$\begin{aligned}
&\int_{-1}^{x} \frac{P_{n-1}'(t)}{t - x_k} dt \\
&= \int_{-1}^{x_{k+1}} \frac{P_{n-1}'(t)}{t - x_k} dt + \int_{x_{k+1}}^{x} \frac{P_{n-1}'(t)}{t - x_k} dt \\
&= O(1) \Big(\frac{1}{x_k - x_{k+1}} + |x - x_{k+1}| \max_{x_{k+1} < t < x_{k-1}} |P_{n-1}''(t)| \Big) \\
&= O(1) \Big(\frac{n}{\sqrt{1 - x_k^2}} + \frac{\sqrt{1 - x_k^2}}{n} \cdot n^{-1} \Big(\frac{\sqrt{1 - x_k^2}}{n} \Big)^{-\frac{5}{2}} \Big) \\
&= O(1) \frac{n}{\sqrt{1 - x_k^2}},
\end{aligned}$$

then, using the following equalities

$$(1 - x_k^2) P_{n-1}''(x_k) = -n(n-1) P_{n-1}(x_k),$$

$$\pi_n'(x_k) = -n(n-1) P_{n-1}(x_k)$$

and (2.9), we obtain that for $x < x_k$

$$\rho_k(x) = O(1)(|l_k(x)| n^{-3/2} (1 - x_k^2)^{5/4} + |\pi_n(x)| n^{-1} \Delta_n^{3/2}(x_k))$$

and for $x \in (x_{k+1}, x_k)$.

$$\rho_k(x) = O(1) |\pi_n(x)| \frac{1 - x_k^2}{n^2}.$$

For $-1 \leq x_k < x$, the estimation runs similarly, and for $x = x_k$, the estimation obviously holds. Lemma 2.1 is proved.

Lemma 2.2. *Let $f \in C_{[-1,1]}$. Then for $n = 1, 2, \cdots$, there exists a polynomial $q_n(x)$ of degree $\leq 2n - 1$ such that*

$$q_n(x) - f(x) = O(1) w_2\Big(f, \frac{1}{n}\Big),$$

$$q_n''(x) = O(1)n^2 w_2\Big(f, \frac{1}{n}\Big)$$

and $q_n(\pm 1) = f(\pm 1)$.

Proof. In 1982, D. Levitan proved that for every $n \geq 1$ there exists an n-th degree polynomial $q_n^*(x)$ such that

$$f(x) - q_n^*(x) = O(1)w_2\Big(f, \frac{1}{n}\Big),$$

$$q_n^{*\prime\prime}(x) = O(1)n^2 w_2\Big(f, \frac{1}{n}\Big).$$

Let us consider

$$q_n(x) = q_n^*(x) + \frac{1+x}{2}(f(1) - q^*(1)) + \frac{1-x}{2}(f(-1) - q_n^*(-1)).$$

It is easy to see that $q_n(x)$ satisfies the requirements of Lemma 2.2.

§3. Some New Results

Now we can prove our main result.

Theorem 3.1. *If* $f \in C_{[-1,1]}$, *then*

$$f(x) - R_n(f, O, x) = O(1)\Big(\frac{|\pi_n(x)|}{\sqrt{n}} n w_2\Big(f, \frac{1}{n}\Big) + w\Big(f, \frac{|\pi_n(x)|}{n}\Delta_n^{1/2}(x)\Big)\Big). \tag{3.1}$$

Proof. Set

$$g(x) = f(x) - \frac{1+x}{2}f(1) - \frac{1-x}{2}f(-1).$$

We have $w_2(g, \delta) = w_2(f, \delta)$, $g(\pm 1) = 0$ and

$$R_n(f, O, x) - f(x) = R_n(g, O, x) - g(x). \tag{3.2}$$

By Lemma 2.2, there exists a polynomial $q_n(x)$ of degree $\leq n$ such that

$$q_n(x) - g(x) = O(1)w_2\Big(f, \frac{1}{n}\Big), \tag{3.3}$$

$$q_n''(x) = O(1)n^2 w_2\Big(f, \frac{1}{n}\Big), \tag{3.4}$$

and

$$q_n(\pm 1) = 0. \tag{3.5}$$

Obviously, equality (3.5) implies that there is a point $\xi \in (-1, 1)$ such that $q_n'(\xi) = 0$. Thus

$$q_n'(x) = \int_\xi^x q_n''(t)dt,$$

and by (3.4) we obtain

$$q_n'(x) = O(1)n^2 w_2\Big(f, \frac{1}{n}\Big). \tag{3.6}$$

Owing to the uniqueness theorem mentioned in section 1, we have

$$q_n(x) = \sum_{k=1}^n q_n(x_k)\gamma_k(x) + \sum_{k=1}^n q_n''(x_k)\rho_k(x).$$

Hence

$$\begin{aligned}&R_n(g,O,x)-g(x)\\&=(R_n(g-q_n,O,x)-g(x)+q_n(x))-\sum_{k=1}^{n}q_n''(x_k)\rho_k(x)\\&=I_1-I_2.\end{aligned}\tag{3.7}$$

Let $x\geq 0$ and

$$|x-x_\nu|=\min_{1\leq k\leq n}|x-x_k|.$$

Form (2.3), (2.4), (2.7), (2.9) and (3.4), it follows[2] that

$$\begin{aligned}I_2=&O(1)n^2w_2\Big(f,\frac{1}{n}\Big)\times\\&\times\Big\{|\pi_n(x)|\frac{1}{n^2}+\sum_{k\neq\nu}(|l_k(x)|n^{-3/2}(1-x_k^2)^{5/4}+|\pi_n(x)|n^{-1}\Delta^{3/2}(x_k))\Big\}\\=&O(1)n^2w_2\Big(f,\frac{1}{n}\Big)|\pi_n(x)|\Big\{\frac{1}{n^2}+\sum_{k\neq\nu}\Big(\frac{k^3n^{-4}}{|(k-\nu)(k+\nu)|}+k^{3/2}n^{-4}\Big)\Big\}.\end{aligned}$$

Thus

$$I_2=O(1)nw_2\Big(f,\frac{1}{n}\Big)\frac{|\pi_n(x)|}{\sqrt{n}}.\tag{3.8}$$

Now we are going to estimate I_1. Using Lemma 2.1, we have

$$\begin{aligned}I_1=&\Big\{[f(x_\nu)-f(x)+q_n(x)-q_n(x_\nu)]+\frac{1}{2}(x-x_\nu)[f(1)-f(-1)]\Big\}l_\nu(x)\\&+O(1)w_2\Big(f,\frac{1}{n}\Big)\Big\{\sum_{k\neq\nu}|l_k(x)|+|\pi_n(x)|\sum_{k=1}^{n}\frac{1}{\sqrt{n}(1-x_k^2)^{\frac14}+1}\Big\}\\=&I_{11}+I_{12}.\end{aligned}\tag{3.9}$$

Since $|l_\nu(x)|\leq 1$, we have

$$\begin{aligned}&(f(x_\nu)-f(x))l_\nu(x)\\&=O(1)w(f,|x-x_\nu||l_\nu(x)|)\\&=O(1)w(f,|\pi_n(x)|n^{-1}\Delta_n^{1/2}(x))\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}(x-x_\nu)[f(1)-f(-1)]l_\nu(x)\\&=O(1)w(f,|(x-x_\nu)l_\nu(x)|)\\&=O(1)w(f,|\pi_n(x)|n^{-1}\Delta_n^{1/2}(x)).\end{aligned}$$

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[2]Here, we use a well known estimation $\frac{1}{|x-x_k|}=O(1)\frac{n^2}{|k-\nu||k+\nu|}$ $(k=1,2,\cdots,n)$.

By (3.6) and (2.9)

$$\begin{aligned}&(q_n(x)-q_n(x_\nu))l_\nu(x)\\&=O(1)\frac{|\pi_n(x)|}{n^2}n^2w_2\Big(f,\frac{1}{n}\Big)|P_{n-1}(x_\nu)|\\&=O(1)\frac{|\pi_n(x)|}{\sqrt{n}}nw_2\Big(f,\frac{1}{n}\Big).\end{aligned}$$

Thus

$$I_{11}=O(1)\Big(w(f,|\pi_n(x)|n^{-1}\Delta_n^{1/2}(x))+\frac{|\pi_n(x)|}{\sqrt{n}}nw_2\Big(f,\frac{1}{n}\Big)\Big). \tag{3.10}$$

Finally, it is easy to see that

$$\begin{aligned}I_{12}&=O(1)w_2\Big(f,\frac{1}{n}\Big)|\pi_n(x)|\Big\{\sum_{k\neq\nu}\frac{\sqrt{k}}{|k-\nu||k+\nu|}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\Big\}\\&=O(1)nw_2\Big(f,\frac{1}{n}\Big)|\pi_n(x)|n^{-1/2}.\end{aligned} \tag{3.11}$$

Concerning (3.9), (3.10) and (3.11) we obtain

$$I_1=O(1)\Big(\frac{|\pi_n(x)|}{\sqrt{n}}nw_2\Big(f,\frac{1}{n}\Big)+w(f,|\pi_n(x)|n^{-1}\Delta_n^{1/2}(x))\Big),$$

which together with (3.7), (3.8) gives (3.1). For $x<0$, the estimation runs similarly.

Theorem 3.1 is proved.

Theorem 3.2. *If $f\in C_{[-1,1]}$, then*

$$\|f(x)-R_n(f,0,x)\|=O\Big(nw_2\Big(f,\frac{1}{n}\Big)\Big).$$

Proof. From the proof of Theorem 3.1 we can see that

$$\begin{aligned}f(x)-R_n(f,0,x)=&(g(x_\nu)-q_n(x_\nu)-g(x)+q_n(x))l_\nu(x)\\&+O(1)\frac{|\pi_n(x)|}{n}nw_2\Big(f,\frac{1}{n}\Big).\end{aligned}$$

Thus,

$$g(x)-q_n(x)=O(1)w_2\Big(f,\frac{1}{n}\Big)$$

and $l_\nu(x)=O(1)$ imply

$$f(x)-R_n(f,0,x)=O(1)\Big(\frac{|\pi_n(x)|}{\sqrt{n}}nw_2\Big(f,\frac{1}{n}\Big)+w_2\Big(f,\frac{1}{n}\Big)\Big),$$

and $\pi_n(x)=O(1)\sqrt{n}$ gives

$$f(x)-R_n(f,0,x)=O(1)nw_2\Big(f,\frac{1}{n}\Big).$$

The theorem is proved.

Note. By the method of proving Theorem 3.1, we can easily prove that if $f\in C_{[-1,1]}$ and

$$\beta_{kn}=O\Big(\Delta_n^{-2}(x_k)w_2\Big(f,\frac{1}{n}\Big)\Big),$$

then

$$R_n(f,\beta,x)-f(x)=O(1)\Big(nw_2\Big(f,\frac{1}{n}\Big)\Big).$$

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